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# ON THE POINTS OF BILATERAL QUASICONTINUITY OF FUNCTIONS

**Abstract**

The set of all bilateral quasicontinuty points of real functions is investigated.

## 1. Introduction

A real function $f : \mathbb{R} \to \mathbb{R}$ is said to be quasicontinuous (cliquish) at $x \in \mathbb{R}$ if for every neighbourhood $U$ of $x$ and every $\varepsilon > 0$ there is a nonempty open set $G \subset U$ such that $|f(x) - f(y)| < \varepsilon$ for each $y \in G$ ($|f(y) - f(z)| < \varepsilon$ for each $y, z \in G$) [10].

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be left (right) hand sided quasicontinuous at a point $x \in \mathbb{R}$ if for every $\delta > 0$ and for every open neighbourhood $V$ of $f(x)$ there exists an open nonempty set $W \subset (x - \delta, x) \cap f^{-1}(V)$ $(W \subset (x, x + \delta) \cap f^{-1}(V))$. A function $f$ is bilaterally quasicontinuous at $x$ if it is both left and right hand sided quasicontinuous at this point [8].

Denote by $C(f), BQ(f), Q^-(f), Q^+(f), Q(f)$ and $A(f)$ the set of all continuity, bilateral quasicontinuity, left hand side quasicontinuity, right hand side quasicontinuity, quasicontinuity and cliquishness points of a function $f : \mathbb{R} \to \mathbb{R}$, respectively. The sets $Q(f)$ and $A(f)$ have been characterized in [9] and the triplet $(C(f), Q(f), A(f))$ has been characterized in [3], [4] and [5] (for a more general domain of $f$).

In the present paper we shall characterize the sixtuple $(C(f), BQ(f), Q^-(f), Q^+(f), Q(f), A(f))$. It is well-known that $C(f) \subset Q(f) \subset A(f)$, $C$ is a $G_\delta$ set, $A(f)$ is closed [9] and $A(f) \setminus C(f)$ is of the first category [12] (see also [3], [4], [5]). From the definition we have

$$C(f) \subset BQ(f) = Q^-(f) \cap Q^+(f) \subset Q^-(f) \cup Q^+(f) = Q(f).$$

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The letters $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{N}$ stand for the set of real, rational and natural numbers, respectively. For a subset $A$ of $\mathbb{R}$ denote by Cl $A$ and Int $A$ the closure and the interior of $A$, respectively. If $A$ is a subset of $\mathbb{R}$ and $x \in \mathbb{R}$ then $d(x, A) = \inf\{|x-a| : a \in A\}$.

## 2. Points of Bilateral Quasicontinuity

**Lemma 2.1** *Let $f : \mathbb{R} \to \mathbb{R}$. Then the set $Q(f) \setminus BQ(f)$ is countable.*

PROOF. Let $x \in Q(f) \setminus Q^+(f)$. Then there are $\varepsilon(x), \delta(x) > 0$ such that for every nonempty set $W \subset (x, x+\delta(x))$ there exists $z_W \in W$ with

$$|f(z_W) - f(x)| \geq 2\varepsilon(x). \tag{1}$$

Let $a(x), b(x), c(x)$ be rational numbers such that $x < c(x) < x + \delta(x)$ and $f(x) - \varepsilon(x) < a(x) < f(x) < b(x) < f(x) + \varepsilon(x)$ and let $\pi : Q(f) \setminus Q^+(f) \to \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$ be a mapping defined by

