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THE LATTICE GENERATED BY DERIVATIVES

Abstract

In [4] Z. Grande asked, "What is the smallest lattice of functions containing all derivatives?" In this paper I prove that the answer is the family of all Baire one functions and that this family is the smallest lattice of functions containing all non-degenerate derivatives. Moreover it is proved that the lattice generated by bounded (non-degenerate) derivatives is the family of all bounded Baire one functions.

First we need some notation. The real line $(-\infty, \infty)$ is denoted by $\mathbb{R}$ and the set of positive integers by $\mathbb{N}$. Throughout this article m is a fixed positive integer. The word *function* means mapping from $\mathbb{R}^m$ into $\mathbb{R}$ unless otherwise explicitly stated. The words *measure, summable* etc. refer to Lebesgue measure and integral in $\mathbb{R}^m$. We denote by $a \vee b$ ($a \wedge b$) the larger (the smaller) of the real numbers a and b. The Euclidean metric in $\mathbb{R}^m$ will be denoted by ϱ. For every set $A \subset \mathbb{R}^m$, let $\operatorname{diam} A$ be its diameter (i.e., $\operatorname{diam} A = \sup\{\varrho(x,y) : x, y \in A\}$), χ_A its characteristic function and $|A|$ its outer Lebesgue measure. The symbol $\int_A f$ will always mean the Lebesgue integral. We say that f is a *Baire one function* if it is a pointwise limit of some sequence of continuous functions. For any function f we write $\|f\|$ for $\sup\{|f(t)| : t \in \mathbb{R}^m\}$.

The word *interval* (resp. *cube*) will always mean a non-degenerate compact interval (resp. cube) in $\mathbb{R}^m$, i.e., the Cartesian product of m non-degenerate compact intervals (resp. compact intervals of equal length) in $\mathbb{R}$. By an *interval function* we mean a mapping from the family of all intervals into $\mathbb{R}$.

We say that the intervals I, J are *contiguous* if they do not overlap (i.e., $I \cap J$ is not a non-degenerate interval) and $I \cup J$ is an interval. We say that

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an interval function F is *additive* if $F(I \cup J) = F(I) + F(J)$ whenever I and J are contiguous intervals.

We say that a sequence of intervals $\{I_n : n \in \mathbb{N}\}$ is *o-convergent* to a point $x \in \mathbb{R}^m$ if

1. $x \in \bigcap_{n=1}^{\infty} I_n$,
2. $\lim_{n\to\infty} \operatorname{diam} I_n = 0$,
3. $\limsup_{n\to\infty} \dfrac{(\operatorname{diam} I_n)^m}{|I_n|} < \infty$.

We will write $I_n \overset{o}{\Rightarrow} x$. (Cf., e.g., [4].)

Let F be an arbitrary interval function and $x \in \mathbb{R}^m$. We define

$$o\text{-}\limsup_{I\Rightarrow x} F(I) = \sup\left\{\limsup_{n\to\infty} F(I_n) : I_n \overset{o}{\Rightarrow} x\right\}.$$

Similarly we define

$$o\text{-}\liminf_{I\Rightarrow x} F(I) = \inf\left\{\liminf_{n\to\infty} F(I_n) : I_n \overset{o}{\Rightarrow} x\right\}.$$

If the two limits above coincide, we denote their value by $o\text{-}\lim_{I\Rightarrow x} F(I)$.

We say that the function f is an *o-derivative* if there exists an additive interval function F (called the *primitive* of f) such that for each $x \in \mathbb{R}^m$

$$o\text{-}\lim_{I\Rightarrow x} \frac{F(I)}{|I|} = f(x).$$

Recall that o-derivatives are Baire one functions. (Cf. [4, Lemma 2.1, p. 151] and [4, Lemma 3.1].)

We say that $x \in \mathbb{R}^m$ is an *o-Lebesgue point* of the function, f, if f is locally summable at x and $o\text{-}\lim_{I\Rightarrow x} \dfrac{\int_I |f - f(x)|}{|I|} = 0$. We say that f is an *o-Lebesgue function* if each $x \in \mathbb{R}^m$ is an o-Lebesgue point of f.

