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CHARACTERIZATIONS OF $AC^*G \cap C$, $\underline{AC}^*G \cap C_i$, AC AND $\underline{AC}$ FUNCTIONS

In connection with the study of AC^*G functions, Lee Peng Yee introduced a condition which lies somewhere between AC and Lusin's condition (N), and it is called the strong Lusin condition. This condition also appears in Bordaon's Lemma 2 of [4]. (In [7] Lee and Vyborny mentioned that this condition was also studied by Kurzweil, Jarnik and Schwabik.) Denoting this condition by Y_{D°, we show that $Y_{D^\circ} = AC^*G \cap C$ on a closed interval.

There are also given several characterizations for the classes $\underline{AC}^*G \cap C_i$, AC and $\underline{AC}$. For these tasks we have developed a study of various interesting conditions, such as: $VB, \underline{VB}, VB^*, \underline{VB}^*, AC, \underline{AC}, AC^*, \underline{AC}^*, AC^{**}, \underline{AC}^{**}, AC_{D\#}, \underline{AC}_{D\#}, AC_{D^\circ}, \underline{AC}_{D^\circ}, AC_D, \underline{AC}_D, Y_{D\#}, \underline{Y}_{D\#}, Y_{D^\circ}, \underline{Y}_{D^\circ}, Y_D, \underline{Y}_D$ (AC_{D° and AC_D were introduced by Gordon in [4]).

1. Preliminaries

For convenience, if T is a property for functions defined on a certain domain, we will also use T to denote the class of all functions having this property. We denote by C the class of all continuous functions. We denote by $\bar{A}$ the closure of the set A. Let $O(F;X)$ denote the oscillation of F on the set X.

Definition 1 *Let $F : [a,b] \to \mathbb{R}$ and let P be a subset of $[a,b]$. F will be said to be TG on P, if P can be expressed as the union of a countable sequence of sets P_i, over each of which F satisfies property T.*

Definition 2 *Let $F : [a,b] \to \mathbb{R}$ and let $\phi \neq X \subseteq Y \subseteq [a,b]$. Let*

$$\Omega(F;Y \wedge X) = \sup\{|F(y) - F(x)| : x \le y,\ x,y \in Y \text{ and } \{x,y\} \cap X \neq \emptyset\};$$
$$\Omega_-(F;Y \wedge X) = \inf\{F(y) - F(x) : x \le y,\ x,y \in Y \text{ and } \{x,y\} \cap X \neq \emptyset\};$$
$$\Omega_+(F;Y \wedge X) = \sup\{F(y) - F(x) : x \le y,\ x,y \in Y \text{ and } \{x,y\} \cap X \neq \emptyset\}.$$

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Remark 1 *If $x, y \in P$, $x < y$, then*

$$O(F;[x,y]) \leq F(x) - F(y) + 2\Omega_+(F;[x,y] \wedge (P \cap [x,y])).$$

Definition 3 *Let $P \subset [a,b]$, $x \in P$ and let $F : P \to \mathbb{R}$. F is said to be C_i at x, if for each $\varepsilon > 0$ there exists a $\delta(x) > 0$ such that $\Omega_-(F;(P \cap (x-\delta(x), x+\delta(x)) \wedge \{x\}) > -\varepsilon$. F is said to be C_i on P if F is C_i at each $x \in P$.*

Let $C_d = \{F : -F \in C_i\}$. Clearly $C = C_d \cap C_i$ on P.

Lemma 1 *([11], p.236). Let $F : [a,b] \to \mathbb{R}$. Then the set $\{x : F'(x) = +\infty\}$ is of measure zero.*

Conditions $VB, \underline{VB}, VB^*, \underline{VB}^*$

Following Ridder (see [9], pp.235,236,251), it is natural to define conditions $\underline{VB}$ and $\underline{VB}^*$.

Definition 4 *Let $F : [a,b] \to \mathbb{R}$ and let P be a subset of $[a,b]$. F is said to be VB (respectively $\underline{VB}$) on P, if there exists $M \in (0,+\infty)$ such that*

$$\sum_{k=1}^{n} |F(b_k) - F(a_k)| < M \text{ (respectively } \sum_{k=1}^{n} (F(b_k) - F(a_k)) > -M),$$

whenever $\{[a_k,b_k]\}$, $k = 1,2,\ldots,n$ is a finite set of nonoverlapping closed intervals with endpoints in P. F is said to be $\overline{VB}$ on P, if $-F \in \underline{VB}$ on P. Clearly $VB = \underline{VB} \cap \overline{VB}$. We define VBG using Definition 1.

Definition 5 *Let $F : [a,b] \to \mathbb{R}$, $P \subset [a,b]$. Let $V(F;P) = \sup\{\sum_i |F(b_i) - F(a_i)| : \{[a_i,b_i]\}_i$ is a sequence of nonoverlapping closed intervals, with $a_i, b_i \in P\}$. If $F \in VB$ on P, then $V(F;P) = \inf\{M : M$ is given by the fact that $F \in VB$ on $P\}$. Let $\underline{V}(F;P) = \inf\{\sum_i (F(b_i) - F(a_i)) : \{[a_i,b_i]\}_i$ is a sequence of nonoverlapping closed intervals, with $a_i, b_i \in P\}$. If $F \in \underline{VB}$ on P, then $\underline{V}(F;P) = \inf\{M : M$ is given by the fact that $F \in \underline{VB}$ on $P\}$.*

Definition 6 *Let $F : [a,b] \to \mathbb{R}$ and let $P \subset [a,b]$. F is said to be VB^* on P (respectively $\underline{VB}^*$ on P) if there exists $M \in (0,+\infty)$ such that*

$$\sum_{k=1}^{n} O(F;[a_k,b_k]) < M \text{ (respectively } \sum_{k=1}^{n} \Omega_-(F;[a_k,b_k] \wedge (P \cap [a_k,b_k])) > -M),$$

whenever $\{[a_k,b_k]\}$, $k = 1,2,\ldots,n$, is a finite set of nonoverlapping closed intervals with $a_k, b_k \in P$. Let $\overline{VB}^* = \{F : -F \in \underline{VB}^*\}$. Clearly $VB^* = \underline{VB}^* \cap$

$\overline{VB}^*$. *We define* VB^*G *using Definition 1. Let* $V^*(F;P) = \sup\{\sum_i O(F;[a_i,b_i$ $\{[a_i,b_i]\}_i$ *is a sequence of nonoverlapping closed intervals with* $a_i, b_i \in P\}$. *If* $F \in VB^*$, *then* $V^*(F;P) = \inf\{M : M$ *is given by the fact that* $F \in VB^*$ *on* $P\}$.

Theorem 1 *Let* $F : [a,b] \to \mathbb{R}$, $P \subset [a,b]$, $c = \inf(P)$, $d = \sup(P)$. *Then the following assertions are equivalent:*

(i) $F \in VB$ *on* P;

(ii) $F \in \underline{VB} \cap \overline{VB}$ *on* P;

(iii) *there exists* $M \in (0,+\infty)$ *such that* $\sum_{i=1}^{n-1} |F(x_i) - F(x_{i-1})| < M$, *whenever* $c = x_0 < x_1 < \cdots < x_{n-1} < x_n = d$ *and* $x_i \in P$, $i = 1,2,\ldots,n-1$;

(iv) $F \in \underline{VB}$ *on* $P \cup \{c,d\}$;

(v) F *is bounded and* $\underline{VB}$ *on* P.

PROOF. (i) $\Rightarrow$ (ii) is evident.

(ii) $\Rightarrow$ (i) There exists $M \in (0,+\infty)$ which satisfies both definitions, $\underline{VB}$ and $\overline{VB}$ on P. Let $\{[a_k,b_k]\}$, $k = 1,2,\ldots,n$, be a finite set of nonoverlapping closed intervals, $a_k, b_k \in P$. Let $A_1 = \{k : F(b_k) \geq F(a_k)\}$ and $A_2 = \{k : F(b_k) < F(a_k)\}$. Then $A_1 \cup A_2 = \{1,2,\ldots,n\}$. We have $\sum_{k\in A_1}(F(b_k) - F(a_k)) < M$ and $\sum_{k\in A_2}(F(b_k) - F(a_k)) > -M$. Hence $\sum_{k=1}^{n} |F(b_k) - F(a_k)| < 2M$. Thus $F \in VB$ on P.

(i) $\Rightarrow$ (iii) Let $M \in (0,+\infty)$ be a constant given by the fact that $F \in VB$ on P. Let $x_0 \in P$. Then for each $x \in P$ we have $|F(x) - F(x_0)| < M$. Hence F is bounded on P. Since $F(c)$ and $F(d)$ are real numbers, it follows that F is bounded on $P \cup \{c,d\}$. Let $\alpha \in (0,+\infty)$ such that $|F(x)| < \alpha$, for each $x \in P \cup \{c,d\}$, and let $c = x_0 < x_1 < \cdots < x_{p-1} < x_p = d$, $x_1,\ldots,x_{p-1} \in P$. Then we have

$$\sum_{i=1}^{p-1} |F(x_{i+1}) - F(x_i)| = |F(x_1) - F(x_0)| + \sum_{i=1}^{p-2} |F(x_{i+1}) - F(x_i)|$$
$$+|F(x_p) - F(x_{p-1})| < 2\alpha + M + 2\alpha = 4\alpha + M.$$

(iii) $\Rightarrow$ (i) is evident.

