Real Analysis Exchange
Vol. 19(2), 1993/94, pp. 510–515

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A FIRST RETURN CHARACTERIZATION FOR BAIRE ONE FUNCTIONS

Abstract

A new characterization of the class of Baire 1 functions is given in terms of the notion of first return recoverability.

In a previous paper [0] by the present authors, a first return characterization was given for the class of Darboux Baire 1 functions utilizing the notion of first return continuity. In this paper we broaden the concept of first return continuity to first return recoverability to give a characterization of the class of Baire 1 functions. The functions considered here are real-valued and defined on $[0,1]$. However, the characterization actually holds for Baire 1 functions $f : X \to Y$, where X is a compact metric space and Y is a separable metric space [0]. The proof presented here for the special case utilizes the ordering properties of the line and, consequently, is markedly simpler than the proof for the general theorem. We begin by defining our terminology.

By a *trajectory* we mean any sequence $\{x_n\}_{n=0}^{\infty}$ of distinct points in $[0,1]$, which is dense in $[0,1]$. Let $\{x_n\}$ be a fixed trajectory and let $y \in [0,1]$. We define what we shall mean by the *first return route to y based on the trajectory* $\{x_n\}$. If $\rho > 0$, we use $B_\rho(y)$ to denote $\{x \in [0,1] : |x-y| < \rho\}$. We let $r\left(B_\rho(y)\right)$ denote the first element of the trajectory in $B_\rho(y)$. The *first return route to* y, $\mathcal{R}_y = \{y_k\}_{k=1}^{\infty}$, is defined recursively via

$$y_1 = x_0,$$

$$y_{k+1} = \begin{cases} r\left(B_{|y-y_k|}(y)\right) & \text{if } y \neq y_k \\ y_k & \text{if } y = y_k. \end{cases}$$

Key Words: Baire 1, first return

Mathematical Reviews subject classification: 26A21

We say that $f : [0,1] \to \mathbb{R}$ is *first return recoverable with respect to* $\{x_n\}$ provided that for each $y \in [0,1]$ we have

$$\lim_{k\to\infty} f(y_k) = f(y),$$

and that f is *first return recoverable* if there exists a trajectory $\{x_n\}$ such that f is first return recoverable with respect to $\{x_n\}$.

Theorem 1 *A function $f : [0,1] \to \mathbb{R}$ is first return recoverable if and only if it is of Baire class one.*

Proof. Suppose that f is a Baire 1 function. Without loss of generality we may assume that f is bounded. For otherwise, we take $\arctan(f)$ and obtain a trajectory $\{x_n\}$ so that $\arctan(f)$ is first return recoverable with respect to $\{x_n\}$. Then, it is an easy matter to verify that f is first return recoverable with respect to $\{x_n\}$ as well.

Since f is of Baire class one, the set of points of discontinuity of f is a first category $\mathcal{F}_\sigma$ set. Enlarge this set to a first category $\mathcal{F}_\sigma$ set F which is dense in $[0,1]$. Since f is bounded and of Baire class one, we may find, using techniques of Kuratowski, a sequence of Baire class one characteristic functions $\{h_j\}$ and a sequence of real numbers $\{a_j\}$ such that the series $\sum_{j=1}^{\infty} a_j h_j$ converges uniformly to f on $[0,1]$. (See Theorem 3, p. 388 in [0].) For each j we let $M_i^j = h_j^{-1}(i)$, $i = 0,1$.

We are going to define a collection of closed sets

$$\mathcal{G} = \{G_\nu : \nu \text{ is a finite sequence of natural numbers.}\}$$

(We shall denote the length of such a finite sequence, ν, of natural numbers by $|\nu|$. We denote the k^{th} term of ν by $\nu(k)$, and if ν has length at least n, we let $\nu|_n$ denote the truncated sequence $\{\nu(1), \nu(2), \ldots, \nu(n)\}$. If $\tau = \nu|_n$ for some n, then we say that ν is an extension of τ. Finally, if $|\nu| = n$ and i is a natural number, we let νi denote the sequence $\{\nu(1), \nu(2), \ldots, \nu(n), i\}$.) We shall inductively define our collection so that

1) Each G_ν is a closed set.

2) For each natural number n, $\bigcup_{|\nu|=n} G_\nu = F$.

3) If $\nu \neq \tau$, and neither is an extension of the other, then $G_\nu \cap G_\tau = \emptyset$,

4) If τ is an extension of ν, then $G_\tau \subseteq G_\nu$.

5) For each n, if $|\nu| = n$, then h_j is constant on G_ν for $j = 1, 2, \ldots, n$.