$$\pi(x) = (a(x), b(x), c(x)).$$

Suppose that $\pi(x_1) = \pi(x_2)$ for some $x_1, x_2 \in Q(f) \setminus Q^+(f)$, $x_1 < x_2$. Then $x_1 < x_2 < c(x_2) = c(x_1)$ and $f(x_1), f(x_2) \in (a(x_1), b(x_1))$. This yields $x_2 \in (x_1, x_1 + \delta(x_1))$ and $|f(x_1) - f(x_2)| < \varepsilon(x_1)$. Since $x_2 \in Q(f)$ there exists an open nonempty $G \subset (x_1, x_1 + \delta(x_1))$ such that $|f(t) - f(x_2)| < \varepsilon(x_1)$ for each $t \in G$. Hence for each $t \in G$ we have $|f(t) - f(x_1)| \leq |f(t) - f(x_2)| + |f(x_2) - f(x_1)| < 2\varepsilon(x_1)$, a contradiction with (1). Hence the mapping $\pi$ is injective and the set $Q(f) \setminus Q^+(f)$ is countable. Similarly we can prove that $Q(f) \setminus Q^-(f)$ is countable and hence $Q(f) \setminus BQ(f) = (Q(f) \setminus Q^+(f)) \cup (Q(f) \setminus Q^-(f))$ is countable. □

By a standard way we can prove

**Lemma 2.2** *If $f_1 : \mathbb{R} \to \mathbb{R}$ is left (right) hand sided continuous at $x \in \mathbb{R}$ and $f_2 : \mathbb{R} \to \mathbb{R}$ is left (right) hand sided quasicontinuous at $x$, then $f_1 + f_2$ is left (right) hand sided quasicontinuous at $x$.*
*If $f_1$ is continuous at $x$ and $f_2$ is cliquish at $x$, then $f_1 + f_2$ is cliquish at $x$.*

**Theorem 2.3** *Let $C, D, D_1, D_2, E, A$ be subsets of $\mathbb{R}$. Then $C = C(f)$, $D = BQ(f)$, $D_1 = Q^+(f)$, $D_2 = Q^-(f)$, $E = Q(f)$ and $A = A(f)$ for some $f : \mathbb{R} \to \mathbb{R}$ if and only if $C \subset D = D_1 \cap D_2 \subset D_1 \cup D_2 = E \subset A$, $C$ is a $G_\delta$ set, $A$ is closed, $A \setminus C$ is of the first category and $E \setminus D$ is countable.*

PROOF. Sufficiency follows from previous remarks and Lemma 2.1.
Necessity. The set $A \setminus C$ is a $F_\sigma$ set of the first category, hence by [13] we

can write $A \setminus C = \bigcup_{n=1}^{\infty} F_n$, where $F_n$ are closed nowhere dense and pairwise disjoint. For each $n \in \mathbb{N}$ we define $g_n : \mathbb{R} \to \mathbb{R}$,

$$g_n(x) = \begin{cases} 2^{-n} \cdot \sin \frac{1}{d(x,F_n)}, & \text{if } x \notin F_n, \\ 0, & \text{if } x \in F_n; \end{cases}$$

(and $g_n(x) = 0$ for $F_n = \emptyset$) and put

$$g = \sum_{n=1}^{\infty} g_n.$$

Then $g$ is continuous at each $x$ in $C \cup (\mathbb{R} \setminus A)$ and for $x \in F_n$, the function $\sum_{j \neq n} g_j$ is continuous at $x$. Moreover, we have for $x \in F_n$
$\liminf\limits_{u \to x^-} g(u) = \liminf\limits_{u \to x^+} g(u) = g(x) - 2^{-n}$,
$\limsup\limits_{u \to x^-} g(u) = \limsup\limits_{u \to x^+} g(u) = g(x) + 2^{-n}$.

Let $S, T$ be dense disjoint subsets of $\mathbb{R}$ such that $S \cup T = \mathbb{R}$. Define $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) = \begin{cases} d(x,A), & \text{if } x \in S, \\ 0, & \text{if } x \in T. \end{cases}$$

Then $h$ is continuous at each $x \in A$.

