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S. K. Mukhopadhyay and S. N. Mukhopadhyay, Department of Mathematics, The University of Burdwan, Burdwan-713104, W. B., India

CONVERGENCE THEOREMS FOR APPROXIMATE MEAN CONTINUOUS INTEGRAL

1. Introduction

We prove three convergence theorems for the approximate mean continuous integral, the D_1-integral, which was recently introduced in [5] by the present authors and which is more general than the C_1D-integral of Sargent [8]. Also in three other theorems results analogous to those for the C_1D-integral are deduced.

2. Preliminaries

The Lebesgue measure will be denoted by μ. The general Denjoy integral and the special Denjoy integral will be denoted by D and D^* respectively.

Definition 1 *A function $F : E \to \mathbb{R}$, where $\mathbb{R}$ is the set of reals and $E \subset \mathbb{R}$, is said to be generalized absolutely continuous or ACG on E if E can be expressed as countable union of closed sets on each of which F is absolutely continuous and is written $F \in ACG(E)$.*

This definition of ACG differs from [7, page 223] in that we are not using continuity.

Definition 2 *Let F be a real valued function defined on $[a, b]$ and let $c \in [a, b]$. Let F be D-integrable in some neighborhood of c. If there is a finite real number L and a measurable set $E_c \subset [a, b]$ having c as a point of density (one sided point of density if $c = a$ or $c = b$) such that for $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$*

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such that $\left|\frac{1}{x-c}(D)\int_c^x F(t)\,dt - L\right| < \varepsilon$ *whenever* $x \in E_c$ *and* $0 < |x-c| < \delta$, *then* L *is said to be* D_1*-limit of* F *at* c *and we write* $D_1\text{-}\lim_{t\to c} F(t) = L$. *The function* F *is said to be* D_1*-continuous at* c *if* $D_1\text{-}\lim_{t\to c} F(t) = F(c)$. *In other words* F *is* D_1*-continuous at* $c \in [a,b]$ *if* F *is* D*-integrable in some neighborhood of* c *and* $F(c)$ *is the approximate derivative at* c *of its indefinite* D*-integral.* F *is said to be* D_1*-continuous on* $[a,b]$ *if it is* D_1*-continuous at every point of* $[a,b]$.

Definition 3 *Let a sequence of functions* $\{F_n\}$ *be defined on* $[a,b]$. *If* $E \subset [a,b]$, *then* $\{F_n\}$ *is said to be absolutely continuous on* E *uniformly in* n *or* UAC *on* E *if for* $\varepsilon > 0$ *there is* $\delta = \delta(\varepsilon) > 0$ *such that for every sequence of non-overlapping intervals* $\{(\alpha_k, \beta_k)\}$ *with end points on* E *and* $\sum(\beta_k - \alpha_k) < \delta$ *we have* $\sum_k |F_n(\beta_k) - F_n(\alpha_k)| < \varepsilon$, *for all* n. *Clearly if* $\{F_n\}$ *is* UAC *on* E, *then it is* UAC *on every subset of* E.

The sequence $\{F_n\}$ *is said to be* $UACG$ *on* E *if* $E = \cup_{i=1}^{\infty} X_i$, X_i *closed and* $\{F_n\}$ *is* UAC *on each* X_i. *Clearly if* $\{F_n\}$ *is* $UACG$ *on* E, *then every closed subset of* E *has a portion on which the sequence* $\{F_n\}$ *is* UAC.

If $c \in [a,b]$ *and if for every* $\varepsilon > 0$ *there is* $\delta = \delta(\varepsilon) > 0$ *such that*

$$|F_n(x) - F_n(c)| < \varepsilon \text{ whenever } x \in [a,b],\ |x-c| < \delta,$$

for all n, *then the sequence* $\{F_n\}$ *is said to be equicontinuous at* c. *It is clear that if the sequence* $\{F_n\}$ *is equicontinuous at each point of* $[a,b]$, *then by the compactness of* $[a,b]$, *for every* $\varepsilon > 0$ *there is* $\delta > 0$ *such that*

$$|F_n(x') - F_n(x'')| < \varepsilon \text{ whenever } x', x'' \in [a,b] \text{ and } |x' - x''| < \delta,$$

for all n.

Let each F_n *be* D*-integrable in* $[a,b]$ *and let* $c \in [a,b]$. *If there is a measurable set* $E_c \subset [a,b]$ *having* c *as a point of density (one sided point of density if* $c = a$ *or* $c = b$*) such that for* $\varepsilon > 0$ *there is* $\delta = \delta(\varepsilon) > 0$ *such that*

$$\left|\frac{1}{x-c}(D)\int_c^x F_n(t)\,dt - F_n(c)\right| < \varepsilon \text{ whenever } x \in E_c \text{ and } 0 < |x-c| < \delta,$$

for all n, *then the sequence* $\{F_n\}$ *is said to be equi-*D_1*-continuous at* c.

Definition 4 [5] *A function* $f : [a,b) \to \mathbb{R}$ *is said to be* D_1*-integrable on* $[a,b]$ *if there is a* D_1*-continuous,* ACG *function* $\phi : [a,b] \to \mathbb{R}$ *such that* $\phi'_{ap} = f$ *almost everywhere in* $[a,b]$. *The function* ϕ *is said to be an indefinite* D_1*-integral of* f *and* $\phi(b) - \phi(a)$ *is the definite integral of* f *on* $[a,b]$. *The definite integral is denoted by* $(D_1)\int_a^b f(t)\,dt$ *or simply* $(D_1)\int_a^b f$.

The function f is said to be D_1-integrable on a measurable subset E of $[a,b]$ if f_E is D_1-integrable on $[a,b]$ where f_E is defined by

$$f_E(x) = \begin{cases} f(x) \text{ if } x \in E \\ 0 \text{ if } x \notin E \end{cases}$$

and we write $f \in D_1(E)$. We shall take $(D_1)\int_E f = (D_1)\int_a^b f_E$.

It follows that the D_1-integral is strictly more general than the GM_1-integral of Ellis and the C_1D-integral of Sargent (cf. [5]).

The following theorems will be needed later.

Theorem C. (Cauchy property of the D_1-integral). *If f is D_1-integrable in $[a,\beta]$ for every β, $a < \beta < b$, and if D_1-$\lim_{\beta\to b^-}(D_1)\int_a^\beta f = L$, then f is D_1-integrable in $[a,b]$ and $(D_1)\int_a^b f = L$.*

Theorem H. (Harnack property of the D_1-integral). *Let $E \subset [a,b]$ be a closed set with complementary intervals $I_k = (a_k, b_k)$, $k = 1,2,\ldots$. Let $f \in D_1(E)$ and $f \in D_1([a_k,b_k])$ for each k with $F_k(x) = (D_1)\int_{a_k}^x f$, $a_k \le x \le b_k$. Let (if there are infinite number of intervals I_k)*

(i) $\sum_{k=1}^{\infty} |(D_1)\int_{a_k}^{b_k} f| < \infty$

(ii) $\lim_{k\to\infty} \sup_{x\in(a_k,b_k]} |\frac{1}{x-a_k}\int_{a_k}^x F_k(t)\,dt| = 0.$

Then f is D_1-integrable in $[a,b]$ and $(D_1)\int_a^b f = (D_1)\int_E f + \sum_k (D_1)\int_{a_k}^{b_k} f$.

Theorems C and H are proved in [5].

Remark 1 *It may be noted that Sargent [8] has obtained the Harnack property for the C_1D-integral with the conditions (i) and (ii) replaced by*

$$(\alpha) \qquad \sum_{k=1}^{\infty} \sup_{a_k<x<b_k} \left| \frac{1}{x-a_k}\int_{a_k}^x F_k(t)\,dt \right| < \infty$$

$$(\beta) \qquad \sum_{k=1}^{\infty} \sup_{a_k<x<b_k} \left| \frac{1}{b_k-x}\int_x^{b_k} F_k(t)\,dt - F_k(b_k) \right| < \infty$$

(see [8, property B]). But (α) and (β) together imply (i) and (ii) and so our conditions (i) and (ii) are more general. In fact from [8, Lemma III] we get

that $\sum_{k=1}^{\infty}\left|\int_{a_k}^{b_k} f(t)\,dt\right| = \sum_{k=1}^{\infty}|F_k(b_k)-F_k(a_k)| \le H\sum\omega_k(a_k,b_k)$ *where* H *is a constant and*

$$\omega_k(a_k,b_k) = \max\cdot\Big[\sup_{a_k<x<b_k}\left|\frac{1}{x-a_k}\int_{a_k}^{x}F_k(t)\,dt - F_k(a_k)\right|, \sup_{a_k<x<b_k}\left|\frac{1}{b_k-x}\int_{x}^{b_k}F_k(t)\,dt - F_k(b_k)\right|\Big].$$

Since $F_k(a_k)=0$, (α) *and* (β) *imply* $\sum_{k=1}^{\infty}|\int_{a_k}^{b_k} f(t)\,dt| < \infty$ *implying (i). Also convergence of the series in* (α) *implies (ii).*

We also need the following theorem of Romanovskii whose proof can be found in [1, page 36, Theorem 46].