(iii) $\Rightarrow$ (iv) follows by the definition of $\underline{VB}$.

(iv) $\Rightarrow$ (v) Let $M \in (0,+\infty)$ be a constant given by the fact that $F \in \underline{VB}$ on $P \cup \{c,d\}$. Let $x \in P$. Then $-M < F(x) - F(c)$ and $-M < F(d) - F(x)$. It follows that $F(c) - M \leq F(x) \leq M + F(d)$, for each $x \in P$. Hence F is bounded on P.

(v) $\Rightarrow$ (iii) Let $M \in (0,+\infty)$ be a constant given by the fact that F is $\underline{VB}$ on P, and let $\alpha \in (0,+\infty)$ such that $|F(x)| < \alpha$, for each $x \in P$. Let $c = x_0 < x_1 < \cdots < x_{n-1} < x_n = d$, $x_i \in P$, $i = 1,2,\dots,n-1$. Let $A_1 = \{i \in \{2,3,\dots,n-2\} : F(x_i) - F(x_{i-1}) \geq 0\}$ and $A_2 = \{i \in \{2,3,\dots,n-2\} : F(x_i) - F(x_{i-1}) < 0\}$. Then $A_1 \cup A_2 = \{2,3,\dots,n-2\}$ and $A_1 \cap A_2 = \emptyset$. We have $F(x_{n-2}) - F(x_1) = \sum_{i=2}^{n-2}(F(x_i) - F(x_{i-1})) = \sum_{i\in A_1} |F(x_i) - F(x_{i-1})| - \sum_{i\in A_2} |F(x_i) - F(x_{i-1})|$. It follows that $\sum_{i=0}^{n-1} |F(x_{i+1}) - F(x_i)| = |F(x_1) - F(x_0)| + |F(x_n) - F(x_{n-1})| + \sum_{i=1}^{n-2} |F(x_{i+1}) - F(x_i)| \leq 4\alpha + F(x_{n-1}) - F(x_1) - 2\sum_{i\in A_2}(F(x_i) - F(x_{i-1})) < 6\alpha + 2M$.

Theorem 2 *Let $F : [a,b] \to \mathbb{R}$, $P \subset [a,b]$. The following assertions are equivalent:*

(i) *$F \in \underline{VB}^*$ on P;*

(ii) *there exists $M \in (0,+\infty)$ such that $\sum_{k=1}^{n}(F(x_k) - F(a_k)) \geq -M$ and $\sum_{k=1}^{n}(F(b_k) - F(x_k)) \geq -M$, whenever $\{[a_k,b_k]\}$, $k = 1,2,\dots,n$, is a finite set of nonoverlapping closed intervals, with $a_k, b_k \in P$ and $x_k \in [a_k,b_k]$;*

(iii) *there exists $M \in (0,+\infty)$ such that $\sum_{k=1}^{n} \Omega_-(F;[a_k,b_k] \wedge \{a_k,b_k\}) \geq -M$, whenever $\{[a_k,b_k]\}$, $k = 1,2,\dots,n$, is a finite set of nonoverlapping closed intervals, with $a_k, b_k \in P$.*

PROOF. (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) are evident.

(ii) $\Rightarrow$ (i) We may suppose without loss of generality that $\alpha_k = \Omega_-(F;[a_k,b_k] \wedge (P \cap [a_k,b_k])) < 0$, whenever $\{[a_k,b_k]\}$, $k = 1,2,\dots,n$ are as in (i). Then there exist $x_k, y_k \in [a_k,b_k]$, $x_k < y_k$, such that at least one of them belongs to P and $\frac{1}{2}\alpha_k > F(y_k) - F(x_k)$. We consider only the case when all $x_k \in P$ (the other situations are similar). Clearly $[x_k,b_k]$, $k = 1,2,\dots,n$, are nonoverlapping closed intervals, with $x_k, b_k \in P$. Hence by (ii), it follows that $\frac{1}{2}\sum_{k=1}^{n} \alpha_k > \sum_{k=1}^{n}(F(y_k) - F(x_k)) > -M$.

Theorem 3 *Let $F : [a,b] \to \mathbb{R}$, $P \subset [a,b]$, $c = \inf(P)$, $d = \sup(P)$. Then the following assertions are equivalent:*

(i) *$F \in VB^*$ on P;*

(ii) *$F \in VB^*$ on $\overline{P}$;*

(iii) *$F \in \overline{VB}^* \cap \underline{VB}^*$ on P;*

(iv) *$F \in VB \cap \underline{VB}^*$ on P;*

(v) *$F \in \overline{VB} \cap \underline{VB}^*$ on P;*

(vi) $F \in \underline{VB}^*$ *on* $P \cup \{c,d\}$;

(vii) $F \in \underline{VB}^*$ *on* P *and* F *is bounded on* P.

PROOF. By Theorem 2, (i), (ii), $\underline{VB}^* \subset \underline{VB}$ on P.

(i) $\Leftrightarrow$ (ii) See [11] (p.229).

(i) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) are evident.

(v) $\Rightarrow$ (i) Clearly $-F \in \underline{VB} \cap \overline{VB}^*$. We show that $-F \in VB^*$. Let $\{[a_k, b_k]\}$, $k = 1, 2, \dots, n$, be a finite set of nonoverlapping closed intervals $a_k, b_k \in P$. Let $M_1, M_2 \in (0, +\infty)$ be constants given by the facts that $-F \in \underline{VB}$ on P and $-F \in \overline{VB}^*$ on P, respectively. By Remark 1, $O(-F; [a_k, b_k]) < -F(a_k) + F(b_k) + 2\Omega_+(-F; [a_k, b_k] \wedge (P \cap [a_k, b_k]))$. Hence $\sum_{k=1}^{n} O(-F; [a_k, b_k]) < M_1 + 2M_2$ and $-F \in VB^*$ on P. It follows that $F \in VB^*$ on P.

(ii) $\Rightarrow$ (vi) Let $F \in VB^*$ on $\overline{P}$. Then $F \in \underline{VB}^*$ on $\overline{P}$. Hence $F \in \underline{VB}^*$ on $P \cup \{c,d\}$.

(vi) $\Rightarrow$ (iv) Let $F \in \underline{VB}^*$ on $P \cup \{c,d\}$. Then $F \in \underline{VB}$ on $P \cup \{c,d\}$. By Theorem 1, (iv), (i), $F \in VB$ on P.

(ii) $\Rightarrow$ (vii) Let $F \in VB^*$ on $\overline{P}$. Then F is bounded on P and $F \in \underline{VB}^*$ on $\overline{P}$. Hence $F \in \underline{VB}^*$ on P.

(vii) $\Rightarrow$ (iv) Let $F \in \underline{VB}^*$ on P, F bounded on P. Then $F \in \underline{VB}$ on P. By Theorem 1, (v), (i), $F \in VB$ on P.

Theorem 4 *Let* $F : [a,b] \to \mathbb{R}$, $P \subset [a,b]$. *Then* $\underline{VB}^* \subset VB^*G$ *on* P.

PROOF. If F is bounded on P, see Theorem 3, (i), (vii). Suppose that F is not bounded on P and let $P_n = \{x \in P : |F(x)| \leq n\}$, $n = 1, 2, \dots$. Then $P = \cup P_n$ and F is bounded on each P_n. Hence $F \in VB^*$ on each P_n. It follows that $F \in VB^*G$ on P.

Theorem 5 *Let* $F : [a,b] \to \mathbb{R}$, *and let* P *be a closed subset of* $[a,b]$. *Then the following assertions are equivalent:*

(i) $F \in VB^*G$ *on* P;

(ii) *For each perfect subset* S *of* P *there exists a portion* $S \cap (c,d)$, *such that* $F \in VB^*$ *on* $S \cap (c,d)$;

(iii) $F \in VB^*G$ *on each* $Z \subset P$, *whenever* $|Z| = 0$.

PROOF. (i) $\Leftrightarrow$ (ii) See [11] (Theorem 9.1, p.233).

(i) $\Rightarrow$ (iii) is evident.

(iii) $\Rightarrow$ (ii) Let S be a closed subset of P. Let $Z \subset S$ be a G_δ-set, such that $|Z| = 0$ and $\overline{Z} = S$ (this is possible, indeed: let $Z_1 = \{x \in S : x$ is a rational number or x is an endpoint of some interval contiguous to $P\} = \{x_1, x_2, \dots\}$.