Since $E \setminus D$ is a countable set, the sets $D_1 \setminus D_2$ and $D_2 \setminus D_1$ are countable disjoint sets. Let $D_1 \setminus D_2 = \{a_1, a_2, \dots, a_n, \dots\}$, $D_2 \setminus D_1 = \{b_1, b_2, \dots, b_n, \dots\}$ (one-to-one sequences). For each $n \in \mathbb{N}$ define $k_n, m_n : \mathbb{R} \to \mathbb{R}$ by

$$k_n(x) = \begin{cases} 2^{-n}, & \text{if } x \geq a_n, \\ 0, & \text{otherwise}; \end{cases}$$

$$m_n(x) = \begin{cases} 2^{-n}, & \text{if } x \leq b_n, \\ 0, & \text{otherwise}; \end{cases}$$

and put

$$k = \sum_{n=1}^{\infty} k_n, \qquad m = \sum_{n=1}^{\infty} m_n.$$

Then $k$ is continuous at each point different from $a_n$ and $m$ is continuous at each point different from $b_n$. Moreover, $k$ is right hand sided continuous and $m$ is left hand sided continuous.

Now define $p : \mathbb{R} \to \mathbb{R}$ as follows

$$p(x) = \begin{cases} 0, & \text{if } x \in C \cup (\mathbb{R} \setminus A), \\ 2^{-n}, & \text{if } x \in F_n \cap E, \\ 2^{-n+1}, & \text{if } x \in F_n \setminus E. \end{cases}$$

Since $p(y) < 2^{-n+1}$ for each $y \in \mathbb{R} \setminus (F_1 \cup F_2 \cup \ldots \cup F_n)$, so $p$ is continuous at each point in $C \cup (\mathbb{R} \setminus A)$.

Finally, define $f : \mathbb{R} \to \mathbb{R}$ by

$$f = g + h + k + m + p.$$

We shall show that $f$ is our function.

1. Let $x \in \mathbb{R} \setminus A$. For each nonempty open $G \subset (x - \frac{1}{2}d(x, A), x + \frac{1}{2}d(x, A))$ we have $\operatorname{diam}(h(G)) \geq \frac{1}{2}d(x, A)$ and therefore $h$ is not cliquish at $x$. Since the functions $p, k, m, g$ are continuous at $x$, by Lemma 2.2 we deduce

$$\mathbb{R} \setminus A \subset \mathbb{R} \setminus A(f). \tag{2}$$

2. Let $x \in C$. Then all functions $p, k, m, h, g$ are continuous at $x$ and hence

$$C \subset C(f). \tag{3}$$

3. Let $x \in E \setminus C$. Then $x \in F_n \cap E$ for some $n \in \mathbb{N}$. Let $\varepsilon, \delta > 0$. Let $i \in \mathbb{N}$ be such that $2^{-i} < \frac{\varepsilon}{3}$. Since $\limsup_{u \to x^+} g_n(u) = 2^{-n}$, there is $w \in (x, x + \delta)$ such that $\max\{0, 2^{-n} - \frac{\varepsilon}{3}\} < g_n(w) \leq 2^{-n} = p(x)$. Then $w \notin F_n$ and hence there is an open neighbourhood $H$ of $w$ such that $|g_n(w) - g_n(y)| < \frac{\varepsilon}{3}$ for each $y \in H$. Now $G = H \cap (x, x+\delta) \setminus (F_1 \cup \ldots \cup F_i)$ is an open nonempty subset of $(x, x+\delta)$. Let $y \in G$. Then $0 \leq p(y) \leq 2^{-i}$ and hence $|p(y) + g_n(y) - p(x) - g_n(x)| \leq |p(y)| + |g_n(y) - g_n(w)| + |g_n(w) - p(x)| + |g_n(x)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + 0 = \varepsilon$. Therefore $g_n + p$ is right hand sided quasicontinuous at $x$. Similarly we can prove that $g_n + p$ is left hand sided quasicontinuous at $x$.