Theorem R. *Let* F *be a non-empty system of open subintervals of the bounded open interval* (a,b) *that has the following four properties:*

(1) *if* (α,β) *and* (β,γ) *are in* F *then so is* (α,γ)*;*

(2) *if* $(\alpha,\beta)\in F$ *then every open subinterval of* (α,β) *is also in* F*;*

(3) *if every proper open subinterval of* (α,β) *is in* F *then* $(\alpha,\beta)\in F$*;*

(4) *if all the contiguous intervals in* (a,b) *of a non-empty perfect subset* E *of* (a,b) *are in* F*, then* F *contains some interval* (α,β) *such that* $(\alpha,\beta)\cap E\neq\emptyset$*.*

Then $(a,b)\in F$*.*

3. Main Results

Lemma 1 *Let* $f_n\in D_1([a,b])$ *and* $F_n(x)=(D_1)\int_a^x f_n$, $a\le x\le b$*, for each* n *and let* $\{F_n\}$ *be UAC on a closed set* $E\subset[a,b]$*. Let* $\{(\alpha_k,\beta_k)\}$ *be the contiguous intervals of* E *on* $[a,b]$, $F_{k,n}=(D_1)\int_{\alpha_k}^{x} f_n$, $\alpha_k\le x\le\beta_k$ *and if there are infinitely many intervals* $\{(\alpha_k,\beta_k)\}$*, let*

$$\lim_{k\to\infty}\sup_{x\in(\alpha_k,\beta_k]}\left|\frac{1}{x-\alpha_k}(D)\int_{\alpha_k}^{x}F_{k,n}(t)\,dt\right| = 0 \tag{1}$$

for all n*. Then* f_n *is Lebesgue integrable on* E *for all* n *and for* $\varepsilon>0$ *there is* $\delta=\delta(\varepsilon)>0$ *such that for all measurable set* A, $A\subset E$ *with* $\mu(A)<\delta$*, we have* $|(L)\int_A f_n|<\varepsilon$ *for all* n*.*

PROOF. Since F_n is absolutely continuous on the closed set E, the function ϕ_n where $\phi_n = F_n$ on E and ϕ_n is linear in the closure of all contiguous intervals $[\alpha_k, \beta_k]$, is absolutely continuous on $[a, b]$. Since $\phi_n' = (F_n)'_{ap}$ almost everywhere on E and $(F_n)'_{ap} = f_n$ almost everywhere on $[a, b]$, $\phi_n' = f_n$ almost everywhere on E and so the first part follows.

Let $\varepsilon > 0$. There is $\delta > 0$ such that for every sequence of non-overlapping intervals $\{[r_p, s_p]\}$ with end points on E, and $\sum_p (s_p - r_p) < \delta$ we have for all n

$$(2) \qquad \sum_p |(D_1) \int_{r_p}^{s_p} f_n| = \sum_p |F_n(s_p) - F_n(r_p)| < \varepsilon/3.$$

Let now A be a measurable subset of E with $\mu(A) < \delta$. We may suppose that each point of A is a limit point of E from both sides. Then there is a sequence of open sets $\{G_m\}$ such that $G_m \supset G_{m+1} \supset A$ and $\lim_{m\to\infty} \mu(G_m) = \mu(A)$. We may further assume that for each m the end points of the constituent open intervals of G_m are in E. Let $G_m = \cup_i (x_{mi}, y_{mi})$, $E_m = \cup_i (E \cap [x_{mi}, y_{mi})$. Denote by $(\alpha_{mij}, \beta_{mij})$ the contiguous intervals of E_m in $[x_{mi}, y_{mi}]$. Then $G_m \sim E_m = \cup_{ij} (\alpha_{mij}, \beta_{mij})$. Clearly α_{mij}, $\beta_{mij} \in E$. Choose m_0 such that if $m > m_0$, then $\mu(G_m) < \delta$. Hence from (2) we have

$$(3) \qquad \sum_i |(D_1) \int_{x_{mi}}^{y_{mi}} f_n| < \varepsilon/3 \text{ and } \sum_{ij} |(D_1) \int_{\alpha_{mij}}^{\beta_{mij}} f_n| < \varepsilon/3$$

whenever $m > m_0$ and for all n. Now if there is only a finite number of contiguous intervals of $E \cap [x_{mi}, y_{mi}]$, then

$$(4) \qquad (D_1) \int_{x_{mi}}^{y_{mi}} f_n = (L) \int_{E\cap[x_{mi}, y_{mi}]} f_n + \sum_j (D_1) \int_{\alpha_{mij}}^{\beta_{mij}} f_n.$$

So suppose there are infinitely many contiguous intervals of $E \cap [x_{mi}, y_{mi}]$. Then from (1) we have for all n

$$(5) \qquad \lim_{j\to\infty} \sup_{x\in(\alpha_{mij}, \beta_{mij}]} |\frac{1}{x - \alpha_{mij}} (D) \int_{\alpha_{mij}}^{x} F_{mij,n}(t)\, dt| = 0$$

where $F_{mij,n} = (D_1) \int_{\alpha_{mij}}^{x} f_n$, $\alpha_{mij} \le x \le \beta_{mij}$. Now from (3) and (5), the conditions (i) and (ii) of Theorem H are satisfied for the set $E \cap [x_{mi}, y_{mi}]$ and the contiguous intervals $(\alpha_{mij}, \beta_{mij})$ of $E \cap [x_{mi}, y_{mi}]$ in $[x_{mi}, y_{mi}]$. Hence by Theorem H we have (4). Thus (4) being true for all cases, by summing the expressions in (4) over i and taking $m > m_0$, we have from (3) that

$$(6) \qquad |(L) \int_{E_m} f_n| = |(L) \int_{\bigcup_i (E\cap(x_{mi}, y_{mi}))} f_n| < 2\varepsilon/3$$

for all n and all $m > m_0$. Since f_n is Lebesgue integrable on E, there is $\delta_n > 0$ such that for every measurable subset B of E with $\mu(B) < \delta_n$, we have

$$\left|(L)\int_B f_n\right| < \varepsilon/3. \tag{7}$$

Since $G_m \supset A$ and $\lim_{m\to\infty}\mu(G_m) = \mu(A)$, $\lim_{m\to\infty}\mu(E_m \sim A) = 0$. So there is $m_1 > m_0$ such that $\mu(E_m \sim A) < \delta_n$ for all $m \geq m_1$. Since $A \subset E_{m_1}$, from (7)

$$\left|(L)\int_{E_{m_1}} f_n - (L)\int_A f_n\right| = \left|(L)\int_{E_{m_1}\sim A} f_n\right| < \varepsilon/3. \tag{8}$$

So from (6) and (8) we have for all n, $|(L)\int_A f_n| < \varepsilon$.

Lemma 2 *Let f_n be Lebesgue integrable on $[a,b]$ and $F_n(x) = (L)\int_a^x f_n$, $a \leq x \leq b$, for each n and let $\{F_n\}$ be UAC on $[a,b]$. Then for $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that for each measurable subset A of $[a,b]$ with $\mu(A) < \delta$, we have $\left|(L)\int_A f_n\right| < \varepsilon$, for all n.*

This follows from Lemma 1.

Lemma 3 *Let $X \subset [a,b]$, X closed and let $\{F_n\}$ be a sequence of functions on $[a,b]$ which is UAC on X and let $\{F_n\}$ converge to F on $X \cup \{a\} \cup \{b\}$. Let G_n be F_n on $X \cup \{a\} \cup \{b\}$ and linear on the closure of each interval of $[a,b] \sim X$. Then $\{G_n\}$ is UAC on $[a,b]$.*

PROOF. Clearly $\{F_n\}$ is UAC on $X\cup\{a\}\cup\{b\}$. Let $\{(a_i,b_i)\}$ be the collection of intervals of $[a,b] \sim X$. Let $\varepsilon > 0$ be given. Then there is $\delta = \delta(\varepsilon) > 0$ such that for every countable collection of non-overlapping intervals $\{(x_j, x'_j)\}$ with x_j, $x'_j \in X \cup \{a\} \cup \{b\}$ and $\sum(x'_j - x_j) < \delta$, we have for all n

$$\sum |F_n(x'_j) - F_n(x_j)| < \varepsilon. \tag{9}$$

Let N be such that $\sum_{i=N}^{\infty}(b_i - a_i) < \delta$. So,

$$\sum_{i=N}^{\infty} |F_n(b_i) - F_n(a_i)| < \varepsilon. \tag{10}$$

Let $M = \max_{1\leq i\leq N-1}\{|\frac{F(b_i)-F(a_i)}{b_i-a_i}|\}$. Then for $1 \leq i \leq N-1$,

$$\begin{aligned}\left|\frac{F_n(b_i)-F_n(a_i)}{b_i-a_i}\right| &\leq \left|\frac{F(b_i)-F(a_i)}{b_i-a_i}\right| + \left|\frac{F(b_i)-F_n(b_i)}{b_i-a_i}\right| \\ &+ \left|\frac{F(a_i)-F_n(a_i)}{b_i-a_i}\right| \leq M + \frac{2L}{b_{i_0}-a_{i_0}},\end{aligned} \tag{11}$$