We say that $x \in \mathbb{R}^m$ is an *o-dispersion point* of a set $A \subset \mathbb{R}^m$ if

$$o\text{-}\lim_{I\Rightarrow x} \frac{|A \cap I|}{|I|} = 0.$$

We say that A is *d_o-open* if each $x \in A$ is an o-dispersion point of $\mathbb{R}^m \setminus A$. The family of all d_o-open sets forms a topology on $\mathbb{R}^m$; the so-called *o-density topology* (cf. [4]). The terms "*d_o-closed*", "*d_o-interior*" (d_o-int) etc. will refer to this topology. We say that the function f is *o-approximately continuous* if it is continuous with respect to this topology. Recall that:

- for every measurable set $A \subset \mathbb{R}^m$, $|A \setminus d_o\text{-int } A| = 0$,
- each o-Lebesgue function is both an o-approximately continuous function and an o-derivative,
- each bounded o-approximately continuous function is an o-Lebesgue function.

We say that the function f is *o-non-degenerate* at a point $x \in \mathbb{R}^m$ if x is an o-dispersion point of the pre-image of the set $\big(f(x) - \varepsilon, f(x) + \varepsilon\big)$ by f for no $\varepsilon > 0$. We say that f is *o-non-degenerate* if it is o-non-degenerate at each point $x \in \mathbb{R}^m$.

We will need a few lemmas. The first one is the well-known Lusin-Menchoff property of the o-density topology (cf. [4]).

Lemma 1 *Given a measurable set B and a closed set $A \subset d_o$-int B, we can find an o-approximately continuous function g such that $0 \le g \le 1$ on $\mathbb{R}^m$, $g = 1$ on A and $g = 0$ off of B.*

The next three lemmas are proved in [4].

Lemma 2 *Assume that a sequence of pairwise disjoint sets $\{H_n : n \in \mathbb{N}\}$, a sequence of o-approximately continuous functions $\{h_n : n \in \mathbb{N}\}$ and $c \in (0,1]$ satisfy the following conditions:*

i) $h_n(x) = 0$ *if* $x \notin H_n$, $n \in \mathbb{N}$,

ii) $\big|\{x \in H_n : h_n(x) = 0\}\big| \ge c \cdot |H_n|$, $n \in \mathbb{N}$,

iii) for every $x \notin \cup_{n=1}^{\infty} H_n$ and every $\tau > 0$ there exists a cube $I \ni x$ such that $\operatorname{diam} I < \tau$ and for each $n \in \mathbb{N}$ either $|H_n \cap I| = 0$ or $H_n \subset I$,

iv) for each $j \in \mathbb{N}$ and each $x \in H_j$ there is a $p > j$ such that for each $n > p$, $\operatorname{diam} H_n < \big[\varrho(x, H_n)\big]^2$.

Set $h = \sum_{n=1}^{\infty} h_n$. Then h is o-non-degenerate.
[4, Lemma 4]

Lemma 3 *The sum of an o-approximately continuous function with an o-non-degenerate function is o-non-degenerate.*
[4, Lemma 5]

Lemma 4 *Whenever u is a Baire one function there exist a Baire one function v, a sequence of pairwise disjoint, compact sets $\{H_n : n \in \mathbb{N}\}$ and a sequence $\{c_n\}$ of non-negative real numbers such that the following conditions are satisfied:*

i) $u - v$ *is an o-Lebesgue function,*

ii) v *is o-approximately continuous at all points of* $\bigcup_{n=1}^{\infty} H_n$,

iii) $v(x) = 0$ *if* $x \in H_n \setminus d_o\text{-}int\, H_n$ *for some* $n \in \mathbb{N}$,

iv) $|v| \le \sum_{n=1}^{\infty} c_n \cdot \chi_{H_n}$,

v) $\sum_{n=1}^{\infty} c_n \cdot \chi_{H_n}$ *is a Baire one function,*

vi) v *is bounded provided that* u *is bounded,*

vii) for every $x \notin \bigcup_{n=1}^{\infty} H_n$ *and every* $\tau > 0$ *there exists a cube* $I \ni x$ *such that* $diam\, I < \tau$ *and for each* $n \in \mathbb{N}$ *either* $H_n \cap I = \emptyset$ *or* $H_n \subset I$,

viii) for each $j \in \mathbb{N}$ *and each* $x \in H_j$ *there is a* $p > j$ *such that for each* $n > p$, $diam\, H_n < \left[\varrho(x, H_n)\right]^2$.