Now, if $x \in D_1 \setminus C$, then $m, h, \sum_{j \neq n} g_j$ and $k$ are right hand sided continuous at $x$ and $g_n + p$ is right hand sided quasicontinuous at $x$. Hence by Lemma 2.2 we have

$$D_1 \setminus C \subset Q^+(f). \tag{4}$$

If $x \in D_2 \setminus C$, then $k, h, \sum_{j \neq n} g_j$ and $m$ are left hand sided continuous at $x$ and $g_n + p$ is left hand sided quasicontinuous at $x$ and hence

$$D_2 \setminus C \subset Q^-(f). \tag{5}$$

4. Let $x \in A \setminus E$. Then $x \in F_n \setminus E$ for some $n \in \mathbb{N}$. Let $\varepsilon, \delta > 0$. Let $i \in \mathbb{N}$ be such that $2^{-i} < \frac{\varepsilon}{4}$. By the definition of $g_n$, there is an open nonempty set $H \subset (x - \delta, x + \delta)$ such that $|g_n(y)| < \frac{\varepsilon}{4}$ for each $y \in H$. Now $G = H \setminus (F_1 \cup \ldots \cup F_i)$ is an open nonempty subset of $(x - \delta, x + \delta)$ and for each $y \in G$ we have $0 \leq p(y) \leq 2^{-i}$. Hence for $y, z \in G$ we have $|g_n(y) + p(y) - g_n(z) - p(z)| \leq |g_n(y)| + |p(y)| + |g_n(z)| + |p(z)| < \varepsilon$. Therefore

$g_n + p$ is cliquish at $x$. Since $h, k, m, \sum_{j\neq n} g_j$ are continuous at $x$, by Lemma 2.2 we have

$$A \setminus E \subset A(f). \tag{6}$$

Now let $\alpha$ be such that $2^{-n} < \alpha < 2^{-n+1}$. Let $G$ be an arbitrary nonempty open set. Then there is $y \in G$ such that $p(y) = 0$ (because $A \setminus C$ is of the first category) and hence $g_n(y) + p(y) \leq 2^{-n} < \alpha$. The set $(\alpha, \infty)$ is a neighbourhood of $p(x) = 2^{-n+1}$ and $(g_n + p)(y) \notin (\alpha, \infty)$, thus $g_n + p$ is not quasicontinuous at $x$. Since the functions $k, m, h, \sum_{j\neq n} g_j$ are continuous at $x$, by Lemma 2.2 we deduce

$$A \setminus E \subset \mathbb{R} \setminus Q(f). \tag{7}$$

5. Let $x \in D \setminus C$. Then $x \in F_n \cap E$ for some $n \in \mathbb{N}$. Since $\liminf\limits_{u\to x} p(u) = 0$ and $\liminf\limits_{u\to x} g(u) = g(x) - 2^{-n}$ so $\liminf\limits_{u\to x}(p+g)(u) = g(x) - 2^{-n} \neq g(x) + 2^{-n} = (g+p)(x)$ and $g + p$ is not continuous at $x$. Since $k, m, h$ are continuous at $x$ we have

$$D \setminus C \subset \mathbb{R} \setminus C(f). \tag{8}$$

6. Let $x \in D_1 \setminus D_2$. Then there are $n, i \in \mathbb{N}$ such that $x = a_i \in F_n \cap E$. Let $\alpha$ be such that $2^{-n} < \alpha < 2^{-n} + 2^{-i}$. Let $G \subset (-\infty, x)$ be an arbitrary nonempty open set. Then there is $y \in G$ with $p(y) = 0$. The interval $(\alpha, \infty)$ is a neighbourhood of $(k_i + p + g_n)(x) = 2^{-i} + 2^{-n}$ and $(k_i + p + g_n)(y) \leq 2^{-n} < \alpha$, hence $k_i + p + g_n$ is not left hand sided quasicontinuous at $x$. Since $h, m, \sum_{j\neq n} g_j$ and $\sum_{j\neq i} k_j$ are continuous at $x$, by Lemma 2.2 we have

$$D_1 \setminus D_2 \subset \mathbb{R} \setminus Q^-(f). \tag{9}$$

Similarly we can prove

$$D_2 \setminus D_1 \subset \mathbb{R} \setminus Q^+(f). \tag{10}$$

Combining (3), (7), (8), (9), (10) and (2) we get $C = C(f)$.
Combining (3), (4), (5), (9), (10), (7) and (2) we get $D = BQ(f)$.
Combining (3), (4), (10), (7) and (2) we obtain $D_1 = Q^+(f)$.
Combining (3), (5), (9), (7) and (2) we obtain $D_2 = Q^-(f)$.
The conditions (3), (4), (5), (7) and (2) imply $E = Q(f)$.
Finally, (3), (2), (6), (4) and (5) imply $A = A(f)$. □