where $L = \sup_n \max_{1\le i\le N-1}\{|F(b_i)-F_n(b_i)|, |F(a_i)-F_n(a_i)|\}$ and $b_{i_0}-a_{i_0} = \min_{1\le i\le N-1}(b_i - a_i)$. Let $\delta_0 = \min(\delta, \frac{\varepsilon(b_{i_0}-a_{i_0})}{M(b_{i_0}-a_{i_0})+2L})$. Let $\{(x_j, x'_j)\}$ be any collection of non-overlapping intervals with x_j, $x'_j \in [a,b]$ and $\sum(x'_j - x_j) < \delta_0$. We may suppose (if necessary, by breaking the interval (x_j, x'_j) into two or three subintervals) that either x_j, $x'_j \in X \cup \{a\} \cup \{b\}$ or $(x_j, x'_j) \subset [a_i, b_i]$ for some i. Let $\sum_1$ denote summation over all j for which x_j, $x'_j \in X \cup \{a\} \cup \{b\}$ and $\sum_2$ denote summation over the rest. Now for $\sum_2$ each j corresponds to an i by the correspondence $(x_j, x'_j) \subset [a_i, b_i]$. We further break $\sum_2$ into two parts $\sum_3$ and $\sum_4$ such that $\sum_3$ denotes summation over all j in $\sum_2$ for which the corresponding $i \ge N$ and $\sum_4$ denotes summation over all j in $\sum_2$ for which $i < N$. Then using (9), (10), (11),

$$
\begin{aligned}
&\textstyle\sum_j |G_n(x'_j) - G_n(x_j)| \\
= &\textstyle\sum_1 |G_n(x'_j) - G_n(x_j)| + \sum_2 |G_n(x'_j) - G_n(x_j)| \\
= &\textstyle\sum_1 |G_n(x'_j) - G_n(x_j)| + \sum_3 |G_n(x'_j) - G_n(x_j)| \\
+ &\textstyle\sum_4 |\frac{G_n(x'_j)-G_n(x_j)}{x'_j - x_j}| \cdot (x'_j - x_j) \\
\le &\textstyle\sum_1 |F_n(x'_j) - F_n(x_j)| + \sum_{i=N}^{\infty} |F_n(b_i) - F_n(a_i)| \\
+ &\textstyle\sum_{i=1}^{N-1} |\frac{F_n(b_i)-F_n(a_i)}{b_i - a_i}| \cdot (x'_j - x_j) \\
&< 2\varepsilon + \frac{M(b_{i_0}-a_{i_0})+2L}{b_{i_0}-a_{i_0}} \cdot \delta_0 < 3\varepsilon.
\end{aligned} \tag{12}
$$

Hence the result.

Lemma 4 *If for each n, $F_n(x)$ is D_1-continuous at $c \in [a,b]$ and $F_n(x)$ tends uniformly to $F(x)$, then $F(x)$ is D_1-continuous at c.*

PROOF. Let $\varepsilon > 0$. There is n such that

$$|F_n(x) - F(x)| < \varepsilon/3 \text{ for } x \in [a,b]. \tag{13}$$

Since $F_n(x)$ is D_1-continuous at c, it is D-integrable in some neighborhood of c. Let $\phi_n(x)$ be the indefinite D-integral of F_n. By the D_1-continuity of F_n there is a set $E = E_n$ having 0 as a point of density and a $\delta = \delta(\varepsilon, n) > 0$ such that

$$\left|\frac{\phi_n(c+h) - \phi_n(c)}{h} - F_n(c)\right| < \varepsilon/3 \text{ for } h \in E,\ 0 < |h| < \delta. \tag{14}$$

Let $h \in E$ and $0 < |h| < \delta$. From (13), since $F_n(x)$ is D-integrable, $F(x)$ is D-integrable and

$$
\begin{aligned}
&\left|\tfrac{\phi_n(c+h)-\phi_n(c)}{h} - \tfrac{1}{h}(D)\int_c^{c+h} F(t)\,dt\right| \\
&= \left|\tfrac{1}{h}(D)\int_c^{c+h}\{F_n(t) - F(t)\}\,dt\right| \le \varepsilon/3.
\end{aligned} \tag{15}
$$

Hence from (13), (14) and (15),

$$\left|\frac{1}{h}(D)\int_c^{c+h} F(t)\,dt - F(c)\right| < \varepsilon \text{ for } h \in E,\ 0 < |h| < \delta. \tag{16}$$

Letting $h \to 0$ we get from (16)

$$\begin{aligned} F(c) - \varepsilon &\le \liminf_{h\to 0} \frac{1}{h}(D)\int_c^{c+h} F(t)\,dt \le \limsup_{h\to 0} \frac{1}{h}(D)\int_c^{c+h} F(t)\,dt \\ &\le F(c) + \varepsilon. \end{aligned}$$

Since ε is arbitrary, $\lim_{h\to 0} ap \frac{1}{h}(D)\int_c^{c+h} F(t)\,dt = F(c)$. So $F(x)$ is D_1-continuous at c.

Theorem 1 *Let $\{f_n\}$ be a sequence of D_1-integrable functions on $[a,b]$ and let $F_n(x) = (D_1)\int_a^x f_n$, $\phi_n = (D)\int_a^x F_n$, $a \le x \le b$. Let*

(i) $\lim_{n\to\infty} f_n = f$ *almost everywhere in* $[a,b]$,

(ii) $\{F_n\}$ *be equi-D_1-continuous and* $\{\phi_n\}$ *be equicontinuous at every point of* $[a,b]$,

(iii) $\{F_n\}$ *and* $\{\phi_n\}$ *be UACG on* $[a,b]$ *and* $\{F_n\}$ *be pointwise bounded on* $[a,b]$,

(iv) *for every perfect set in* $[a,b]$ *having infinitely many complementary intervals* $\{(\alpha_k,\beta_k)\}$

$$\lim_{k\to\infty} \sup_{x\in(\alpha_k,\beta_k]} \left|\frac{1}{x-\alpha_k}(D)\int_{\alpha_k}^x F_{k,n}(t)\,dt\right| = 0, \text{ uniformly in } n,$$

where $F_{k,n}(x) = (D_1)\int_{\alpha_k}^x f_n$, $\alpha_k \le x \le \beta_k$.

Then f is D_1-integrable in $[a,b]$ and $\lim_{n\to\infty}(D_1)\int_a^b f_n = (D_1)\int_a^b f$.

Remark 2 *For D-integral this is the result of [1, page 40, Theorem 47]. The last part of the condition (iii) i.e. $\{F_n\}$ is pointwise bounded on $[a,b]$, and the condition (iv) are absent there since they are redundant for D-integral. In fact for D-integral, $\{F_n\}$ is equicontinuous with $F_n(a) = 0$ implies that there is $\delta > 0$ such that $|F_n(x') - F_n(x'')| < 1$ whenever $|x'-x''| < \delta$, $x', x'' \in [a,b]$ and for all n. Divide $[a,b]$ into subintervals $a = c_0 < c_1 < \cdots < c_N = b$*

such that $c_i - c_{i-1} \leq \delta/2$ *for each* $i, i = 1, 2, \ldots, N$. *Let* $x \in [a,b]$. *Then* $x \in [c_{i-1}, c_i]$ *for some* i. *So*

$$|F_n(x)| \leq |F_n(x) - F_n(c_{i-1})| + \sum_{k=2}^{i} |F_n(c_{k-1}) - F_n(c_{k-2})| \leq N$$

and hence $\{F_n\}$ *is uniformly bounded on* $[a,b]$. *For condition (iv),* $\{F_n\}$ *being equicontinuous, we have for* $\varepsilon > 0$ *there is* k_0 *such that for* $k > k_0$, *we have* $0(F_n; \alpha_k, \beta_k) < \varepsilon$, *for all* n *and so, for* $k > k_0$ *and for all* n *and* $x \in (\alpha_k, \beta_k]$ $\left|\frac{1}{x-\alpha_k} \int_{\alpha_k}^{x} F_{k,n}(t)\,dt\right| \leq 0(F_n; \alpha_k, \beta_k) < \varepsilon$. *Hence*

$$\lim_{k\to\infty} \sup_{x\in(\alpha_k,\beta_k]} \left|\frac{1}{x-\alpha_k} \int_{\alpha_k}^{x} F_{k,n}(t)\,dt\right| = 0, \text{ uniformly in } n.$$

PROOF. Let H be the collection of all subintervals (α, β) of $[a,b]$ such that the theorem holds on $[\alpha, \beta]$ and on all of its compact subintervals. Since $\{F_n\}$ is $UACG$ on $[a,b]$, there is a subinterval I of $[a,b]$ on which $\{F_n\}$ is UAC and hence by Lemma 2 and Vitali's theorem [6, page 152],

$$\lim_{n\to\infty} \int_I f_n = \int_I f. \tag{17}$$

So H is non-empty. Clearly if (α, β) and (β, γ) are in H, then so is (α, γ) and if $(\alpha, \beta) \in H$, then every open subinterval of (α, β) is also in H.

Now we shall show that if every proper open subinterval of (α, β) is in H, then (α, β) is also in H. Suppose every proper open subinterval of (α, β) is in H and $\alpha < x < \beta$. Let $\psi_n(t) = (D_1)\int_x^t f_n$, $x \leq t \leq \beta$. Since $\{F_n\}$ is equi-D_1-continuous at β, $\{\psi_n\}$ is equi-D_1-continuous at β and so, for $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$ and a measurable set $E_\beta \subset [a,b]$ having β as a point of density such that for all n

$$\frac{1}{\beta - y}(D)\int_y^\beta \psi_n - \varepsilon < \psi_n(\beta) < \frac{1}{\beta - y}(D)\int_y^\beta \psi_n + \varepsilon \tag{18}$$

whenever $y \in E_\beta \cap (\beta - \delta, \beta)$.