We induct on the length of ν: Using the well known fact that a first category $\mathcal{F}_\sigma$ subset of the line can be written as a countable union of disjoint closed sets, we first express $M_0^1 \cap F$ as a countable union of disjoint closed sets, and similarly express $M_1^1 \cap F$ as a countable union of disjoint closed sets. We combine these two collections into one collection of disjoint closed sets and enumerate it as $\{G_i\}_{i=1}^\infty$. (Without loss of generality we shall assume that $G_1 \neq \emptyset$.) Note that conditions 1) – 5) hold for $n = 1$.

Next, assume that G_ν has been defined for all ν of length at most k and that 1)–5) are satisfied for $n = k$. Fix a ν of length k. Since both $M_0^{k+1} \cap G_\nu$ and $M_1^{k+1} \cap G_\nu$ are first category $\mathcal{F}_\sigma$ sets, we may express each as a countable union of disjoint closed sets. We combine these two collections to form the collection $\{G_{\nu i}\}_{i=1}^\infty$. Doing this for each ν of length k, we obtain the collection $\{G_\tau\}$, where each τ has length $k+1$. It is a straightforward matter to see that 1)–5) are valid for $n = k+1$. In this manner we have completed the definition of the collection $\mathcal{G}$. Throughout the following, we let $\mathcal{G}_k = \{G_\nu : |\nu| \leq k$ and each term of ν is at most $k\}$.

Next, we select the required trajectory $\{x_n\}$ from points in F. We shall do this inductively by stages, selecting a natural number n_k at the k^{th} stage, partitioning $[0,1]$ into intervals of length $1/2^{n_k}$, then selecting and ordering some points from F in some of these intervals. At the end of the k^{th} stage, we want $\{x_l\}_{l=0}^{m_k}$, the trajectory defined up to this point, to satisfy the following properties:

i) If $x \in G_\nu \in \mathcal{G}_k$, then the nearest point of $\{x_l\}_{l=0}^{m_k}$ to x is in G_ν, and

ii) If $x \in G_\nu \in \mathcal{G}_{k-1}$, x_l is in the first return route to x, and $m_{k-1} < l \leq m_k$, then $x_l \in G_\nu$.

At the first stage ($k = 1$), we set $n_1 = 1$ and partition $[0,1]$ into two intervals of length $1/2$. If $G_1 \cap [0, 1/2] \neq \emptyset$, we select both the maximum and the minimum of this intersection. Similarly, if $G_1 \cap [1/2, 1] \neq \emptyset$, then we select maximum and the minimum of this intersection. Thus, we have selected at least one and at most four distinct points, which we label from left to right as $x_0, x_1, \ldots, x_{m_1}$. Note that both conditions i) and ii) are satisfied at this stage.

Assume that stage k has been completed, that the points $x_0, x_1, \ldots, x_{m_k}$ have been specified and conditions i) and ii) are satisfied at this stage. Choose $n_{k+1} > n_k$ so large that

a) Each partition interval of length $1/2^{n_{k+1}}$ contains at most one of the points $x_0, x_1, \ldots, x_{m_k}$, and

b) For each $1 \leq l \leq k+1$, each partition interval of length $1/2^{n_{k+1}}$ contains points from at most one of $\{G_\nu \in \mathcal{G}_{k+1} : |\nu| = l\}$.

We describe how to select the points to be added to the trajectory at this stage and then we shall explain how to order these newly selected points.

Look at any partition interval of length $1/2^{n_{k+1}}$ which intersects $\bigcup\{G_\nu : G_\nu \in \mathcal{G}_{k+1}\}$. Fix this interval. First, note that if both G_ν and G_τ intersect this interval, then either τ is an extension of ν or ν is an extension of τ. Let G_ν be the unique element of $\mathcal{G}_{k+1}$ having the longest ν such that G_ν intersects this interval. Say $|\nu| = m$, and note that $m \leq k+1$. Select the maximum and minimum of G_ν in this interval, and do the same for $G_{\nu|_{m-1}}, G_{\nu|_{m-2}}, \ldots, G_{\nu|_1}$. Now, repeat this for each interval in the partition. If a partition interval misses $\bigcup\{G_\nu : G_\nu \in \mathcal{G}_{k+1}\}$, we don't select any points from that interval at this stage.