If we put $D_1 = D$ and $D_2 = E$ we obtain

**Corollary 2.4** *Let $C, D, E$ and $A$ be subsets of $\mathbb{R}$. Then $C = C(f)$, $D = BQ(f)$, $E = Q(f)$ and $A = A(f)$ for some $f : \mathbb{R} \to \mathbb{R}$ if and only if $C \subset D \subset E \subset A$, $C$ is a $G_\delta$ set, $A$ is closed, $A \setminus C$ is of the first category and $E \setminus D$ is countable.*

We recall that a set $A$ is quasiopen (by some authors semi-open) if $A \subset \operatorname{Cl} \operatorname{Int} A$ [11].

**Theorem 2.5** *Let $D$ be a subset of reals. Then the following conditions are equivalent:*

*(a) There is $f : \mathbb{R} \to \mathbb{R}$ such that $D = BQ(f)$.*

*(b) There is $f : \mathbb{R} \to \mathbb{R}$ such that $D = Q(f)$.*

*(c) The set $\operatorname{Cl} D \setminus D$ is of the first category .*

*(d) The set $\operatorname{Int} \operatorname{Cl} D \setminus D$ is of the first category .*

*(e) There are a $G_\delta$ set $C$ and a closed set $A$ such that $A \setminus C$ is of the first category and $C \subset D \subset A$.*

*(f) There is a decreasing sequence $\{W_n\}$ of open sets such that $\bigcap_{n=1}^{\infty} W_n \subset D \subset \bigcap_{n=1}^{\infty} \operatorname{Cl} W_n$.*

*(g) There is a sequence $\{D_n\}$ of quasiopen sets such that $D = \bigcap_{n=1}^{\infty} D_n$.*

Proof.

(a)$\Rightarrow$(e): We put $C = C(f)$ and $A = A(f)$.

(e)$\Rightarrow$(c): We have $\operatorname{Cl} D \setminus D \subset A \setminus C$.

(c)$\Rightarrow$(d): We have $\operatorname{Int} \operatorname{Cl} D \setminus D \subset \operatorname{Cl} D \setminus D$.

(d)$\Rightarrow$(b): It follows from Theorem 3 in [9].

(b)$\Rightarrow$(f): It follows from Theorem 1 in [5].

(f)$\Rightarrow$(g): We put $D_n = W_n \cup D$. Then we have $D = \bigcap_{n=1}^{\infty} D_n$. Further, $W_n \subset D_n$ implies $\operatorname{Cl} W_n \subset \operatorname{Cl} \operatorname{Int} D_n$ and hence $D_n = W_n \cup D \subset \operatorname{Cl} W_n \subset \operatorname{Cl} \operatorname{Int} D_n$, thus the sets $D_n$ are quasiopen.

(g)$\Rightarrow$(a). Put $C = \bigcap_{n=1}^{\infty} \operatorname{Int} D_n$, $E = D$, $A = \bigcap_{n=1}^{\infty} \operatorname{Cl} \operatorname{Int} D_n$. Then we have $C \subset D \subset E$, $C$ is a $G_\delta$ set and $A$ is a closed set. Since $A \setminus C \subset \bigcup_{n=1}^{\infty} (\operatorname{Cl} \operatorname{Int} D_n \setminus \operatorname{Int} D_n)$, the set $A \setminus C$ is of the first category. Finally, since $D_n \subset \operatorname{Cl} \operatorname{Int} D_n$, we have $D = E = \bigcap_{n=1}^{\infty} D_n \subset \bigcap_{n=1}^{\infty} \operatorname{Cl} \operatorname{Int} D_n = A$. Now we apply Corollary 2.4. $\square$