$G_j = \cup_{j=1}^{\infty}(x_i - \frac{1}{2^{j+1}}, x_i + \frac{1}{2^{j+1}})$, $j = 1, 2, \dots$. Let $Z = \cap_{j=1}^{\infty} G_j$. Then $Z_1 \subset Z$, $|Z| = 0$ and $\overline{Z}_1 = S$. Hence $\overline{Z} = S$). Since $F \in VB^*G$ on Z, there exists a sequence of sets $\{Z_i\}$, $i \geq 1$, such that $Z = \cup_{i=1}^{\infty} Z_i$ and $F \in VB^*$ on Z_i. By Theorem 3, (i), (ii), $F \in VB^*$ on $\overline{Z}_i$. By the Baire Category Theorem (see [11], p.54), there exists an open interval I, such that $\phi \neq I \cap Z \subset \overline{Z}_i$, for some i. It follows that $F \in VB^*$ on $I \cap Z$. Hence $F \in VB^*$ on $\overline{I \cap Z}$. But $I \cap S = I \cap \overline{Z} \subset \overline{I \cap Z}$. (Indeed, let $x_0 \in I \cap \overline{Z}$ and suppose to the contrary that $x_0 \notin \overline{I \cap Z}$; then there exists $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \cap (a, b) \cap Z = \emptyset$; let $\delta_1 = \min\{\delta; x_0 - \inf(I); \sup(I) - x_0\}$; then $(x_0 - \delta_1, x_0 + \delta_1) \cap (a, b) \cap Z = (x_0 - \delta_1, x_0 + \delta_1) \cap Z = \emptyset$, a contradiction, since $x_0 \in \overline{Z}$.) Hence $F \in VB^*$ on $I \cap S$.

Conditions $AC, \underline{AC}, AC^*, \underline{AC}^*, AC^{**}, \underline{AC}^{**}, \dots$

Definition 7 *Let $F : [a, b] \to \mathbb{R}$, $P \subset [a, b]$. F is said to be AC (respectively $\underline{AC}$) on P, if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $\sum_{k=1}^{n} |F(b_k) - F(a_k)| < \varepsilon$ (respectively $\sum_{k=1}^{n}(F(b_k) - F(a_k)) > -\varepsilon$), whenever $\{[a_k, b_k]\}$, $k = 1, 2, \dots, n$, is a finite set of nonoverlapping closed intervals, with $a_k, b_k \in P$ and $\sum_{k=1}^{n}(b_k - a_k) < \delta$. Let $\overline{AC} = \{F : -F \in \underline{AC}\}$. ($\underline{AC}, \overline{AC}$ - Ridder's conditions, see [9] p.235,236). We define $ACG, \underline{AC}G$ and $\overline{AC}G$ using Definition 1.*

Definition 8 *Let $F : [a, b] \to \mathbb{R}$, $P \subset [a, b]$. F is said to be AC' (respectively $\underline{AC}'$) on P, if for each $\varepsilon > 0$, there exists $\delta > 0$, such that $\sum_{k=1}^{n} |F(b_k) - F(a_k)| < \varepsilon$ (respectively $\sum_{k=1}^{n}(F(b_k) - F(a_k)) > -\varepsilon$), whenever $\{[a_k, b_k]\}$, $k = 1, 2, \dots, n$, set of nonoverlapping closed intervals, with $a_k \in P \cup P_+$, $b_k \in P \cup P_-$ and $\sum_{k=1}^{n}(b_k - a_k) < \delta$, where $P_- = \{x \in P : x$ is a left accumulation point$\}$ and $P_+ = \{x \in P : x$ is a right accumulation point$\}$. F is said to be $\overline{AC}'$ on P if $-F \in \underline{AC}'$ on P.*

Definition 9 *Let $F : [a, b] \to R$, $P \subset [a, b]$, $c = \inf(P)$, $d = \sup(P)$. F is said to be AC^* (respectively $\underline{AC}^*$) on P, if for each $\varepsilon > 0$ there exists a $\delta > 0$, such that $\sum_{k=1}^{n} O(F; [a_k, b_k]) < \varepsilon$ (respectively $\sum_{k=1}^{n} \Omega_-(F; [a_k, b_k] \wedge (P \cap [a_k, b_k])) > -\varepsilon$), whenever $\{[a_k, b_k]\}$, $k = 1, 2, \dots, n$, is a finite set of nonoverlapping closed intervals, with $a_k, b_k \in P$ and $\sum_{k=1}^{n}(b_k - a_k) < \delta$. Let $\overline{AC}^* = \{F : -F \in \underline{AC}^*\}$. ($\underline{AC}^*, \overline{AC}^*$ - Ridder's conditions, see [9], p.251).*

If in addition F is bounded on $[c, d]$, then we obtain the conditions: bAC^*, $b\underline{AC}^*, b\overline{AC}^*$. We define $AC^*G, \underline{AC}^*G, \overline{AC}^*G, bAC^*G, b\underline{AC}^*$ and $b\overline{AC}^*G$ using Definition 1.

Definition 10 *Let $F : [a, b] \to \mathbb{R}$, $P \subset [a, b]$, $c = \inf(P)$, $d = \sup(P)$. F is said to be AC^{**} (respectively $\underline{AC}^{**}$) on P, if for each $\varepsilon > 0$, there exists a*

$\delta > 0$, such that $\sum_{k=1}^{n} O(F;[a_k,b_k]) < \varepsilon$ (respectively $\sum_{k=1}^{n} \Omega_-(F;[a_k,b_k] \wedge (P \cap [a_k,b_k])) > -\varepsilon$), whenever $\{[a_k,b_k]\}$, $k = 1,2,\dots,n$, is a finite set of nonoverlapping closed intervals, with $P \cap [a_k,b_k] \neq \emptyset$ and $\sum_{k=1}^{n}(b_k - a_k) < \delta$. Let $\overline{AC}^{**} = \{F : -F \in \underline{AC}^{**}\}$. We define $AC^{**}G, \underline{AC}^{**}G$ and $\overline{AC}^{**}G$ using Definition 1.

Remark 2 *In [6], Lee introduced a condition called AC^{**}. We do not know if it is equivalent to our condition AC^{**}. However, Lee's condition $AC^{**}G$ is equivalent to our condition $AC^{**}G$ (see Theorem 3 of [6] and our Corollary 1).*

Remark 3 *Let $F : [a,b] \to \mathbb{R}$, $P \subset [a,b]$. Then we have:*

(i) $AC = \underline{AC} \cap \overline{AC}$ *on P;*

(ii) $AC^* = \underline{AC}^* \cap \overline{AC}^*$ *on P;*

(iii) $AC^{**}\underline{AC}^{**} \cap \overline{AC}^{**}$ *on P;*

(iv) $AC' \subset AC$ *and* $\underline{AC}' \subset \underline{AC}$ *on P;*

(v) $\underline{AC}^{**} \subset \underline{AC}^* \subset \underline{AC}$ *and* $\underline{AC}^{**}G \subset \underline{AC}^*G \subset \underline{AC}G$ *on P;*

(vi) $AC^{**} \subset AC^* \subset AC$ *and* $AC^{**}G \subset AC^*G \subset ACG$ *on P.*

Definition 11 *(Saks, [10], p.128). Let $F : [a,b] \to R$, $P \subset [a,b]$. F is said to be $N^{-\infty}$ on P, if $|F(\{x \in P : (F|_P)'(x) = -\infty\})| = 0$. Let $N^{+\infty} = \{F : -F \in N^{-\infty}\}$. Let $N^{\infty} = N^{-\infty} \cap N^{+\infty}$.*

Theorem 6 *Let $F : [a,b] \to \mathbb{R}$, $P \subset [a,b]$.*

(i) *If $P_+ = [a,b)$ and $P_- = (a,b]$, and $F \in \underline{AC}'$ on P, then $F \in \underline{AC}$ on $[a,b]$;*

(ii) *If $F|_{\overline{P}} \in C_i$ on P and $F \in \underline{AC}$ on P, then $F \in \underline{AC}'$ on P;*

(iii) *If $F \in \underline{AC}$ on P, then $F|_P \in C_i$ on P;*

(iv) $\underline{AC} = VB \cap N^{-\infty} \cap C_i$ *on $[a,b]$.*

PROOF. (i) is evident.