Since every proper open subinterval of (α, β) is in H, we have for every proper open subinterval $(x', y') \subset (\alpha, \beta)$, f is D_1-integrable on $[x', y']$ and

$$\lim_{n\to\infty} \int_{x'}^{y'} f_n = \int_{x'}^{y'} f. \tag{19}$$

Let $\psi(t) = (D_1)\int_x^t f$, $x \leq t \leq \beta$. Then $\{\psi_n\}$ is a sequence of D-integrable function, and by (19) ψ_n converges to ψ everywhere on $[x, \beta)$. Now for $x \leq$

$t \leq \beta$,

$$\begin{aligned}\phi_n(t) &= \int_a^t F_n(\xi)d\xi = \int_a^x F_n(\xi)d\xi + \int_x^t F_n(\xi)d\xi \\ &= \int_a^x F_n(\xi)d\xi + \int_x^t F_n(x)d\xi + \int_x^t \psi_n(\xi)d\xi \\ &= \int_a^x F_n(\xi)d\xi + F_n(x)(t-x) + \int_x^t \psi_n(\xi)d\xi.\end{aligned}$$

So

$$\begin{aligned}\left|\int_{t_1}^{t_2} \psi_n\right| &= |\phi_n(t_2) - \phi_n(t_1) - F_n(x)(t_2 - t_1)| \\ &\leq |\phi_n(t_2) - \phi_n(t_1)| + |F_n(x)||t_2 - t_1|,\end{aligned} \tag{20}$$

for $t_1, t_2 \in [x, \beta]$.

Since $\{F_n\}$ is pointwise bounded on $[a, b]$, there is $M_x > 0$ such that

$$|F_n(x)| \leq M_x \text{ for all } n, \tag{21}$$

and also since $\{\phi_n\}$ is equicontinuous at every point on $[a, b]$, by the compactness of $[a, b]$, for $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that for all n

$$|\phi_n(x') - \phi_n(x'')| < \varepsilon/2 \text{ whenever } x', x'' \in [a, b] \tag{22}$$

and $|x' - x''| < \delta$. Let $\delta' = \min(\delta, \varepsilon/2M_x)$. Then from (20), (21) and 22), $|\int_{t_1}^{t_2} \psi_n| < \varepsilon/2 + M_x\varepsilon/2M_x = \varepsilon$ whenever $t_1, t_2 \in [x, \beta]$ and $|t_2 - t_1| < \delta'$, and for all n. So the sequence of indefinite integrals of ψ_n is equicontinuous at every point of $[x, \beta]$. Also since $\{\phi_n\}$ is $UACG$ on $[a, b]$, from (20) and (21) we see that the sequence of indefinite integrals of ψ_n is $UACG$ on $[x, \beta]$. Hence by [1, page 40, Theorem 47], we get

$$\lim_{n\to\infty} (D) \int_x^\beta \psi_n = (D) \int_x^\beta \psi. \tag{23}$$

Also (23) is true for every subinterval of $[x, \beta]$. That is

$$\lim_{n\to\infty} (D) \int_y^\beta \psi_n = (D) \int_y^\beta \psi \text{ for } x < y < \beta. \tag{24}$$

From (18) and (24), we get

$$\begin{aligned}\tfrac{1}{\beta - y}(D) \textstyle\int_y^\beta \psi - \varepsilon &\leq \liminf_{n\to\infty} \psi_n(\beta) \\ &\leq \limsup_{n\to\infty} \psi_n(\beta) \leq \tfrac{1}{\beta - y}(D) \textstyle\int_y^\beta \psi + \varepsilon\end{aligned} \tag{25}$$

whenever $y \in E_\beta \cap (\beta - \delta, \beta)$.

Letting $y \to \beta$ first and then $\varepsilon \to 0$ we see from (25) that $\lim_{n\to\infty} \psi_n(\beta)$ and $D_1 - \lim_{t\to\beta} \psi(t)$ exist and they are equal. Hence by Theorem C, we have $\int_x^\beta f = D_1 - \lim_{t\to\beta} \psi(t) = \lim_{n\to\infty} \psi_n(\beta) = \lim_{n\to\infty} \int_x^\beta f_n$. Similarly $\int_\alpha^x f = \lim_{n\to\infty} \int_\alpha^x f_n$. Hence $\int_\alpha^x f + \int_x^\beta f = \lim_{n\to\infty} \left[\int_\alpha^x f_n + \int_x^\beta f_n\right]$. That is $\int_\alpha^\beta f = \lim_{n\to\infty} \int_\alpha^\beta f_n$. So $(\alpha, \beta) \in H$.

Let E be a non-empty perfect subset of $[a, b]$ and let all the contiguous intervals of E be in H. We shall show that H contains some interval (α, β) such that $(\alpha, \beta) \cap E \neq \emptyset$. Since $\{F_n\}$ is $UACG$ on $[a, b]$, there is a portion $P = E \cap [\alpha, \beta]$ of E on which $\{F_n\}$ is UAC. Therefore if $\{(\alpha_k, \beta_k)\}$ are the contiguous intervals of P in (α, β), then for all n $\sum |(D_1) \int_{\alpha_k}^{\beta_k} f_n| < \infty$, and also from the hypothesis of the theorem (if there are infinitely many intervals $\{(\alpha_k, \beta_k)\}$), we have for all n, $\lim_{k\to\infty} \sup_{x\in(\alpha_k,\beta_k]} \left|\frac{1}{x-\alpha_k} \int_{\alpha_k}^x F_{k,n}(t)\, dt\right| = 0$. Hence by Theorem H,

$$(D_1) \int_\alpha^\beta f_n = (L) \int_P f_n + \sum_k (D_1) \int_{\alpha_k}^{\beta_k} f_n \tag{26}$$

for all n. Since $\{F_n\}$ is UAC on P, by Lemma 1, f_n is Lebesgue integrable on P for all n and the family $\{f_n\}$ has equi-absolutely continuous integrals on P [6, p.152]. Hence by Vitali's theorem [6, page 152, Theorem 2], we get

$$\lim_{n\to\infty} \int_P f_n = \int_P f. \tag{27}$$

From the condition (iv) of the theorem, for $\varepsilon > 0$ there is k_0, independent of n, such that for $k > k_0$ and for all n

$$\sup_{x\in(\alpha_k,\beta_k]} \left|\frac{1}{x-\alpha_k}(D) \int_{\alpha_k}^x F_{k,n}(t)\, dt\right| < \varepsilon \tag{28}$$

where $F_{k,n}(t) = (D_1) \int_{\alpha_k}^t f_n, \quad \alpha_k \le t \le \beta_k$. Since $(\alpha_k, \beta_k) \in H$, f is D_1-integrable on $[\alpha_k, \beta_k]$ and on all of its compact subintervals and

$$\lim_{n\to\infty} (D_1) \int_{\alpha_k}^t f_n = \int_{\alpha_k}^t f, \quad \alpha_k \le t \le \beta_k. \tag{29}$$

Let $T_k = \int_{\alpha_k}^t f, \quad \alpha_k \le t \le \beta_k$. Then from (29), $F_{k,n} \to T_k$ on $[\alpha_k, \beta_k]$. Since

$$\phi_n(t) = \int_a^t F_n(\xi)d\xi = \int_a^{\alpha_k} F_n(\xi)d\xi + F_n(\alpha_k)(t - \alpha_k) + \int_{\alpha_k}^t F_{k,n}(\xi)d\xi,$$

we have for $t_1, t_2 \in [\alpha_k, \beta_k]$

$$|\int_{t_1}^{t_2} F_{k,n}(\xi)| \le |\phi_n(t_2) - \phi_n(t_1)| + |F_n(\alpha_k)||t_2 - t_1|. \tag{30}$$

Just as we deduced, using (20), (21) and (22), the equicontinuity of the sequence of the indefinite integrals of ψ_n, we deduce, using (30) and two other relations that the sequence of the indefinite integrals of $F_{k,n}$ for fixed k is equicontinuous at every point of $[\alpha_k, \beta_k]$ and is also $UACG$ on $[\alpha_k, \beta_k]$. Hence by [1, page 40, Theorem 47], T_k is D-integrable on $[\alpha_k, \beta_k]$ and

$$\lim_{n\to\infty} (D) \int_{\alpha_k}^{\beta_k} F_{k,n} = (D) \int_{\alpha_k}^{\beta_k} T_k. \tag{31}$$

Also (31) is true if β_k is replaced by t, $\alpha_k \le t \le \beta_k$. Hence from (28), for $k > k_0$ $\sup_{x\in(\alpha_k,\beta_k]} \left|\frac{1}{x-\alpha_k}(D) \int_{\alpha_k}^{x} T_k(t)\, dt\right| \le \varepsilon$. Since ε is arbitrary,

$$\lim_{k\to\infty} \sup_{x\in(\alpha_k,\beta_k]} \left|\frac{1}{x-\alpha_k}(D) \int_{\alpha_k}^{x} T_k(t)\, dt\right| = 0. \tag{32}$$

Further as $\{F_n\}$ is UAC on P, for $\varepsilon > 0$ there is a k_1 such that

$$\sum_{k=k_1}^{\infty} |(D_1) \int_{\alpha_k}^{\beta_k} f_n| < \varepsilon, \quad \text{for all } n. \tag{33}$$