We have now selected all the points which we wish to add to the trajectory at this stage, and have yet to describe how to order these points, or rather those which have not already appeared in the trajectory construction. We first define an ordering on the ν's for which $G_\nu \in \mathcal{G}_{k+1}$. For each $j \leq k$ order the ν's of length j whose terms involve only $1, 2, \ldots, k$ in any manner as

$$\nu_1^j < \nu_2^j < \ldots < \nu_{i_j}^j.$$

Also, arbitrarily order those ν's for which $G_\nu \in \mathcal{G}_{k+1}$ and which either are of length $k+1$ or have $k+1$ as one of their terms as

$$\nu_1 < \nu_2 < \ldots < \nu_{i_{k+1}}.$$

Our ordering scheme on the ν's for which $G_\nu \in \mathcal{G}_{k+1}$ is then

$$\begin{gathered} \nu_1^k < \nu_2^k < \ldots < \nu_{i_k}^k < \\ < \nu_1^{k-1} < \nu_2^{k-1} < \ldots < \nu_{i_{k-1}}^{k-1} < \\ < \nu_1^{k-2} < \nu_2^{k-2} < \ldots < \nu_{i_{k-2}}^{k-2} < \\ \vdots \\ < \nu_1^1 < \nu_2^1 < \ldots < \nu_{i_1}^1 < \\ < \nu_1 < \nu_2 < \ldots < \nu_{i_{k+1}}. \end{gathered}$$

Now add the points selected at this stage $k+1$ to the trajectory in the following order. First look at the newly selected points of $G_{\nu_1^k}$ and order them from left to right and label them as x_n's beginning with x_{m_k+1}. Then look at the newly selected points of $G_{\nu_2^k}$ and order them from left to right, etc., continuing to follow the ordering of the ν's shown above. (Keep in mind that a point only gets listed in the trajectory once.)

Let us now show that conditions i) and ii) are satisfied at the end of stage $k+1$. To show that condition i) holds, assume that $x \in G_\tau \in \mathcal{G}_{k+1}$. If $x = x_l$ for some $l \leq m_{k+1}$ then we are done. If not, let J be the interval containing x of length $1/2^{n_{k+1}}$ from the partition considered at stage $k+1$. Let G_ν be the unique element of $\mathcal{G}_{k+1}$ having the longest ν which intersects J. Note that ν is an extension of τ. Let $|\nu| = m$. Since the only points of J we pick are the max and min of J intersected with each of $G_\nu, G_{\nu|_{m-1}}, G_{\nu|_{m-2}}, \ldots, G_{\nu|_1}$ and the sequence $\{G_\nu, G_{\nu|_{m-1}}, G_{\nu|_{m-2}}, \ldots, G_{\nu|_1}\}$ is monotonically increasing, we have that i) holds.

To show that condition ii) holds, assume that $x \in G_\tau \in \mathcal{G}_k$. If $x = x_l$ for some $1 \leq l \leq m_k$ then we are done. If that is not the case, let I be the interval of length $1/2^{n_k}$ of the partition considered at stage k which contains x and the nearest element of $\mathcal{R}_x$ restricted to $\{x_l\}_{l=0}^{m_k}$. Let G_ν be the unique element of $\mathcal{G}_k$ having the longest ν which intersects I. Note that ν is an extension of τ. At stage $k+1$, interval I is subdivided into a finite collection $\mathcal{H}$ of intervals of length $1/2^{n_{k+1}}$. Suppose μ is such that $G_\mu \in \mathcal{G}_{k+1}$ and G_μ intersects some $J \in \mathcal{H}$. Then, the points picked at stage $k+1$ from J are either in G_τ or they are labeled after the nearest point to x is labeled. Thus, condition ii) is satisfied.

This completes the construction of the trajectory $\{x_n\}$. (That $\{x_n\}$ is dense in $[0,1]$ follows from the fact that F is dense there.)

Now we must show that f is first return recoverable with respect to $\{x_n\}$. Clearly, the only points in doubt are those in $F \setminus \{x_n\}$. Let x be such a point and let $\epsilon > 0$. Choose N so large that

$$\left| f(t) - \sum_{j=1}^{N} a_j h_j(t) \right| < \frac{\epsilon}{2}$$

for all $t \in [0,1]$. Let $\mathcal{R}_x$ denote the first return route to x based on the trajectory $\{x_n\}$. Let ν be that unique sequence of length N such that $x \in G_\nu$. Let

$$M = \max\{N, \nu(1), \nu(2), \ldots, \nu(N)\}.$$

Let $t \in \mathcal{R}_x$ be such that t is added to the trajectory at stage $M+1$ or later. By properties i) and ii) it follows that $t \in G_\nu$ as well. Since each of h_i is constant on G_ν for all $i = 1, 2, \ldots, N$, we have that

$$h_i(t) = h_i(x) \text{ for all } i = 1, 2, \ldots, N.$$

Thus for all such $t \in \mathcal{R}_x$ we have

$$|f(x) - f(t)| \leq \left| \sum_{i=1}^{N} a_i h_i(x) - \sum_{i=1}^{N} a_j h_j(t) \right| + \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

completing the proof for one direction of the theorem.

The proof for the other direction is much shorter and more straightforward, even in the general situation considered in [0]. Thus, we do not repeat the argument here.

References

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