(ii) Suppose that $F \in \underline{AC}$ on P. For $\varepsilon > 0$, let $\delta > 0$ be given by the fact that F is $\underline{AC}$ on P. Let $\{[a_k,b_k]\}$, $k = 1,2,\dots,n$, be a finite set of nonoverlapping closed intervals, with $a_k \in P \cup P_+$, $b_k \in P \cup P_-$, such that

$\sum(b_k - a_k) < \delta$. We may suppose without loss of generality that $F(b_k) < F(a_k)$, for each $k = 1, 2, \ldots, n$. Let $A_1 = \{k : a_k, b_k \in P\}$. Clearly

$$\sum_{k \in A_1} (F(b_k) - F(a_k)) > -\varepsilon. \tag{1}$$

Let $A_2 = \{k : a_k \in P, b_k \in P_- \setminus P\}$. Since $F \in \mathcal{C}_i$ on $\overline{P}$, there exists $t_k \in (a_k, b_k) \cap P$, such that $F(t_k) < F(b_k) + \varepsilon/2^k$. Hence

$$\sum_{k \in A_2} (F(b_k) - F(a_k)) > \sum_{k \in A_2} (F(t_k) - F(a_k) - \varepsilon/2^k) > -2\varepsilon. \tag{2}$$

Let $A_3 = \{k : a_k \in P_+ \setminus P, b_k \in P\}$. Since $F \in \mathcal{C}_i$ on $\overline{P}$, there exists $s_k \in (a_k, b_k) \cap P$, such that $F(a_k) < F(s_k) + \varepsilon/2^k$. Hence

$$\sum_{k \in A_3} (F(b_k) - F(a_k)) > \sum_{k \in A_3} (F(b_k) - F(s_k) - \varepsilon/2^k) > -2\varepsilon. \tag{3}$$

Let $A_4 = \{k : a_k \notin P, b_k \notin P\}$. Since $F \in \mathcal{C}_i$ on $\overline{P}$, there exist $a_k < s_k < t_k < b_k$, $s_k, t_k \in P$, such that $F(a_k) < F(s_k) + \varepsilon/2^k$ and $F(t_k) < F(b_k) + \varepsilon/2^k$. Hence

$$\sum_{k \in A_4} (F(b_k) - F(a_k)) > \sum_{k \in A_4} (F(t_k) - F(s_k) - \varepsilon/2^k) > -3\varepsilon. \tag{4}$$

By (1), (2), (3), (4), it follows that $\sum_{k=1}^{n}(F(b_k) - F(a_k)) > -\varepsilon - 2\varepsilon - 2\varepsilon - 3\varepsilon = -8\varepsilon$, hence $F \in \underline{AC}'$on P.

(iii) is evident.

(iv) See [3] (Corollary 5, p.398).

Theorem 7 *Let $F : [a, b] \to \mathbb{R}$, $P \subset [a, b]$. Then we have:*

(i) *$\underline{AC}^* \subset VB^*G$ and $\underline{AC}^*G \subset VB^*G$ on P;*

(ii) *$b\underline{AC}^* \subset VB^*$ and $b\underline{AC}^*G \subset VB^*G$ on P;*

(iii) *$\underline{AC}^*G = b\underline{AC}^*G$ on P.*

PROOF. Let $c = \inf(P)$, $d = \sup(P)$.

(i) For $\varepsilon = 1$, let $\delta > 0$ be given by the fact that $F \in \underline{AC}^*$ on P. Then $F \in \underline{VB}^*$ on $I \cap P$ with constant 1, whenever I is an interval, with $I \cap P \neq \emptyset$ and $|I| < \delta$. By Theorem 4, $F \in VB^*G$ on $P \cap I$. Since P can be covered by a finite sequence of nonoverlapping intervals J_i, $i = 1, 2, \ldots, p$, $|J_i| < \delta$, it follows that $F \in VB^*G$ on each $P \cap J_i$. Hence $F \in VB^*G$ on P.

(ii) Suppose that F is bounded on $[c,d]$. By Theorem 3, (i), (vii), $F \in VB^*$ on $P \cap J_i$. Let $M > 0$, such that $|F(x)| < M$ on $[c,d]$. Then $V^*(F;P) \leq \sum_{k=1}^{p} V^*(F; P \cap J_k) + 2Mp < +\infty$. Hence $F \in VB^*$ on P.

(iii) $b\underline{AC}^*G \subset \underline{AC}^*G = \underline{AC}^*G \cap VB^*G = (\underline{AC}^* \cap VB^*)G \subset b\underline{AC}^*G$. These follow by (i), and the fact that any function which is VB^* on a set E, is bounded on the interval $[\inf(E), \sup(E)]$.

Theorem 8 *Let $F : [a,b] \to \mathbb{R}$, $P \subset [a,b]$. The following assertions are equivalent:*

(i) *$F \in b\underline{AC}^*$ on $\overline{P}$;*

(ii) *$F \in b\underline{AC}^*$ on P and $F \in \mathcal{C}_i$ on $\overline{P}$.*

PROOF. (i) $\Rightarrow$ (ii) follows by definitions.

(ii) $\Rightarrow$ (i) Since $F \in b\underline{AC}^*$ on P, by Theorem 7, (ii), $F \in VB^*$ on P. By Theorem 3, (i), (ii), $F \in VB^*$ on $\overline{P}$. Since $F \in b\underline{AC}^*$ on P, it follows that $F \in \underline{AC}$ on P. By Theorem 6, (ii), $F \in \underline{AC}'$ on P. We show that $F \in \underline{AC}$ on $\overline{P}$. For $\varepsilon/2$, let $\delta > 0$, be given by the fact that $F \in \underline{AC}'$ on P. Let $\{[c_k, d_k]\}$, $k = 1, 2, \ldots$, be the intervals contiguous to $\overline{P}$. Since $F \in VB^*$ on $\overline{P}$, there exists a natural number p, such that $\sum_{k=p+1}^{\infty} O(F; [c_k, d_k]) < \varepsilon/2$. Let $\eta = \inf\{\delta : d_1 - c_1; d_2 - c_2, \cdots, d_p - c_p\}$. Let $\{[a_i, b_i]\}$, $i = 1, 2, \ldots, n$, be a finite set of nonoverlapping closed intervals, with $a_i, b_i \in \overline{P}$ and $\sum_{i=1}^{n}(b_i - a_i) < \eta$. We may suppose without loss of generality, the following nontrivial case: $a_i \in P$ and $b_i \notin P \setminus P_-$, for each $i = 1, \ldots, n$. Then there exists a natural number k_i such that $b_i = d_{k_i}$. Clearly $c_{k_i} \in P \cup P_-$ and $a_i < c_{k_i} < d_{k_i} = b_i$. Then $\sum_{i=1}^{n}(F(d_{k_i}) - F(c_{k_i}) + F(c_{k_i}) - F(a_i)) > -\varepsilon/2 - \varepsilon/2 = -\varepsilon$. Hence $F \in \underline{AC} \cap VB^*$ on $\overline{P}$, and by [2] (see Proposition 2), $F \in b\underline{AC}^*$ on $\overline{P}$.

Theorem 9 *Let $F : [a,b] \to \mathbb{R}$ and let P be a closed subset of $[a,b]$. Then the following assertions are equivalent:*

(i) *$F \in \underline{AC}^*G \cap \mathcal{C}_i$ on P;*

(ii) *$F \in \mathcal{C}_i$ on P, and for each perfect set $S \subset P$ there exists a portion $S \cap (c,d)$ such that $F \in b\underline{AC}^*$ on $S \cap (c,d)$;*

(iii) *$F \in \mathcal{C}_i$ on P and $F \in \underline{AC}^*G$ on Z, whenever $Z \subset P$, $|Z| = 0$.*

PROOF. By Theorem 7, (iii), $\underline{AC}^*G = b\underline{AC}^*G$ on P. Now the proof is similar to that of Theorem 5, using Theorem 8 instead of Theorem 3, (i), (ii).

Theorem 10 *Let $F : [a,b] \to \mathbb{R}$. Then $\mathcal{C}_i \cap VB^*G \cap N^{-\infty} \subset \underline{AC}^*G$ on $[a,b]$.*

PROOF. Let $F \in \mathcal{C}_i \cap VB^*G \cap N^{-\infty}$ on $[a,b]$. Then by Theorem 3, (i), (ii), there exist $P_n = \overline{P}_n$, $n = 1, 2, \ldots$, such that $[a,b] = \cup P_n$ and $F \in \mathcal{C}_i \cap VB^*$ on P_n. Let $F_n : [a,b] \to R$, such that $F_n(x) = F(x)$, $x \in P_n$, and F_n is linear on the closure of each interval contiguous to P_n. By [3] (see Proposition 2), $F_n \in \mathcal{C}_i$ on $[a,b]$. Clearly $F_n \in VB$ on $[a,b]$. By [1] (see Lemma 2, p.432), it follows that $F_n \in N^{-\infty}$ on $[a,b]$ (similarly to Theorem 6 of [1], p.433). By Theorem 6, (iv), $F_n \in \underline{AC}$ on $[a,b]$. Hence $F \in \underline{AC} \cap VB^*$ on P_n. By [2] (see Proposition 2), $F \in \underline{AC}^*$ on P_n. Thus $F \in \underline{AC}^*G$ on $[a,b]$.