Since $(\alpha_k, \beta_k) \in H$,

$$\lim_{n\to\infty} (D_1) \int_{\alpha_k}^{\beta_k} f_n = (D_1) \int_{\alpha_k}^{\beta_k} f, \tag{34}$$

and so, from (33)

$$\sum_{k=k_1}^{\infty} |(D_1) \int_{\alpha_k}^{\beta_k} f| \le \varepsilon. \tag{35}$$

So from (35) and (32), all the conditions of Theorem H are satisfied for the set P and the contiguous intervals $\{(\alpha_k, \beta_k)\}$ of P on $[\alpha, \beta]$ and hence by Theorem H, f is D_1-integrable in $[\alpha, \beta]$ and

$$\int_{\alpha}^{\beta} f = \sum_k \int_{\alpha_k}^{\beta_k} f + \int_P f. \tag{36}$$

From (33) and (35) $\left|\sum_{k=k_1}^{\infty} \int_{\alpha_k}^{\beta_k} f_n - \sum_{k=k_1}^{\infty} \int_{\alpha_k}^{\beta_k} f\right| < 2\varepsilon$ for all n and hence

$$\lim_{n\to\infty} \sum_{k=k_1}^{\infty} \int_{\alpha_k}^{\beta_k} f_n = \sum_{k=k_1}^{\infty} \int_{\alpha_k}^{\beta_k} f. \tag{37}$$

So from (26), (27), (34), (37) and (36) $\lim_{n\to\infty} \int_\alpha^\beta f_n = \int_\alpha^\beta f$. Thus H satisfies all the conditions of Theorem R. Hence $(a, b) \in H$. So the result.

Theorem 2 *Let $\{f_n\}$ be a sequence of D_1-integrable functions on $[a, b]$ and let $F_n(x) = (D_1) \int_a^x f_n$, $a \le x \le b$. Let*

(i) $\lim f_n(x) = f(x)$ *almost everywhere in* $[a, b]$,

(ii) $\{F_n\}$ *is UACG on* $[a, b]$,

(iii) $\{F_n\}$ *converges on* $[a, b]$ *to a* D_1*-continuous function.*

Then f is D_1-integrable on $[a, b]$ and $\lim_{n\to\infty}(D_1) \int_a^b f_n = (D_1) \int_a^b f$.

PROOF. Let $[a, b] = \cup_i X_i$ where $\{X_i\}$ is a sequence of closed sets such that $\{F_n\}$ is UAC on X_i for each i. From the condition (iii), if $F(x)$ is the limit of $F_n(x)$ then $F(x)$ is D_1-continuous on $[a, b]$. Also from (ii), F is ACG on $[a, b]$. We will show that $F'_{ap}(x) = f(x)$ almost everywhere on $[a, b]$. Let $G_n : [a, b] \to \mathbb{R}$ be such that $G_n(x) = F_n(x)$ for $x \in X_i \cup \{a\} \cup \{b\}$ and G_n is linear in the closure of each interval of $[a, b] \sim X_i$. Then $\{G_n\}$ is UAC on $[a, b]$ by Lemma 3. Let $G(x) = F(x)$ for $x \in X_i \cup \{a\} \cup \{b\}$ and G be linear in the closure of each interval of $[a, b] \sim X_i$. Then $G_n(x)$ converges to $G(x)$ everywhere in $[a, b]$. Write $g_n(x) = G'_n(x)$ for almost all x in $[a, b]$ and $g(x) = f(x)$ when $x \in X_i$ and $g(x) = G'(x)$ elsewhere in $[a, b]$. Then $g_n = G'_n = (F_n)'_{ap}$ almost everywhere in X_i and since f_n is D_1-integrable on $[a, b]$ for each n, we have $(F_n)'_{ap} = f_n$ almost everywhere in $[a, b]$ and hence $g_n = G'_n = (F_n)'_{ap} = f_n$ almost everywhere in X_i. Since f_n converges to f almost everywhere, $g_n(x)$ converges to $f(x) = g(x)$ almost everywhere in X_i. Also for almost all $x \in [a, b] \sim X_i$, $g_n(x) = G'_n(x) \to G'(x) = g(x)$. Hence $g_n(x)$ converges to $g(x)$ almost everywhere on $[a, b]$. Now each g_n is Lebesgue integrable in $[a, b]$ and the primitives G_n of g_n are UAC on $[a, b]$. Hence by Lemma 2, $\{G_n\}$ is a family of equi-absolutely continuous integrals on $[a, b]$ (cf. [6, p.152]) and hence by Vitali's theorem [6, page 152, Theorem 2] g is Lebesgue integrable in $[a, b]$ and for $a \le x \le b$

$$(L) \int_a^x g = \lim_{n\to\infty} (L) \int_a^x g_n = \lim_{n\to\infty} G_n(x) = G(x).$$

So $G'(x) = g(x)$ almost everywhere in $[a,b]$. Thus $F'_{ap}(x) = G'(x) = g(x) = f(x)$ for almost all x in X_i. So $F'_{ap}(x) = f(x)$ almost everywhere in X_i. Since $[a,b] = \bigcup_i X_i$ and $F'_{ap}(x) = f(x)$ almost everywhere in $[a,b]$ and since F is D_1-continuous, ACG on $[a,b]$, f is D_1-integrable on $[a,b]$ and F is an indefinite D_1-integral of f on $[a,b]$. Hence $\lim_{n\to\infty}(D_1)\int_a^b f_n = \lim_{n\to\infty} F_n(b) = F(b) = (D_1)\int_a^b f$.

Theorem 3 *Let $\{f_n\}$ be a sequence of D_1-integrable functions on $[a,b]$ and let $F_n(x) = (D_1)\int_a^x f_n$, $a \le x \le b$. Let*

(i) $\lim_{n\to\infty} f_n = f$ *almost everywhere in* $[a,b]$,

(ii) *for each $i = 1,2,\dots$ there exist closed sets X_i and D-integrable functions G_i, H_i, on $[a,b]$ such that $\bigcup_{i=1}^{\infty} X_i = [a,b], G_i, H_i \in VB(X_i)$, with*

$$G_i(v) - G_i(u) \le F_n(v) - F_n(u) \le H_i(v) - H_i(u)$$

for all $n \ge i$ whenever u or $v \in X_i$ and (in case there are infinite number of contiguous intervals of X_i)

$$\lim_{k\to\infty} \sup_{t\in(a_{ik},b_{ik}]} [\frac{1}{t-a_{ik}}|(D)\int_{a_{ik}}^t (G_i(x) - G_i(a_{ik}))dx|]$$
$$= 0 = \lim_{k\to\infty} \sup_{t\in(a_{ik},b_{ik}]} [\frac{1}{t-a_{ik}}|(D)\int_{a_{ik}}^t (H_i(x) - H_i(a_{ik}))dx|]$$

where $\{(a_{ik}, b_{ik})\}$ are the complementary intervals of X_i,

(iii) $\lim_{n\to\infty} F_n = F$ *where F is D_1-continuous, the convergence being uniform on the set of end points of contiguous intervals (a_{ik}, b_{ik}) of the set X_i, $i = 1,2,\dots$.*

Then f is D_1-integrable in $[a,b]$ and F is the D_1-primitive of f.

PROOF. For each i the approximate derivatives $(G_i)'_{ap}$ and $(H_i)'_{ap}$ exist almost everywhere in X_i and are Lebesgue integrable on X_i and

(38) $$(G_i)'_{ap}(x) \le f_n(x) \le (H_i)'_{ap}(x)$$

for almost all $x \in X_i$ and for all $n \ge i$. Since X_i's are closed and $\cup_i X_i = [a,b]$, by Baire's theorem there is an interval I contained in some X_m. Since D_1-integrable functions are measurable [5], by (38) and by the Lebesgue Dominated Convergence Theorem f_n and f are Lebesgue integrable on I and

(39) $$\lim_{n\to\infty}\int_I f_n = \int_I f.$$

Let a point x be called regular if there is an interval I containing x such that f is D_1-integrable in the closure of I with F its primitive. By (39), and by condition (iii) the set of all regular points is non-empty. Let X be the set of all $x \in [a,b]$ such that x is not a regular point. The theorem will be proved if we show that X is empty. If possible let $X \neq \emptyset$. Since X is closed and since $\cup_{i=1}^{\infty}(X \cap X_i) = X$, by Baire's theorem there is an interval (α, β) such that $(\alpha,\beta) \cap X = (\alpha,\beta) \cap X_p \neq \emptyset$ for some p. Let $[c,d]$ be the smallest interval containing $(\alpha,\beta) \cap X_p$. We shall show that f satisfies the hypothesis of Theorem H in the interval $[c,d]$ with $E = [c,d] \cap X_p$. By (38) and by the Lebesgue Dominated Convergence Theorem f and f_n, $n \geq p$, are Lebesgue integrable in $[c,d] \cap X_p$ and

$$\int_{[c,d]\cap X} f = \lim_{n\to\infty} \int_{[c,d]\cap X} f_n. \tag{40}$$

Let $(c,d) \sim X = \cup_{k=1}^{\infty}(c_k, d_k)$. Note that f is D_1-integrable on each $[u,v] \subset (c_k, d_k)$ with F its primitive. Since F is D_1-continuous, by Theorem C, f is D_1-integrable on each $[c_k, d_k]$ with F its primitive. Since $c_k \in X_p$, $d_k \in X_p$

$$|F_n(d_k) - F_n(c_k)| \leq |G_p(d_k) - G_p(c_k)| + |H_p(d_k) - H_p(c_k)|$$

for $n \geq p$ and since G_p, $H_p \in VB(X_p)$, there is M such that for $n \geq p$

$$\sum_{k=1}^{\infty} |F_n(d_k) - F_n(c_k)| \leq M.$$

Hence for any positive integer K

$$\sum_{k=1}^{K} |F(d_k) - F(c_k)| = \lim_{n\to\infty} \sum_{k=1}^{K} |F_n(d_k) - F_n(c_k)| \leq M.$$