## 3. Limits

If $\mathcal{F}$ is a family of functions $f : \mathbb{R} \to \mathbb{R}$, then $B(\mathcal{F})$, $U(\mathcal{F})$ and $D(\mathcal{F})$ denote the collection of all pointwise, uniform and quasiuniform limits of sequences taken from $\mathcal{F}$, respectively. Recall that a sequence $\{f_n\}$, $f_n : \mathbb{R} \to \mathbb{R}$, quasiuniformly converges to $f : \mathbb{R} \to \mathbb{R}$ [14;p.143] if the sequence $\{f_n\}$ pointwise converges to $f$ and $\forall \varepsilon > 0\ \forall m \in \mathbb{N}\ \exists p \in \mathbb{N}\ \forall x \in \mathbb{R}\ \min\{|f_{m+1}(x) - f(x)|, \ldots, |f_{m+p}(x) - f(x)|\} < \varepsilon$.

Denote by $\mathcal{Q}$, $\mathcal{BQ}$ and $\mathcal{K}$ the family of all quasicontinuous, bilaterally quasicontinuous and cliquish functions $f : \mathbb{R} \to \mathbb{R}$, respectively. It is well-known that $U(\mathcal{Q}) = \mathcal{Q}$ and $U(\mathcal{K}) = \mathcal{K}$ [10], $B(\mathcal{Q}) = \mathcal{K}$ and $B(\mathcal{K})$ is the family of all functions with the Baire property [7] and $D(\mathcal{K}) = \mathcal{K}$ [2]. A standard proof shows that the uniform limit of a sequence of bilaterally quasicontinuous functions is bilaterally quasicontinuous. In [6] it is shown that every cliquish $f : \mathbb{R} \to \mathbb{R}$ is the quasiuniform limit of a sequence of Darboux quasicontinuous functions. From previous remarks and Lemma 3.2 we obtain

**Theorem 3.1** *We have $U(\mathcal{BQ}) = \mathcal{BQ}$, $D(\mathcal{BQ}) = B(\mathcal{BQ}) = \mathcal{K}$.*

**Lemma 3.2** *Let $f : \mathbb{R} \to \mathbb{R}$ be Darboux and quasicontinuous. Then it is bilaterally quasicontinuous.*

PROOF. Let $x \in \mathbb{R}$, $\varepsilon, \delta > 0$. If $f(y) = f(x)$ for each $y \in (x, x+\delta)$, then evidently $f$ is right hand sided quasicontinuous at $x$. Let $f(y) \neq f(x)$ for some $y \in (x, x+\delta)$ and let e.g. $f(x) < f(y)$. If $\alpha \in (f(x), \min\{f(y), f(x)+\varepsilon\})$, then there is $z \in (x, y)$ with $f(z) = \alpha$. Now there is an open nonempty $G \subset (x, y)$ such that $f(G) \subset (f(x), f(x)+\varepsilon))$, thus $f$ is right hand sided quasicontinuous at $x$. □

**Remark 3.1** *The converse of Lemma 3.2 is not true. Let $C$ be the Cantor set. We can arrange the set of all complementary intervals in $[0,1]$ in a one-to-one sequence $\{I_n\}$ such that the sets $\bigcup_{n=1}^{\infty} I_{2n}$ and $\bigcup_{n=1}^{\infty} I_{2n-1}$ are dense in $C$. Now the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 1$ for $x \in \bigcup_{n=1}^{\infty} Cl\, I_{2n}$ and $f(x) = 0$ otherwise, is bilaterally quasicontinuous but not Darboux.*

*However, if $f$ is in Baire class 1, then $f$ is Darboux and quasicontinuous if and only if it is bilaterally quasicontinuous. In fact, if $f$ is bilaterally quasicontinuous then it has the Young property (i.e. for each $x$ there exist sequences $x_n \uparrow x$ and $y_n \downarrow x$ such that $f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n)$ ) and hence by [1;p.9] it is Darboux.*

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