Remark 4 *In [10], Saks showed that $C \cap VB^*G \cap N^{-\infty} \subset \underline{AC}^*G$ on $[a,b]$.*

The Condition Monotone$_*$

Definition 12 *Let $F : [a,b] \to \mathbb{R}$, $P \subset [a,b]$, $c = \inf(P)$, $d = \sup(P)$. F is said to be left increasing$_*$ (respectively right increasing$_*$) on P, if $F(x_1) \le F(x_2)$, whenever $c \le x_1 < x_2 \le d$ and $x_1 \in P$ (respectively $x_2 \in P$). F is said to be increasing$_*$ on P, if it is simultaneously left increasing$_*$ and right increasing$_*$ on P. If $F(x_1) < F(x_2)$, we obtain conditions strictly left increasing$_*$, strictly right increasing$_*$, strictly increasing$_*$. Similarly we define conditions left decreasing$_*$, right decreasing$_*$, etc. We define decreasing$_*G$, strictly decreasing$_*G$, etc. using Definition 1. Clearly monotone$_*$ = monotone on $[a,b]$.*

Theorem 11 *Let $F : [a,b] \to \mathbb{R}$, $P \subset [a,b]$. If $\overline{F}'(x) < 0$ on P then F is strictly decreasing$_*G$ on P.*

PROOF. See [11] (the proof of Theorem 10.1, p.235).

Derivation Bases

Definition 13 *([8], pp.99,101). Let $P \subset [a,b]$ and let $\delta : P \to (0, +\infty)$.*

(i) Let $\beta^{\#}_\delta[P] = \{([y,z];x) : [y,z] \subset (x - \delta(x), x + \delta(x)), x \in P\}$ and $D^{\#}[P] = \{\beta^{\#}[P] : \delta : [a,b] \to (0,+\infty)\}$. $D^{\#}[P]$ is called the sharp derivation basis on the set P. If δ are constant functions, then we obtain the uniform sharp derivation basis $U^{\#}[P]$ on the set P.

(ii) Let $\beta^0_\delta[P] = \{([y,z];x) : x \in P \text{ and } x \in [y,z] \subset (x - \delta(x), x + \delta(x))\}$ and let $D^0[P] = \{\beta^0_\delta[P] : \delta : [a,b] \to (0,+\infty)\}$. $D^0[P]$ is called the ordinary derivation basis on the set P. If δ are constant functions, then we obtain the uniform ordinary derivation basis $U^0[P]$ on the set P.

(iii) Let $\beta_\delta[P] = \{([y,z];x) : x \in P, y = x \text{ or } z = x, \text{ and } [y,z] \subset (x - \delta(x), x + \delta(x))\}$ and let $D[P] = \{\beta_\delta[P] : \delta : [a,b] \to (0,+\infty)\}$. $D[P]$ is called the derivation basis on the set P. If δ are constant functions, then we obtain the uniform derivation basis $U[P]$ on the set P.

Lemma 2 *([5], p.83). Let $\delta : [a,b] \to (0,+\infty)$. Then there exist a partition $a = x_0 < x_1 < \cdots < x_n = b$ and $t_i \in [x_{i-1}, x_i]$, $i = 1,2,\ldots,n$, such that $[x_{i-1}, x_i] \subset (t_i - \delta(t_i), t_i + \delta(t_i))$, $i = 1,2,\ldots,n$.*

Conditions $AC_{D^\#}, AC_{D^o}, AC_D, \ldots$

Definition 14 *Let $F : [a,b] \to R$, $P \subset [a,b]$. F is said to be $\underline{AC}_{D^\#}$ on P, if for every $\varepsilon > 0$ there exist $\eta > 0$ and $\delta : P \to (0,+\infty)$, such that $\sum_{i=1}^n (F(d_i) - F(c_i)) > -\varepsilon$, whenever $[c_i, d_i]$, $i = 1,2,\ldots,n$, are nonoverlapping closed intervals, with $\sum_{i=1}^n (d_i - c_i) < \eta$ and $([c_i,d_i];t_i) \in \beta_\delta^\#[P]$. F is said to be $\overline{AC}_{D^\#}$ on P if $-F \in \underline{AC}_{D^\#}$ on P, i.e., $\sum_{i=1}^n (F(d_i) - F(c_i)) < \varepsilon$. Let $AC_{D^\#} = \overline{AC}_{D^\#} \cap \overline{AC}_{D^\#}$ on P, i.e., $\sum_{i=1}^n |F(d_i) - F(c_i)| < \varepsilon$. If we put D^0 and β_δ^0 instead of $D^\#$ and $\beta_\delta^\#$, we obtain conditions $\underline{AC}_{D^o}, \overline{AC}_{D^o}, AC_{D^o}$ on P. If we put D and β_δ instead of $D^\#$ and $\beta_\delta^\#$, we obtain conditions $\underline{AC}_D, \overline{AC}_D, AC_D$ on P.*

Remark 5 *Let $F : [a,b] \to \mathbb{R}$, $P \subset [a,b]$.*

(i) *$AC_{U^o} \subset AC_{D^o} = AC_D$ and $AC_{U^o}G \subset AC_{D^o}G = AC_D G$ on P.*

(ii) *$\underline{AC}_{U^o} \subset \underline{AC}_{D^o} = \underline{AC}_D$ and $\underline{AC}_{U^o}G \subset \underline{AC}_{D^o}G = \underline{AC}_D G$ on P.*

(iii) *$AC_{D^\#} \subset AC_{D^o}$ and $\underline{AC}_{D^\#} \subset \underline{AC}_{D^o}$ on P.*

(iv) *Conditions AC_{D^o} and AC_D have been defined by Gordon in [4].*

Lemma 3 *Let $F : [a,b] \to \mathbb{R}$, $P \subset [a,b]$. The following assertions are equivalent:*

(i) *$F \in AC_{D^\#}$ (respectively $F \in \underline{AC}_{D^\#}$) on P;*

(ii) *For each $\varepsilon > 0$ there exist $\eta > 0$ and $\delta : P \to (0,+\infty)$ such that $\sum_{i=1}^n V(F;[c_i,d_i]) < \varepsilon$ (respectively $\sum_{i=1}^n \underline{V}(F;[c_i,d_i]) > -\varepsilon$), whenever $[c_i,d_i]$, $i = 1,2,\ldots,n$ are nonoverlapping closed intervals, with $\sum_{i=1}^n (d_i - c_i) < \eta$ and $([c_i,d_i];t_i) \in \beta_\delta^\#[P]$.*

PROOF. (i) $\Rightarrow$ (ii) For $\varepsilon > 0$, let $\eta > 0$ and $\delta : P \to (0,+\infty)$ be given by the fact that $F \in AC_{D^\#}$ (respectively $F \in \underline{AC}_{D^\#}$) on P. Let $([c_i,d_i];t_i) \in \beta_\delta^\#[P]$, $i = 1,2,\ldots,n$, such that $[c_i,d_i]$, $i = 1,2,\ldots,n$ are nonoverlapping closed intervals, with $\sum_{i=1}^n (d_i - c_i) < \eta$. Let $[c_{i,j}, d_{i,j}]$, $j = 1,2,\ldots,k_i$, be a finite set of nonoverlapping closed intervals contained in $[c_i,d_i]$. Then $([c_{i,j},d_{i,j}];t_i) \in \beta_\delta^\#[P]$, $i = 1,2,\ldots,n$, $j = 1,2,\ldots,k_i$. Since $\sum_{i=1}^n \sum_{j=1}^{k_i} (d_{i,j} - c_{i,j}) < \eta$, it follows that $\sum_{i=1}^n \sum_{j=1}^{k_i} |F(d_{i,j}) - F(c_{i,j})| < \varepsilon$ (respectively

$\sum_{i=1}^{n}\sum_{j=1}^{k_i}(F(d_{i,j}) - F(c_{i,j})) > -\varepsilon)$. Hence $\sum_{i=1}^{n} V(F;[c_i,d_i]) < \varepsilon$ (respectively $\sum_{i=1}^{n} \underline{V}(F;[c_i,d_i]) > -\varepsilon)$.

(ii) $\Rightarrow$ (i) For $\varepsilon > 0$ let η and δ be given by (ii). Let $\{[c_i,d_i]\}$, $i = 1,2,\dots,n$, be nonoverlapping closed intervals with $\sum_{i=1}^{n}(d_i - c_i) < \eta$ and $([c_i,d_i];t_i) \in \beta_\delta^{\#}[P]$. Then $\sum_{i=1}^{n} |F(d_i) - F(c_i)| < \sum_{i=1}^{n} V(F;[c_i,d_i]) < \varepsilon$ (respectively $\sum_{i=1}^{n}(F(d_i) - F(c_i)) \geq \sum_{i=1}^{n} \underline{V}(F;[c_i,d_i]) > -\varepsilon)$.