[4, Lemma 10]

Finally we will use a part of Proposition 3 of [4].

Lemma 5 *Let* $\{H_n : n \in \mathbb{N}\}$ *be a sequence of pairwise disjoint compact subsets of* $\mathbb{R}^m$ *and let* $\{K_n\}$ *be a sequence of non-negative numbers such that the function* $\sum_{n=1}^{\infty} K_n \cdot \chi_{H_n}$ *is a Baire one function. Then there is a sequence* $\{\varepsilon_n\}$ *of positive numbers satisfying the following condition:*

(•) *for every sequence of functions* $\{f_n : n \in \mathbb{N}\}$ *if for each* $n \in \mathbb{N}$

i) f_n *is an o-derivative,*

ii) $f_n(x) = 0$ *if* $x \notin H_n$,

iii) $\|f_n\| \le K_n$,

iv) $\left|\int_I f_n\right| \le \varepsilon_n$ *for every interval* I,

then the function $f = \sum_{n=1}^{\infty} f_n$ *is an o-derivative.*

The main tool will be the following lemma.

Lemma 6 *Let* $A \subset \mathbb{R}^m$ *be non-empty, bounded and measurable. Suppose* v *is a function such that* $v \cdot \chi_A$ *is o-approximately continuous and* $\|v \cdot \chi_A\| = c < \infty$, *and suppose* $\varepsilon > 0$. *Then there exist o-approximately continuous functions* f_1, f_2, f_3 *and* f_4 *such that*

$$v \cdot \chi_A = (f_1 \wedge f_2) \vee (f_3 \wedge f_4) \tag{$*$}$$

and for $j \in \{1, 2, 3, 4\}$

i) $f_j(x) = 0$ *if* $x \notin A$,

ii) $\|f_j\| \le 12c$,

iii) $|\{x \in A : f_j(x) = 0\}| \ge |A|/12$,

iv) $|\int_I f_j| \le \varepsilon$ *for every interval* I.

PROOF. Write A as the union $A = \cup_{n=1}^k A_n$ of measurable, pairwise disjoint, non-empty sets of diameter less than

$$\frac{\varepsilon}{24m \cdot (1 \vee c) \cdot (1 \vee \operatorname{diam} A)^{m-1}}.$$

For $n \in \{1, \dots, k\}$ express the set A_n as the union $A_n = \cup_{j=1}^4 A_{n,j}$ of measurable, pairwise disjoint, non-empty sets of equal measure. For $j \in \{1,2,3,4\}$ find closed, disjoint sets $A_{n,j,1}, A_{n,j,2} \subset d_o$-int $A_{n,j}$ such that $|A_{n,j,l}| \ge |A_{n,j}|/3$ $(l \in \{1,2\})$. Use Lemma 1 to find an o-approximately continuous function $\psi_{n,j}$ such that

- $\psi_{n,j}(x) = 1$ if $x \in A_{n,j,2}$,
- $\psi_{n,j}(x) = 0$ if $x \notin A_{n,j}$ or $x \in A_{n,j,1}$,
- $0 \le \psi_{n,j} \le 1$ on $\mathbb{R}^m$.

Construct also an o-approximately continuous function φ_j such that

- $\varphi_j(x) = 0$ if $x \in \cup_{n=1}^k A_{n,j,1}$,
- $\varphi_j(x) = 1$ if $x \notin \cup_{n=1}^k A_{n,j}$,
- $0 \le \varphi_j \le 1$ on $\mathbb{R}^m$.

For $n \in \{1, \dots, k\}$ and $j \in \{1,2,3,4\}$ set

$$\gamma_{n,j} = \begin{cases} \dfrac{\int_{A_n} (v \cdot \varphi_j)}{\int_{A_n} \psi_{n,j}} & \text{if } |A_n| > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Define $f_j = v \cdot \chi_A \cdot \varphi_j - \sum_{n=1}^k \gamma_{n,j} \cdot \psi_{n,j}$. Then clearly f_1, f_2, f_3, f_4 are approximately continuous and condition i) holds. Observe that for each $x \in A$, $f_j(x) \ne v(x)$ for at most one $j \in \{1,2,3,4\}$. So $(*)$ is satisfied.