Since K is arbitrary, this gives $\sum_{k=1}^{\infty} |(D_1)\int_{c_k}^{d_k} f| \leq M$. Finally it follows from (ii) and (iii) that if $x \in (c_k, d_k)$, then

$$G_p(x) - G_p(c_k) \leq F(x) - F(c_k) \leq H_p(x) - H_p(c_k)$$

and hence for $t \in [c_k, d_k]$

$$\begin{aligned} |(D)\int_{c_k}^{t} [F(x) - F(c_k)]| &\leq |(D)\int_{c_k}^{t} [H_p(x) - H_p(c_k)]| \\ &+ |(D)\int_{c_k}^{t} [G_p(x) - G_p(c_k)]| \end{aligned}$$

and so $\lim_{k\to\infty}\sup_{t\in(c_k,d_k]}|\frac{1}{t-c_k}(D)\int_{c_k}^t[F(x)-F(c_k)]|=0$. So by Theorem H, f is D_1-integrable in $[c,d]$ and

$$(D_1)\int_c^d f=(D_1)\int_{[c,d]\cap X} f+\sum_{k=1}^{\infty}(D_1)\int_{c_k}^{d_k} f. \tag{41}$$

Since F is the primitive of f on each $[c_k,d_k]$ and since F_n converges uniformly to F on the set of end points of contiguous intervals (c_k,d_k) of the set X_p, we get from (40) and (41)

(42)

$$\begin{aligned}(D_1)\int_c^d f &= \lim_{n\to\infty}\int_{[c,d]\cap X} f_n+\lim_{n\to\infty}\sum_{k=1}^{\infty}[F_n(d_k)-F_n(c_k)]\\ &= \lim_{n\to\infty}[(D_1)\int_{[c,d]\cap X} f_n+\sum_{k=1}^{\infty}(D_1)\int_{c_k}^{d_k} f_n]\\ &= \lim_{n\to\infty}(D_1)\int_c^d f_n=\lim_{n\to\infty}[F_n(d)-F_n(c)]\\ &= F(d)-F(c).\end{aligned}$$

The relation (42) can also be established for every subinterval $[u,v]\subset[c,d]$. In fact, if u,v are in a single $[c_k,d_k]$ then F being the primitive of f on $[c_k,d_k]$ (42) is obvious for $[u,v]$ and if $u,v\in X_p$, the proof is as above. Otherwise we can break the interval $[u,v]$ into two or three subintervals so that each subinterval will come under one of these two cases. Hence F is the primitive of f on $[c,d]$. But this is a contradiction, since $(c,d)\cap X\neq\emptyset$. So $X=\emptyset$ and the theorem is proved.

Remark 3 *Note that the condition (iii) in Theorems 2 and 3 is weaker than the uniform convergence of F_n to F by Lemma 4.*

Example 1 *Let*

$$f_n(x)=\begin{cases}-\frac{1}{x^2}\cos\frac{1}{x} & \text{if } \frac{1}{n\pi}\le x\le 1\\ 0 & \text{if } 0\le x<1/n\pi,\end{cases}$$

$$f(x)=\begin{cases}-\frac{1}{x^2}\cos\frac{1}{x} & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases}$$

and

$$F_n(x)=\begin{cases}\sin\frac{1}{x} & \text{if } \frac{1}{n\pi}\le x\le 1\\ 0 & \text{if } 0\le x<1/n\pi.\end{cases}$$

Then F_n is a D_1-primitive of f_n. F_n does not converge uniformly in $[0,1]$ but converges pointwise to $F(x)=\sin\frac{1}{x}$, $x\neq 0$, $F(0)=0$. Also it converges uniformly on the set $\{\frac{1}{i\pi},\ i=1,2,3,\dots\}$. Let

$$X_1=[\frac{1}{\pi},1],\ X_i=[\frac{1}{i\pi},\frac{1}{(i-1)\pi}],\ i=2,3,\dots .$$

Then $\{0\} \cup (\cup X_i) = [0,1]$ *and taking* $G_i = H_i = F_i$*, the above theorem can be applied to determine the* D_1*-integrability of* f *on* $[0,1]$.

4. The C_1D-integral

It is known that the C_1P-integral of Burkill, which is equivalent to the C_1D-integral of Sargent [8] is included in the GM_1-integral of Ellis [2]. It is clear that the D_1-integral is more general than the GM_1-integral and hence more general than the C_1D-integral. Therefore the above results also give sufficient conditions for the convergence of the C_1D-integrals to a D_1-integrable function. We deduce here with the help of the above results, that the limit function is also C_1D-integrable.

We refer to [8] for the definition of the C_1D-integral which needed the concepts of AC^* (C_1-sense) and ACG^* (C_1-sense), which also can be found in [8]. See also [9] for an equivalent definition.

Let F be D^*-integrable in $[a,b]$. For convenience we write for $x, y \in [a,b]$, $x \neq y$, $C_1(F;x,y) = \frac{1}{y-x}(D^*)\int_x^y F(t)\,dt$.

Definition 5 *A sequence of functions* $\{F_n\}$ *is said to be* UAC^* *(*C_1*-sense) over a set* $E \subset [a,b]$ *if for all* n, F_n *is* D^**-integrable on* $[a,b]$ *and for each* $\varepsilon > 0$ *there is* $\delta = \delta(\varepsilon) > 0$*, independent of* n*, such that* $\sum_k \omega_n(a_k,b_k) < \varepsilon$ *for every countable collection of non-overlapping intervals* $\{(a_k,b_k)\}$ *with end points on* E *satisfying* $\sum(b_k - a_k) < \delta$*, where*

$$\omega_n(a_k,b_k) = \max\left\{\sup_{a_k<x<b_k}|C_1(F_n;a_k,x) - F_n(a_k)|,\ \sup_{a_k<x<b_k}|C_1(F_n;b_k,x) - F_n(b_k)|\right\}$$

and $\{F_n\}$ *is said to be* $UACG^*$ *(*C_1*-sense) on* $[a,b]$ *if there is a sequence of closed sets* E_i *such that* $[a,b] = \cup E_i$ *and* $\{F_n\}$ *is* UAC^* *(*C_1*-sense) on each* E_i.

Considering the definition of the usual AC^* and ACG^* as in [7] one gets, as in Definition 3, the definition of UAC^* and $UACG^*$ for the sequence of functions $\{F_n\}$.

Definition 6 *For each* n *let* $F_n : [a,b] \to \mathbb{R}$ *be* D^**-integrable and* $\phi_n(t) = (D^*)\int_a^t F_n$, $a \leq t \leq b$. *Let* $x \in [a,b]$. *If for every* $\varepsilon > 0$ *there is* $\delta = \delta(\varepsilon) > 0$ *such that* $|\frac{\phi_n(t)-\phi_n(x)}{t-x} - F_n(x)| < \varepsilon$ *whenever* $0 < |t-x| < \delta$*, for all* n*, then* $\{F_n\}$ *is said to be equi-*C_1*-continuous at* x.

Clearly equi-C_1-continuity implies equi-D_1-continuity.

Lemma 5 *Let $\{F_n\}$ be a sequence of D^*-integrable functions on $[a,b]$ and let $\phi_n(x) = (D^*)\int_a^x F_n$. Let*

(i) $\lim_{n\to\infty} F_n = F$, *where F is a C_1-continuous function, everywhere on $[a,b]$,*

(ii) $\{\phi_n\}$ *and* $\{F_n\}$ *be respectively $UACG^*$ and $UACG^*$-(C_1-sense) on $[a,b]$,*

(iii) ϕ_n *converge to a continuous function (or $\{\phi_n\}$ be equi-continuous) on $[a,b]$.*

Then F is ACG^* (C_1-sense) on $[a,b]$.