Lemma 4 *Let $F : [a,b] \to \mathbb{R}$, $P \subset [a,b]$. The following assertions are equivalent:*

(i) *$F \in AC_{D^\circ}$ (respectively $\underline{AC}_{D^\circ}$) on P;*

(ii) *For each $\varepsilon > 0$ there exist $\eta > 0$ and $\delta : P \to (0,+\infty)$, such that $\sum_{i=1}^{n} \Omega(F;[c_i,d_i] \wedge \{t_i\}) < \varepsilon$ (respectively $\sum_{i=1}^{n} \Omega_-(F;[c_i,d_i] \wedge \{t_i\}) > -\varepsilon$), whenever $[c_i,d_i]$, $i = 1,2,\dots,n$, are nonoverlapping closed intervals, with $\sum_{i=1}^{n}(d_i - c_i) < \eta$ and $([c_i,d_i];t_i) \in \beta_\delta^0[P]$.*

Proof. (i) $\Rightarrow$ (ii) For $\varepsilon > 0$ let $\eta > 0$ and $\delta : P \to (0,+\infty)$ be given by the fact that $F \in AC_{D^\circ}$ (respectively $F \in \underline{AC}_{D^\circ}$) on P. Let $([c_i,d_i];t_i) \in \beta_\delta^0[P]$, $i = 1,2,\dots,n$, such that $[c_i,d_i]$, $i = 1,2,\dots,n$, are nonoverlapping closed intervals, with $\sum_{i=1}^{n}(d_i - c_i) < \eta$. Then $\sum_{i=1}^{n} \Omega(F;[c_i,d_i] \wedge \{t_i\}) = \sum_{i=1}^{n} \sup\{|F(x_i) - F(t_i)| : x_i \in [c_i,d_i]\} \leq \varepsilon$ (respectively $\sum_{i=1}^{n} \Omega_-(F;[c_i,d_i] \wedge \{t_i\}) = \sum_{i=1}^{n} \inf\{F(y_i) - F(t_i)$ and $F(t_i) - F(x_i) : c_i \leq x_i \leq t_i \leq y_i \leq d_i\} < \varepsilon)$.

(ii) $\Rightarrow$ (i) $\sum_{i=1}^{n} |F(d_i) - F(c_i)| \leq \sum_{i=1}^{n}(|F(d_i) - F(t_i)| + |F(t_i) - F(c_i)|) \leq 2\sum_{i=1}^{n} \Omega(F;[c_i,d_i] \wedge \{t_i\}) \leq 2\varepsilon$ (respectively $\sum_{i=1}^{n}(F(d_i) - F(c_i)) = \sum_{i=1}^{n}(F(d_i) - F(t_i) + F(t_i) - F(c_i)) \geq 2\sum_{i=1}^{n} \Omega_-(F;[c_i,d_i] \wedge \{t_i\}) > -2\varepsilon)$.

Theorem 12 *Let $F : [a,b] \to \mathbb{R}$, $P \subset [a,b]$. Then we have:*

(i) *$\underline{AC}^{**} = \underline{AC}_{U^\circ}$ on P;*

(ii) *$AC^{**} = AC_{U^\circ}$ on P.*

Proof. (i) For $\varepsilon > 0$ let $\eta > 0$ be given by the fact that $F \in \underline{AC}^{**}$ on P. Let $\{[a_i,b_i]\}$, $i = 1,2,\dots,n$, be nonoverlapping closed intervals such that $[a_i,b_i] \cap P \neq \emptyset$ and $\sum_{i=1}^{n}(b_i - a_i) < \eta$. Let $\delta : P \to (0,+\infty)$, $\delta(x) = \eta$. Let $t_i \in [a_i,b_i] \cap P$. Then $[a_i,b_i] \subset [t_i - \eta, t_i + \eta]$ and $\sum_{i=1}^{n} \Omega_-(F;[a_i,b_i] \wedge \{t_i\}) \geq \sum_{i=1}^{n} \Omega_-(F;[a_i,b_i] \wedge P) > -\varepsilon$. By Lemma 4, $F \in \underline{AC}_{U^\circ}$ on P. Conversely, for $\varepsilon > 0$, let η and δ be given by the fact that $F \in \underline{AC}_{U^\circ}$ on P. Let $\delta_1 = \min\{\eta,\delta\}$. Let $[a_i,b_i]$, $i = 1,2,\dots,n$, be nonoverlapping closed intervals such that $[a_i,b_i] \cap P \neq \emptyset$ and $\sum_{i=1}^{n}(b_i - a_i) < \delta_1$. Then we have $\sum_{i=1}^{n} \Omega_-(F;[a_i,b_i] \wedge P) = \sum_{i=1}^{n} \inf\{F(y_i) - F(t_i)$ and $F(t_i) - F(x_i) : a_i \leq x_i \leq t_i \leq y_i \leq b_i,\ t_i \in P\} > -\varepsilon$.

(ii) is evident.

Conditions $Y_{D^\#}, Y_{D^0}, Y_D, \ldots$

Definition 15 *Let $F : [a,b] \to \mathbb{R}$. F is said to be $\underline{Y}_{D^\#}$ on $[a,b]$, if for each $Z \subset [a,b]$ with $|Z| = 0$, and for each $\varepsilon > 0$, there exists $\delta : Z \to (0,+\infty)$, such that $\sum_{i=1}^{n}(F(d_i) - F(c_i)) > -\varepsilon$, whenever $[c_i, d_i]$, $i = 1,2,\ldots,n$, are nonoverlapping closed intervals and $([c_i,d_i];t_i) \in \beta_\delta^\#[Z]$. F is said to be $\overline{Y}_{D^\#}$ on $[a,b]$ if $-F \in \underline{Y}_{D^\#}$ on $[a,b]$, i.e., $\sum_{i=1}^{n}(F(d_i) - F(c_i)) < \varepsilon$. Let $Y_{D^\#} = \underline{Y}_{D^\#} \cap \overline{Y}_{D^\#}$ on $[a,b]$, i.e., $\sum_{i=1}^{n}|F(d_i) - F(c_i)| < \varepsilon$.*

If we put D^0 and β_δ^0 instead of $D^\#$ and $\beta_\delta^\#$, we obtain conditions $\underline{Y}_{D^0}, \overline{Y}_{D^0}$ and Y_{D^0} on $[a,b]$. If we put D and β_δ instead of $D^\#$ and $\beta_\delta^\#$, we obtain conditions $\underline{Y}_D, \overline{Y}_D$ and Y_D on $[a,b]$.

Remark 6 *Let $F : [a,b] \to \mathbb{R}$. Then we have:*

(i) *$Y_{D^0} = Y_D$ on $[a,b]$;*

(ii) *$\underline{Y}_{D^0} = \underline{Y}_D$ on $[a,b]$;*

(iii) *$Y_{D^\#} \subset Y_{D^0}$ and $\underline{Y}_{D^\#} \subset \underline{Y}_{D^0}$ on $[a,b]$.*

(iv) *Y_{D^0} was defined by Lee Peng Yee in [6], but he called it "the strong Lusin condition". This condition also appears in Lemma 2 of [4].*

Lemma 5 *Let $F : [a,b] \to \mathbb{R}$. The following assertions are equivalent:*

(i) *$F \in Y_{D^\#}$ (respectively $F \in \underline{Y}_{D^\#}$) on $[a,b]$;*

(ii) *For each $Z \subset [a,b]$, $|Z| = 0$, and for each $\varepsilon > 0$, there exists $\delta : Z \to (0,+\infty)$ such that $\sum_{i=1}^{n} V(F;[c_i,d_i]) < \varepsilon$ (respectively $\sum_{i=1}^{n} \underline{V}(F;[c_i,d_i]) > -\varepsilon$), whenever $[c_i,d_i]$, $i = 1,2,\ldots,n$, are nonoverlapping closed intervals and $([c_i,d_i];t_i) \in \beta_\delta^\#[Z]$.*

PROOF. The proof is similar to that of Lemma 3.

Lemma 6 *Let $F : [a,b] \to R$. The following assertions are equivalent:*

(i) *$F \in Y_{D^0}$ (respectively $F \in \underline{Y}_{D^0}$) on $[a,b]$;*

(ii) *For each $Z \subset [a,b]$, $|Z| = 0$, and for each $\varepsilon > 0$ there exists $\delta : Z \to R_+$, such that $\sum_{i=1}^{n} \Omega(F;[c_i,d_i] \wedge \{t_i\}) < \varepsilon$ (respectively $\sum_{i=1}^{n} \Omega_-(F;[c_i,d_i] \wedge \{t_i\}) > -\varepsilon$), whenever $[c_i,d_i]$, $i = 1,2,\ldots,n$, are nonoverlapping closed intervals and $([c_i,d_i];t_i) \in \beta_\delta^0[Z]$.*

PROOF. The proof is similar to that of Lemma 4.

Theorem 13 *Let $F : [a, b] \to \mathbb{R}$. Then we have:*

(i) $\underline{Y}_{D^0} \in VB^*G \cap C_i \cap N^{-\infty}$ *on* $[a, b]$;

(ii) $\underline{Y}_{D^\#} \subset VB \cap C_i \cap N^{-\infty}$ *on* $[a, b]$.

PROOF. (i) We show that $F \in C_i$ on $[a, b]$. Let $x_0 \in [a, b]$. Then there exists $\delta(x_0) > 0$ such that $\Omega_-(F; [u, v] \wedge \{x_0\}) > -\varepsilon$, whenever $x_0 - \delta(x_0) < u \le x_0 \le v < x_0 + \delta(x_0)$, $u \ne v$. Hence $F \in C_i$ at x_0.