For $n \in \{1, \dots, k\}$ if $|A_n| > 0$, then $|\gamma_{n,j}| \le \dfrac{c \cdot |A_n \setminus A_{n,j,1}|}{|A_{n,j,2}|} \le 11c$. Thus condition ii) is fulfilled. Next observe that for $j \in \{1,2,3,4\}$ we have that

$\{x \in A : f_j(x) = 0\} \supset \cup_{n=1}^k A_{n,j,1}$ and $|A_{n,j,1}| \geq |A_n|/12$ for $n \in \{1, \dots, k\}$. Hence iii) holds. Let I be an arbitrary interval. Let B denote the union of those A_n for which $A_n \setminus I \neq \emptyset$. Observe that

$$|B \cap I| \leq 2m \cdot \max\{\operatorname{diam} A_n : n \in \{1, \dots, k\}\} \cdot (\operatorname{diam} A)^{m-1} \leq \frac{\varepsilon}{12 \cdot (1 \vee c)}.$$

So for $j \in \{1, 2, 3, 4\}$

$$\left|\int_I f_j\right| = \left|\int_{A \cap I} f_j\right| \leq \sum_{n=1}^k \left|\int_{A_n} f_j\right| + \left|\int_{\cup_{A_n \setminus I \neq \emptyset} A_n \cap I} f_j\right| \leq \int_{B \cap I} |f_j| < \varepsilon.$$

(We used that $\int_{A_n} f_j = 0$ for $n \in \{1, \dots, k\}$). This proves iv). □

Theorem 7 *For each Baire one function $u : \mathbb{R}^m \to \mathbb{R}$ there are o-non-degenerate o-derivatives $f_1, f_2, f_3, f_4 : \mathbb{R}^m \to \mathbb{R}$ such that $u = (f_1 \wedge f_2) \vee (f_3 \wedge f_4)$. Moreover if u is bounded, then the o-derivatives f_1, f_2, f_3, f_4 can also be chosen to be bounded.*

PROOF. Let the function v, the sequence of compact sets $\{H_n : n \in \mathbb{N}\}$ and the sequence of non-negative real numbers $\{c_n\}$ be as in Lemma 4. Apply Lemma 5 with $K_n = 12c_n$ $(n \in \mathbb{N})$ and find a sequence of positive numbers $\{\varepsilon_n\}$ satisfying the conditions of this lemma. For each $n \in \mathbb{N}$ use Lemma 6 with $A = H_n$ and $\varepsilon = \varepsilon_n$, to get o-approximately continuous functions $f_{n,1}$, $f_{n,2}$, $f_{n,3}$ and $f_{n,4}$ fulfilling its conclusions. (Conditions ii) and iii) of Lemma 4 imply the o-approximate continuity of $v \cdot \chi_{H_n}$.) Set $\widetilde{f_j} = \sum_{n=1}^\infty f_{n,j}$ $(j \in \{1, 2, 3, 4\})$. By condition (•) of Lemma 5 we get that $\widetilde{f_1}$, $\widetilde{f_2}$, $\widetilde{f_3}$ and $\widetilde{f_4}$ are o-derivatives.

For $j \in \{1, 2, 3, 4\}$ use Lemma 2 for the family $\{f_{n,j} : n \in \mathbb{N}\}$ to prove that $\widetilde{f_j}$ is o-non-degenerate. (The assumptions of this lemma follow by conditions i) and iii) of Lemma 6 and conditions vii) and viii) of Lemma 4.) Since $u - v$ is o-approximately continuous and $\widetilde{f_j}$ is o-non-degenerate, $f_j = (u - v) + \widetilde{f_j}$ is o-non-degenerate. (Also cf. Lemma 3). Clearly by condition $(*)$ of Lemma 6,

$$u = (u - v) + v = (u - v) + \left[(\widetilde{f_1} \wedge \widetilde{f_2}) \vee (\widetilde{f_3} \wedge \widetilde{f_4})\right] = (f_1 \wedge f_2) \vee (f_3 \wedge f_4).$$

If u is bounded, we can choose the function v to be bounded. Then for each $j \in \{1, 2, 3, 4\}$ the functions from the family $\{f_{n,j} : n \in \mathbb{N}\}$ have a common bound. So $\widetilde{f_j}$ and f_j are bounded, which completes the proof. □

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