PROOF. Since $\{F_n\}$ is $UACG^*$ (C_1-sense) on $[a,b]$, $[a,b]$ can be expressed as countable union of closed sets on each of which $\{F_n\}$ is UAC^* (C_1-sense). Let E be such a closed set and $\{(a_k,b_k)\}$ be a sequence of non-overlapping intervals with end points on E. Since $F_n \to F$ where each F_n is D^*-integrable on $[a,b]$, and $\{\phi_n\}$ is $UACG^*$ on $[a,b]$ and ϕ_n converge to a continuous function (or $\{\phi_n\}$ is equi-continuous), then from [3, pages 40–44, Corollary 7.7 or Corollary 7.9] applied on $[a_k,x] \subset [a_k,b_k]$ we have $\lim_{n\to\infty}\int_{a_k}^x F_n = \int_{a_k}^x F$, $\quad a_k < x < b_k$. So $\lim_{n\to\infty}[C_1(F_n;a_k,x) - F_n(a_k)] = C_1(F;a_k,x) - F(a_k)$. Hence

$$\begin{aligned}\liminf_{n\to\infty} \sup_{a_k<x<b_k} &|C_1(F_n;a_k,x) - F_n(a_k)| \\ &\geq \liminf_{n\to\infty} |C_1(F_n;a_k,x) - F_n(a_k)| \\ &= |C_1(F;a_k,x) - F(a_k)|,\end{aligned}$$

and so

$$\begin{aligned}\liminf_{n\to\infty} \sup_{a_k<x<b_k} &|C_1(F_n;a_k,x) - F_n(a_k)| \\ &\geq \sup_{a_k<x<b_k} |C_1(F;a_k,x) - F(a_k)|.\end{aligned}$$

Thus

$$\begin{aligned}\sum_k \sup_{a_k<x<b_k} &|C_1(F;a_k,x) - F(a_k)| \\ &\leq \liminf_{n\to\infty} \sum_k \sup_{a_k<x<b_k} |C_1(F_n;a_k,x) - F_n(a_k)|\end{aligned}$$

and similarly

$$\begin{aligned}\sum_k \sup_{a_k<x<b_k} &|C_1(F;b_k,x) - F(b_k)| \\ &\leq \liminf_{n\to\infty} \sum_k \sup_{a_k<x<b_k} |C_1(F_n;b_k,x) - F_n(b_k)|.\end{aligned}$$

Therefore since $\{F_n\}$ is UAC^* (C_1-sense) on E, F is AC^* (C_1-sense) on E and F being C_1-continuous in $[a,b]$, F is ACG^* (C_1-sense) on $[a,b]$.

Lemma 6 *If $\{F_n\}$ is $UACG^*$ (C_1-sense) in $[a,b]$, then $\{F_n\}$ is $UACG$ in $[a,b]$.*

PROOF. Let $[a,b] = \cup E_i$ where E_i is closed and $\{F_n\}$ is UAC^* (C_1-sense) on E_i for each i. Let $\varepsilon > 0$ be arbitrary and let $\delta = \delta(\varepsilon)$ be obtained by applying Definition 5 on the set E_i. Let $\{(a_k, b_k)\}$ be a countable collection of nonoverlapping intervals with end points on E_i such that $\sum(b_k - a_k) < \delta$. Then $\sum_k \omega_n(a_k, b_k) < \varepsilon$ for all n. Since $|F_n(b_k) - F_n(a_k)| \leq H\omega_n(a_k, b_k)$, where H is a constant independent of n (See [8, Lemma III] and [9, Lemma 1].), we have $\sum_k |F_n(b_k) - F_n(a_k)| < H\varepsilon$. This shows that $\{F_n\}$ is UAC on E_i. This completes the proof.

Lemma 7 *Let $X \subset [a,b]$ be a closed set. Let $F : [a,b] \to \mathbb{R}$ be D-integrable in $[a,b]$ and D_1-continuous on X. Let $F \in VB(X)$ and let*

$$\begin{aligned} &\lim_{k\to\infty} \sup_{a_k<t<b_k} |C_1(F; a_k, t) - F(a_k)| = 0 \\ &\lim_{k\to\infty} \sup_{a_k<t<b_k} |C_1(F; b_k, t) - F(b_k)| = 0 \end{aligned} \tag{43}$$

where $\{(a_k, b_k)\}$ is the collection of contiguous intervals of X. (Here the integral in the definition of $C_1(F, x, y)$ is taken as D-integral). Then F is continuous on X relative to X.

PROOF. Let $x \in X$. If x is an isolated point of X, there is nothing to prove. So we suppose that x is a limit point of X, say, from the left. Since $F \in VB(X)$, $\lim_{t\to x-} F(t)$ exists and is finite, the limit being taken relative to X. We may suppose that $\lim_{t\to x-} F(t) = 0$. We are to show that $F(x) = 0$.

Case I Let x be a limit point of $\cup(a_k, b_k)$ from the left. Let $\varepsilon > 0$. Then there is $\delta > 0$ such that

$$|F(\xi)| < \varepsilon \text{ for } \xi \in (x - \delta, x) \cap X. \tag{44}$$

From the second of the relations (43) there is k_0 such that

$$\left| \int_\xi^{b_k} F - (b_k - \xi)F(b_k) \right| < \varepsilon(b_k - \xi) \text{ for } k \geq k_0 \text{ and } \xi \in (a_k, b_k). \tag{45}$$

Let $\{(a_{k_n}, b_{k_n})\}$ be the subcollection of $\{(a_k, b_k);\ k \geq k_0\}$ such that

$$\cup_{n=1}^{\infty}(a_{k_n}, b_{k_n}) \subset (x - \delta, x).$$

Let $x - \delta_0 = \inf \cup_{n=1}^{\infty}(a_{k_n}, b_{k_n})$. Then $0 < \delta_0 \le \delta$. Let $t \in (x - \delta_0, x)$. If $t \in \cup_{n=1}^{\infty}(a_{k_n}, b_{k_n})$, then for some $n = m$ say, $t \in (a_{k_m}, b_{k_m})$ and so from (45) and (44) we have

$$\left| \int_t^{b_{k_m}} F \right| < 2\varepsilon(b_{k_m} - t) \tag{46}$$

and for all intervals (a_{k_n}, b_{k_n}) with $(a_{k_n}, b_{k_n}) \subset (t, x)$

$$\left| \int_{a_{k_n}}^{b_{k_n}} F \right| < 2\varepsilon(b_{k_n} - a_{k_n}). \tag{47}$$

If $t \in X$, then for all intervals (a_{k_n}, b_{k_n}) with $(a_{k_n}, b_{k_n}) \subset (t, x)$, (47) holds. Also if $E = (t, x) \cap X$, then from (44)

$$\left| \int_E F \right| \le \varepsilon\mu(E). \tag{48}$$

Hence adding all the relations (46), (47), (48) we have by using [7, page 257, Theorem 5.1] $\left|\int_t^x F\right| < 2\varepsilon(x - t)$. This shows that

$$\lim_{t \to x-} \frac{1}{x - t} \int_t^x F = 0. \tag{49}$$

Since F is D_1-continuous at x, $F(x) = 0$.

Case II Suppose x is not a limit point of $\cup(a_k, b_k)$ from the left. Then for $\varepsilon > 0$ there is $\delta > 0$ such that $|F(\xi)| < \varepsilon$ for $\xi \in (x - \delta, x)$. Hence $\left|\frac{1}{x-t}\int_t^x F\right| \le \varepsilon$ for $t \in (x - \delta, x)$ which shows that (49) holds and so $F(x) = 0$ as above.

If x is a limit point of X from the right the proof is similar.

Remark 4 *Lemma 7, which is used in Theorem 6, has some interest in itself, since it gives a reasonably sufficient condition under which a D_1-continuous function (and hence a C_1-continuous function i.e. a derivative function) becomes continuous on a closed set.*

Theorem 4 *Let $\{f_n\}$ be a sequence of C_1D-integrable functions on $[a, b]$ and $F_n(x) = (C_1D)\int_a^x f_n$, $\phi_n(x) = (D^*)\int_a^x F_n$, $a \le x \le b$. Let*

(i) $\lim_{n\to\infty} f_n = f$ *almost everywhere in* $[a, b]$,

(ii) $\{F_n\}$ *be equi-C_1-continuous on* $[a, b]$,

(iii) $\{F_n\}$ *be* $UACG^*$ *(C_1-sense) and* $\{\phi_n\}$ *be* $UACG^*$ *on* $[a,b]$ *and* $\{F_n\}$ *be pointwise bounded on* $[a,b]$,

(iv) *for every perfect set in* $[a,b]$ *having infinitely many complementary intervals* $\{(\alpha_k,\beta_k)\}$

$$\lim_{k\to\infty} \sup_{x\in(\alpha_k,\beta_k]} \left|\frac{1}{x-\alpha_k}(D^*)\int_{\alpha_k}^{x} F_{k,n}(t)\,dt\right| = 0,$$

uniformly in n, *and* $F_{k,n}(x) = (C_1D)\int_{\alpha_k}^{x} f_n$, $\alpha_k \le x \le \beta_k$.

Then f *is* C_1D*-integrable in* $[a,b]$ *and* $\lim_{n\to\infty}(C_1D)\int_a^b f_n = (C_1D)\int_a^b f$.

PROOF. We shall show that under the hypothesis $\{\phi_n\}$ is equicontinuous on $[a,b]$. Let $\xi \in [a,b]$. Since $\{F_n\}$ is equi-C_1-continuous at ξ, for any $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that $|\frac{1}{x-\xi}(D^*)\int_\xi^x F_n(t)\,dt - F_n(\xi)| < \varepsilon$ whenever $0 < |x-\xi| < \delta$, for all n. Hence $|\phi_n(x) - \phi_n(\xi)| < \varepsilon|x-\xi| + |F_n(\xi)||x-\xi|$. Since $\{F_n(\xi)\}$ is bounded, letting $\delta_0 < \min[\delta, 1, \varepsilon/M]$ where $M = M(\xi) = \sup|F_n(\xi)|$, we have $|\phi_n(x) - \phi_n(\xi)| < 2\varepsilon$ whenever $0 < |x-\xi| < \delta_0$, for all n. So $\{\phi_n\}$ is equi-continuous on $[a,b]$. Since $UACG^*$ (C_1-sense) implies $UACG$ by Lemma 6 and also $UACG^*$ implies $UACG$ and since equi-C_1-continuity implies equi-D_1-continuity, by Theorem 1, f is D_1-integrable on $[a,b]$ and

$$\lim_{n\to\infty}(C_1D)\int_a^b f_n = (D_1)\int_a^b f = F(b) \tag{50}$$

where

$$F(x) = (D_1)\int_a^x f, \quad a \le x \le b. \tag{51}$$

To complete the proof we need to show that f is, in fact C_1D-integrable. We will first show that F is C_1-continuous on $[a,b]$. Since (50) is true if b is replaced by any x, $a \le x \le b$, $F_n \to F$ everywhere on $[a,b]$. Also $\{\phi_n\}$ is equi-continuous and $UACG^*$ on $[a,b]$. So, by [1, page 40, Theorem 47], F is D^*-integrable in $[a,b]$ and $\lim_{n\to\infty}(D^*)\int_a^b F_n = (D^*)\int_a^b F$. Since this is true if b is replaced by any x, $a \le x \le b$, we have

$$\lim \phi_n(x) = \phi(x), \tag{52}$$

where $\phi(x) = (D^*)\int_a^x F$.