We show that $F \in VB^*G$ on $[a, b]$. By Theorem 5, (iii), it is sufficient to show that $F \in VB^*G$ on each $Z \subset [a, b]$, whenever $|Z| = 0$. For $\varepsilon > 0$, let $\delta : Z \to (0, +\infty)$, be given by the fact that $F \in \underline{Y}_{D^0}$ on $[a, b]$. Let $Z_n = \{x \in Z : \delta(x) > 1/n\}$, $n = 1, 2, \dots$. Then $Z = \cup_{i=1}^{\infty} Z_n$. Let $Z_{n,i} = Z_n \cap [\frac{1}{n}, \frac{i+1}{n})$. Fix n and i, such that $Z_{n,i} \ne \emptyset$. We show that $F \in \underline{VB}^*$ on $Z_{n,i}$. Let $[c_k, d_k]$, $k = 1, 2, \dots, n$, be nonoverlapping closed intervals with endponts in $Z_{n,i}$. Let $x_k \in [c_k, d_k]$. Then $([c_k, x_k]; c_k)$ and $([x_k, d_k]; d_k)$ belong to $\beta_\delta^0[Z_{n,i}]$. Hence $\sum_{i=1}^{n}(F(x_k) - F(c_k)) > -\varepsilon$ and $\sum_{k=1}^{n}(F(d_k) - F(x_k)) > -\varepsilon$. By Theorem 2, (i), (ii), $F \in \underline{VB}^*$ on $Z_{n,i}$. By Theorem 4, $F \in VB^*G$ on $Z_{n,i}$. Hence $F \in VB^*G$ on Z.

We show that $F \in N^{-\infty}$ on $[a, b]$. Let $E^{-\infty} = \{x : F'(x) = -\infty\}$. By Lemma 1 we have $|E^{-\infty}| = 0$. By Theorem 11, F is strictly decreasing$_*G$ on $E^{-\infty}$, i.e., $E^{-\infty} = \cup_{n=1}^{\infty} E_n$ and F is strictly decreasing$_*$ on each E_n. Clearly $|E_n| = 0$. Let $\varepsilon > 0$ and $\delta : E_n \to (0, +\infty)$ be given by the fact that $F \in \underline{Y}_{D^0}$ on $[a, b]$. For $t \in E_n$ let $A_n(t) = \{[F(v), F(u)] : t - \delta(t) < u \le t \le v < t + \delta(t), u \ne v\}$. Then $F(t) \in [F(v), F(u)]$ and $F(v) \ne F(u)$ (since F is strictly decreasing$_*$ on E_n). Let $\alpha > 0$. Since $F \in C_i$ on $[a, b]$, there exists $t - \delta(t) < u_1 < t < v_1 < t + \delta(t)$, such that $\Omega_-(F; [t, v_1] \wedge \{t\}) > -\alpha/2$ and $\Omega_-(F; [u_1, t] \wedge \{t\}) > \varepsilon - \alpha/2$. But $\Omega_-(F; [t, v_1] \wedge \{t\}) = F(v_1) - F(t)$ and $\Omega_-(F; [u_1, t] \wedge \{t\}) = F(t) - F(u_1)$ (since F is strictly decreasing$_*$ on E_n). Hence $0 < F(u_1) - F(v_1) < \alpha$. Let $A_n = \cup_{t \in E_n} A_n(t)$. Then A_n is a cover in the Vitali sense of the set $F(E_n)$. By the Vitali Covering Theorem, there exists a finite set of pairwise disjoint closed intervals $[u_i, v_i]$, $i = 1, 2, \dots, N$, such that

$$|F(E_n)| \le \sum_{i=1}^{N} |F(v_i) - F(u_i)| + \varepsilon \tag{1}$$

For each $i = 1, 2, \dots, N$, let $t_i \in E_n$ such that $t_i - \delta(t_i) < u_i \le t_i \le v_i < t_i + \delta(t_i)$. Then $F(u_i) - F(v_i) > 0$ and $\sum_{i=1}^{N}(F(v_i) - F(u_i)) > -\varepsilon$. By (1), $|F(E_n)| < 2\varepsilon$. It follows that $|F(E_n)| = 0$. Hence $|F(E^{-\infty})| = 0$. Thus $F \in N^{-\infty}$ on $[a, b]$.

(ii) By (i) and Remark 6, (iii), we need only to prove that $F \in VB$ on $[a,b]$. Let $\varepsilon > 0$ and $t \in [a,b]$. By Lemma 5, for $Z = \{t\}$, there exists $\delta(t) > 0$ such that $\underline{V}(F;[u,v]) > -\varepsilon$, whenever $[u,v] \subset (t-\delta(t), t+\delta(t))$. Hence $F \in \underline{VB}$ on $[u,v]$. By Theorem 1, (i), (iv), $F \in VB$ on $[u,v]$. By Lemma 2 there exist a partition $a = x_0 < x_1 < \cdots < x_n = b$ and $t_i \in [x_{i-1}, x_i]$, $i = 1,2,\ldots,n$, such that $[x_{i-1}, x_i] \subset (t_i - \delta(t_i), t_i + \delta(t_i))$, $i = 1,2,\ldots,n$. It follows that $F \in VB$ on $[x_{i-1}, x_i]$, $i = 1,2,\ldots,n$. Hence $F \in VB$ on $[a,b]$.

Characterizations of $AC^*G \cap \mathcal{C}, \underline{AC}^*G \cap \mathcal{C}_i, AC$ and $\underline{AC}$

Lemma 7 *Let $F : [a,b] \to \mathbb{R}$. Then we have:*

(i) $\underline{AC}_{D^0} \subset \underline{Y}_{D^0}$ *on* $[a,b]$*;*

(ii) $\underline{AC}_{D^\#} \subset \underline{Y}_{D^\#}$ *on* $[a,b]$.

PROOF. Let $Z \subset [a,b]$, $|Z| = 0$. Then $Z = \cup_{n=1}^{\infty} Z_n$, where the sets Z_n, $n = 1,2,\ldots$, are pairwise disjoint, and $F \in \underline{AC}_{D^0}$ (respectively $F \in \underline{AC}_{D^\#}$) on each Z_n. Let $\varepsilon > 0$. For each n there exist $\delta_n : Z_n \to (0,+\infty)$ and a positive number η_n, such that $\sum_{i=1}^{S_n} \Omega_-(F;[c_{n,i}, d_{n,i}] \wedge \{t_{n,i}\}) > -\varepsilon/2^n$ (respectively $\sum_{i=1}^{s_n} \underline{V}(F;[c_{n,i}, d_{n,i}]) > -\varepsilon/2^n$), whenever $[c_{n,i}, d_{n,i}]$, $i = 1,2,\ldots,s_n$, are nonoverlapping closed intervals, with $\sum_{i=1}^{s_n}(d_{n,i} - c_{n,i}) < \eta_n$ and $([c_{n,i}, d_{n,i}], t_{n,i}) \in \beta^0_{\sigma_n}[Z_n]$ (respectively $\beta^\#_{\delta_n}[Z_n]$), see Lemma 3 and Lemma 4. For each n choose an open set U_n, such that $Z_n \subset U_n$ and $|U_n| < \eta_n$. Let $\delta : Z \to (0,+\infty)$, $\delta(t) = \min\{\delta_n(t); d(t; R \setminus U_n)\}$, $t \in Z_n$. Let $[c_j, d_j]$, $j = 1,2,\ldots,m$, be nonoverlapping closed intervals such that $([c_j,d_j];t_j) \in \beta^0_\delta[Z]$ (respectively $\beta^\#_\delta[Z]$). Let $A_n = \{j \in \{1,2,\ldots,m\} : t_j \in Z_n\}$. Then $\sum_{j=1}^m \Omega_-(F;[c_j,d_j] \wedge \{t_j\}) = \sum_{j \in A_n} \Omega_-(F;[c_j,d_j] \wedge \{t_j\}) > \sum_{n=1}^{\infty}(-\varepsilon/2^n) > -\varepsilon$ (respectively $\sum_{j=1}^m \underline{V}(F;[c_j,d_j]) > -\varepsilon$). By Lemma 5 (respectively Lemma 6), $F \in \underline{Y}_{D^0}$ (respectively $F \in \underline{Y}_{D^\#}$) on $[a,b]$.

Remark 7 *In Lemma 2 of [4], Gordon showed that $AC_{D^0}G \subset Y_{D^0}$.*

Theorem 14 *Let $F : [a,b] \to \mathbb{R}$. The following assertions are equivalent:*

(i) $F \in \underline{AC}^*G \cap \mathcal{C}_i$ *on* $[a,b]$*;*

(ii) $F \in \mathcal{C}_i$ *on* $[a,b]$*, and for each perfect subset* $S \subset [a,b]$ *there exists a portion* $S \cap [c,d]$*, such that* $F \in b\underline{AC}^*$ *on* $S \cap [c,d]$*;*

(iii) $F \in \mathcal{C}_i$ *on* $[a,b]$ *and* $F \in \underline{AC}^*G$*, whenever* $Z \subset [a,b]$, $|Z| = 0$*;*

(iv) $F \in \underline{AC}^{**}G$ *on* $[a,b]$*;*

(v) $F \in \underline{AC}_{U^0}G$ *on* $[a,b]$*;*

(vi) $F \in \underline{AC}_{D^0}G$ *on* $[a,b]$*;*

(vii) $F \in \underline{Y}_{D^0}$ *on* $[a,b]$*;*

(viii) $F \in \mathcal{C}_i \cap VB^*G \cap N^{-\infty}$ *on* $[a,b]$*.*

PROOF. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) follow by Theorem 9.

(i) $\Rightarrow$ (iv) First, we show that if $F \in b\underline{AC}^*$ on $P \subset [a,b]$ and $F \in \mathcal{C}_i$ at each point of $\overline{P}$, then $F \in b\underline{AC}^{**}$ on $\overline{P}$. By Theorem 8, $F \in b\underline{AC}^*$ on $\overline{P}$. Let $\{(c_k,d_k)\}$, $k=1,2,\dots$, be the intervals contiguous to $\overline{P}$. For $\varepsilon > 0$ let $\delta > 0$ be given by the fact that $F \in b\underline{AC}^*$ on $\overline{P}$. Let N be a natural number such that $\sum_{k\ge N+1}(d_k-c_k) < \delta$. Then

$$\sum_{k\ge N+1} \Omega(F;[c_k,d_k]\wedge\{c_k,d_k\}) > -\varepsilon. \tag{1}$$

Let $\eta > 0$ such that $\eta < (d_k-c_k)/2$, $k=1,2,\dots,N$ and

$$\sum_{k=1}^{N}(\Omega_-(F;[c_k,c_k+\eta]\wedge\{c_k\}) + \Omega_-(F;[d_k-\eta,d_k]\wedge\{d_k\})) > -\varepsilon. \tag{2}$$

(this is possible since $F \in \mathcal{C}_i$ on P.) Let $\delta_1 = \inf\{\delta,\eta\}$. Let $\{[a_i,b_i]\}$, $i = 1,2,\dots,n$, be a finite set of nonoverlapping closed intervals such that $[a_i,b_i]\cap P \ne \emptyset$ and $\sum_{i=1}^n(b_i-a_i) < \delta_1$. Let $a_i' = \inf([a_i,b_i]\cap\overline{P})$ and $b_i' = \sup([a_i,b_i]\cap\overline{P})$. By (1) and (2), since $F \in b\underline{AC}^*$ on $\overline{P}$, we have $\sum_{i=1}^n \Omega_-(F;[a_i,b_i]\wedge(\overline{P}\cap[a_i,b_i])) \ge \sum_{i=1}^n \Omega_-(F;[a_i',b_i']\wedge(\overline{P}\cap[a_i',b_i'])) + \sum_{i=1}^n \Omega_-(F;[a_i,a_i']\wedge\{a_i'\}) + \sum_{i=1}^n \Omega_-(F;[b_i',b_i]\wedge\{b_i'\}) > -\varepsilon-2\varepsilon-2\varepsilon = -5\varepsilon$. Hence $F \in b\underline{AC}^{**}$ on $\overline{P}$. By Theorem 7, (iii), $\underline{AC}^*G\cap\mathcal{C}_i = b\underline{AC}^*G\cap\mathcal{C}_i \subset b\underline{AC}^{**}G$.

(iv) $\Leftrightarrow$ (v) See Theorem 12, (i).

(v) $\Rightarrow$ (vi) See Remark 3, (ii).

(vi) $\Rightarrow$ (vii) See Lemma 7, (i).

(vii) $\Rightarrow$ (viii) See Theorem 13, (i).

(viii) $\Rightarrow$ (i) See Theorem 10.

Corollary 1 *Let* $F:[a,b]\to\mathbb{R}$*. The following assertions are equivalent:*

(i) $F \in AC^*G\cap C$ *on* $[a,b]$*;*

(ii) $F \in C$ *on* $[a,b]$ *and for each subset* $S \subset [a,b]$ *there exists a portion* $S\cap[c,d]$ *such that* $F \in bAC^*$ *on* $S\cap[c,d]$*;*

(iii) $F \in C$ *on* $[a,b]$ *and* $F \in AC^*G$ *on* Z*, whenever* $Z \subset [a,b]$*,* $|Z| = 0$*;*

(iv) $F \in AC^{**}G$ *on* $[a,b]$*;*

(v) $F \in AC_{U^o}G$ *on* $[a,b]$;

(vi) $F \in AC_{D^o}G$ *on* $[a,b]$;

(vii) $F \in Y_{D^o}$ *on* $[a,b]$;

(viii) $F \in \mathcal{C} \cap VB^*G \cap N^\infty$ *on* $[a,b]$.

Remark 8 *The equivalence between (i) and (vi) in Corollary 1, was already shown in [4] and [6].*

Theorem 15 *Let* $F : [a,b] \to \mathbb{R}$. *The following assertions are equivalent:*

(i) $F \in \underline{AC}$ *on* $[a,b]$;

(ii) $F \in \mathcal{C}_i$ *on* $[a,b]$ *and* $F \in \underline{AC}$ *on* Z, *whenever* $Z \subset [a,b]$, $|Z| = 0$;

(iii) $F \in VB \cap N^{-\infty} \cap \mathcal{C}_i$ *on* $[a,b]$;

(iv) $F \in \underline{AC}_{D^o}$ *on* $[a,b]$;

(v) $F \in \underline{AC}_{U\#}$ *on* $[a,b]$;

(vi) $F \in \underline{AC}_{U\#}G$ *on* $[a,b]$;

(vii) $F \in \underline{AC}_{D\#}$ *on* $[a,b]$;

(viii) $F \in \underline{AC}_{D\#}G$ *on* $[a,b]$;

(ix) $F \in \underline{Y}_{D\#}$ *on* $[a,b]$.

PROOF. (i) $\Rightarrow$ (ii) See Theorem 6, (iii).

(ii) $\Rightarrow$ (i) Let Z be the set of all rational numbers of $[a,b]$. By Theorem 6, (ii), $F \in \underline{AC}'$ on Z. By Theorem 6, (i), since $Z_+ = [a,b)$ and $Z_- = (a,b]$, $F \in \underline{AC}$ on $[a,b]$.

(i) $\Leftrightarrow$ (iii) See Theorem 6, (iv).

(i) $\Rightarrow$ (iv) This follows by definitions.

(iv) $\Rightarrow$ (i) For $\varepsilon > 0$, let $\eta > 0$ and $\delta : [a,b] \to (0,+\infty)$, be given by the fact that $F \in \underline{AC}_{D^o}$ on $[a,b]$. Let $[c_k,d_k]$, $k = 1,2,\dots,n$, be nonoverlapping closed intervals, with $\sum_{k=1}^n (d_k - c_k) < \eta$. By Lemma 2, there exist a partition $c_k = x_{k,0} < x_{k,1} < \dots < x_{k,n} = d_k$ and $t_{k,i} \in [x_{k,i-1}, x_{k,i}] \subset (t_{k,i} - \delta(t_{k,i}), t_{k,i} + \delta(t_{k,i}))$. Then $\sum_{k=1}^n (F(d_k) - F(c_k)) = \sum_{k=1}^n \sum_{i=1}^{n_k} (F(x_{k,i}) - F(x_{k,i-1})) > -\varepsilon$, hence $F \in \underline{AC}$ on $[a,b]$.

(i) $\Rightarrow$ (v) $\Rightarrow$ (vi) $\Rightarrow$ (viii) and (v) $\Rightarrow$ (vii) $\Rightarrow$ (viii) follow by definitions.

(viii) $\Rightarrow$ (ix) See Lemma 7, (ii).

(ix) $\Rightarrow$ (iii) See Theorem 13, (ii).

Corollary 2 *Let $F : [a,b] \to \mathbb{R}$. The following assertions are equivalent:*

(i) $F \in AC$ *on* $[a,b]$;

(ii) $F \in C$ *on* $[a,b]$ *and* $F \in AC$ *on* Z, *whenever* $Z \subset [a,b]$, $|Z| = 0$;

(iii) $F \in VB \cap N^\infty \cap AC$ *on* $[a,b]$;

(iv) $F \in AC_{D^o}$ *on* $[a,b]$;

(v) $F \in AC_{U\#}$ *on* $[a,b]$;

(vi) $F \in AC_{U\#}G$ *on* $[a,b]$;

(vii) $F \in AC_{D\#}$ *on* $[a,b]$;

(viii) $F \in AC_{D\#}G$ *on* $[a,b]$;

(ix) $F \in Y_{D\#}$ *on* $[a,b]$.

Remark 9 *The equivalence between (i) and (iv) in Corollary was already shown in [4] (the proof of Theorem 5).*

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