Let ξ be arbitrary point in $[a,b]$. Since $\{F_n\}$ is equi-C_1-continuous at ξ, for any $\varepsilon > 0$ there is $\delta = \delta(\varepsilon)$ such that for all n

$$\left|\frac{\phi_n(x) - \phi_n(\xi)}{x-\xi} - F_n(\xi)\right| < \varepsilon \text{ whenever } 0 < |x-\xi| < \delta. \tag{53}$$

Since $F_n(\xi) \to F(\xi)$, we have from (52) and (53) letting $n \to \infty$ $|\frac{\phi(x)-\phi(\xi)}{x-\xi} - F(\xi)| \leq \varepsilon$ whenever $0 < |x - \xi| < \delta$. Hence F is C_1-continuous at ξ and so on $[a, b]$. Then by Lemma 5, F is ACG^* (C_1-sense) on $[a, b]$. Hence by [8, Theorem III] we have from (51) $C_1DF(x) = F'_{ap}(x) = f(x)$ almost everywhere on $[a, b]$. So f is C_1D-integrable on $[a, b]$ and from (50) $\lim_{n\to\infty}(C_1D)\int_a^b f_n = (C_1D)\int_a^b f$.

Theorem 5 *Let $\{f_n\}$ be a sequence of C_1D-integrable functions on $[a, b]$ and $F_n(x) = (C_1D)\int_a^x f_n$, $\phi_n(x) = (D^*)\int_a^x F_n$, $a \leq x \leq b$. Suppose*

(i) *$\lim_{n\to\infty} f_n = f$ almost everywhere on $[a, b]$.*

(ii) *$\{F_n\}$ and $\{\phi_n\}$ be respectively $UACG^*$ (C_1-sense) and $UACG^*$ on $[a, b]$,*

(iii) *F_n converge pointwise to a C_1-continuous function and ϕ_n converge pointwise to a continuous function (or $\{\phi_n\}$ be equi-continuous on $[a, b]$).*

Then f is C_1D-integrable on $[a, b]$ and $\lim_{n\to\infty}(C_1D)\int_a^b f_n = (C_1D)\int_a^b f$.

PROOF. From Lemma 5 we have F is ACG^* (C_1-sense) on $[a, b]$ where F is the limit of F_n. Since $UACG^*$ (C_1-sense) implies $UACG$, by Lemma 6 and C_1-continuity implies D_1-continuity and C_1D-integrability implies D_1-integrability with integrals equal, by Theorem 2, f is D_1-integrable on $[a, b]$ and

$$\lim_{n\to\infty}(C_1D)\int_a^b f_n = (D_1)\int_a^b f. \tag{54}$$

We are to show that f is indeed C_1D-integrable on $[a, b]$. By the given condition the left hand limit is $F(b)$. Hence $(D_1)\int_a^b f = F(b)$. This is also true if b is replaced by any x, $a \leq x \leq b$. Clearly F is ACG and D_1-continuous on $[a, b]$ and so F is a D_1-primitive of f and hence $F'_{ap} = f$ almost everywhere on $[a, b]$ and since F is ACG^* (C_1-sense) on $[a, b]$, by [8, Theorem III], we have $C_1DF(x) = F'_{ap}(x) = f(x)$ almost everywhere on $[a, b]$. So f is C_1D-integrable on $[a, b]$ and hence from (54) $\lim_{n\to\infty}(C_1D)\int_a^b f_n = (C_1D)\int_a^b f$.

Theorem 6 *Let $\{f_n\}$ be a sequence of C_1D-integrable functions on $[a, b]$ and let $F_n(x) = (C_1D)\int_a^x f_n$, $a \leq x \leq b$. Suppose*

(i) *$\lim_{n\to\infty} f_n = f$ almost everywhere in $[a, b]$,*

(ii) *for each $i = 1, 2, \ldots$ there exists a closed set X_i and D^*-integrable functions G_i, H_i on $[a, b]$ such that $[a, b] = \cup X_i$, G_i, $H_i \in VB(X_i)$ with*

$$G_i(v) - G_i(u) \leq F_n(v) - F_n(u) \leq H_i(v) - H_i(u) \tag{55}$$

for $n \geq i$ whenever u or $v \in X_i$ and (in case there are infinitely many contiguous intervals of X_i)

$$(56)\qquad \begin{array}{l} \sum_{k=1}^{\infty} \sup_{x\in(a_{ik},b_{ik}]} |C_1(G_i;a_{ik},x) - G_i(a_{ik})| < \infty \\ \sum_{k=1}^{\infty} \sup_{x\in[a_{ik},b_{ik})} |C_1(G_i;b_{ik},x) - G_i(b_{ik})| < \infty \end{array}$$

where $\{(a_{ik}, b_{ik})\}$ are the complementary intervals of X_i, with similar relations holding when G_i is replaced by H_i,

(iii) *$\lim_{n\to\infty} F_n = F$ where F is C_1-continuous, the convergence being uniform on the set of end points of contiguous intervals (a_{ik}, b_{ik}) of the sets X_i, $i = 1, 2, \ldots$.*

Then f is C_1D-integrable in $[a, b]$ and F is the CD_1-primitive of f.

PROOF. Since (56) implies $\lim_{k\to\infty} \sup_{a_{ik}<x<b_{ik}} |C_1(G_i, a_{ik}, x) - G_i(a_{ik})| = 0$ with similar remark for H_i, we conclude that all the hypothesis of Theorem 3 are satisfied and so by Theorem 3, f is D_1-integrable in $[a, b]$ and F is the D_1-primitive of f.

It follows from (ii) that if $t \in (a_{ik}, b_{ik})$, then

$$G_i(t) - G_i(a_{ik}) \leq F(t) - F(a_{ik}) \leq H_i(t) - H_i(a_{ik})$$

and therefore for $x \in (a_{ik}, b_{ik})$

$$\begin{aligned} C_1(G_i;a_{ik},x) - G_i(a_{ik}) &\leq C_1(F;a_{ik},x) - F(a_{ik}) \\ &\leq C_1(H_i;a_{ik},x) - H(a_{ik}). \end{aligned}$$

Hence

$$(57)\qquad \begin{array}{rl} & \sum_{k=1}^{\infty} \sup_{a_{ik}<x<b_{ik}} |C_1(F;a_{ik},x) - F(a_{ik})| \\ \leq & \sum_{k=1}^{\infty} \sup_{a_{ik}<x<b_{ik}} |C_1(G_i;a_{ik},x) - G_i(a_{ik})| \\ + & \sum_{k=1}^{\infty} \sup_{a_{ik}<x<b_{ik}} |C_1(H_i;a_{ik},x) - H_i(a_{ik})| \quad < \quad \infty. \end{array}$$

Similarly

$$(58)\qquad \sum_{k=1}^{\infty} \sup_{a_{ik}<x<b_{ik}} |C_1(F;b_{ik},x) - F(b_{ik})| < \infty,$$

From (55) and (iii), $F \in VB(X_i)$ for each i. Since F is C_1-continuous in $[a, b]$, F is D_1-continuous in $[a, b]$. The conditions (57) and (58) show that the conditions (43) of Lemma 7 are also satisfied for the set X_i and the contiguous intervals $\{(a_{ik}, b_{ik})\}$. Hence by Lemma 7, F is continuous on X_i (relative to

X_i) for each i. Since F is a D_1-primitive of f, F is ACG and hence F satisfies Lusin condition (N) on $[a, b]$ (cf. [7, page 225, Theorem 6.1]). Hence $F \in AC(X_i)$ for each i [7, page 227, Theorem 6.7]. So from (57), (58) and [8, Theorem II] we conclude that F is AC^* (C_1-sense) on X_i for each i. Since $[a, b] = \cup_{i=1}^{\infty} X_i$ and F is C_1-continuous on $[a, b]$, F is ACG^* (C_1-sense) on $[a, b]$. Since F is D_1-primitive of f, $F'_{ap} = f$ almost everywhere in $[a, b]$. Hence by [8, Theorem III] $C_1DF = f$ almost everywhere in $[a, b]$. So f is C_1D-integrable in $[a, b]$ and F is its C_1D-primitive.

Remark 5 *Analogous results hold for the GM_1-integral of Ellis [2]. It may be noted that these results are known for the D^*-integral (see [1], [3], [4] respectively).*

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