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Milan Matejdes, Department of Mathematics, Zvolen Technical University, Masarykova 24, 960 53 Zvolen, Slovakia, matejdesvsld.tuzvo.sk

CONTINUITY OF MULTIFUNCTIONS

Abstract

This paper summarizes the results related to the study of the continuity of multifunctions, separate and joint continuity, selection theorems, limits of multifunctions and generalized derivatives. The concept is based on the notion of Baire continuity and its connection to the topics mentioned above is given.

1. Introduction

In recent years a considerable amount of research has been devoted to questions involving many types of generalized continuity. Perhaps the notion of quasi-continuity has been studied most intensively. The main reason for this study is a close connection to other continuity types and various applications in topology, mathematical analysis, and probability. Among the papers let us mention the topical survey of T. Neubrunn [0], giving comprehensive information concerning quasi-continuity. Making a slight generalization in the definition of quasi-continuity we obtain a new notion of continuity (so called $\mathcal{B}$-continuity or Baire-continuity) which was introduced and studied for its deep connection to the following topics:

- continuity points
- the Baire property
- selection theorems
- dense representation of multifunctions
- inclusion relations

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- limits of multifunctions
- separate and joint continuity

Our goal is to give a survey of results about Baire-continuity of multifunctions mainly those which concern the topics mentioned above. Despite the fact that Baire-continuity is more general than quasi-continuity we can consider this work as a continuation of those papers which are devoted to the latter because both are very closely related. We hope that our survey will give new information about approaches to the study of generalized types of continuity.

The proofs are usually included in cases when, according to our opinion, the results are not known. Otherwise the reader is referred to the corresponding papers as well as to the papers which contain further information concerning a given topic.

2. Notation and Basic Definitions

Throughout this work, X, Y denote topological spaces and M denotes a metric space. If $A \subset X$, we use the notation $cl(A), int(A), D(A)$ for the closure, the interior and the set of all points at which A is of second category, respectively. By $S_\epsilon(A)$ we denote an ϵ - neighborhood of $A \subset M$, $\epsilon > 0$ i.e., $S_\epsilon(A) = \{z \in M : d(z, A) < \epsilon\}$ where d is a metric for M. If $A = \{z\}$, we briefly write $S_\epsilon(z)$. If (M, d) is separable, by (M^o, d^o) we denote a metrizable compactification of (M, d) (see [7, p.-328, Th.-3, p.-337, Corollary of Th. 19]).

A multifunction $F : X \to \mathcal{P}(Y)$ is a set valued mapping which assigns to each element x of X a set $F(x) \in \mathcal{P}(Y) = \{A \subset Y : A \neq \emptyset\}$. For a function f (i.e., a single valued mapping) as a rule we write $f : X \to Y$. If we consider a closed (compact) valued multifunction, we use notation $F : X \to \mathcal{C}(Y)$ $(F : X \to \mathcal{K}(Y))$ where $\mathcal{C}(Y)$ $(\mathcal{K}(Y))$ denotes the set of all non-empty closed (non-empty compact) sets.

The upper (lower) inverse image $F^+(A)$ $(F^-(A))$ is defined for any set $A \subset Y$ as $F^+(A) = \{x \in X : F(x) \subset A\}$, $F^-(A) = \{x \in X : F(x) \cap A \neq \emptyset\}$. Identifying $\{f(x)\}$ with $f(x)$ we have $f^+(A) = f^-(A) = f^{-1}(A) = \{x : f(x) \in A\}$. A selection of $F : X \to \mathcal{P}(Y)$ is any function $f : X \to Y$ such that $f(x) \in F(x)$ for any $x \in X$.

Definition 1 *(The Baire continuity) Let $\mathcal{B}$ be a family of subsets of X such that $\mathcal{O} \subset \mathcal{B} \subset \mathcal{B}r \cup \mathcal{O}$ where $\mathcal{O} = \{A \subset X : A$ is non-empty open$\}$ and $\mathcal{B}r = \{A \subset X : A$ is of second category having the Baire property $\}$. A multifunction $F : X \to \mathcal{P}(Y)$ is lower-$\mathcal{B}$-continuous (upper-$\mathcal{B}$-continuous) (briefly l-$\mathcal{B}$-continuous (u-$\mathcal{B}$-continuous)) at a point x if for any open sets*

V, U with $F(x) \cap V \neq \emptyset$ ($F(x) \subset V$) there is a set $B \in \mathcal{B}$ such that $B \subset F^-(V) \cap U$ ($B \subset F^+(V) \cap U$). F is $\mathcal{B}$-continuous at x if it is lower and upper $\mathcal{B}$-continuous at x. F is lower $\mathcal{B}$-continuous, upper $\mathcal{B}$-continuous, $\mathcal{B}$-continuous if it is so at any point, respectively.

For $\mathcal{B} = \mathcal{O}$ we have the well-known notion of upper (lower) quasi-continuity. If F is both lower and upper quasi-continuous, then we say that F is quasi-continuous. Supposing X is Baire, $\mathcal{O} \subset \mathcal{B}r$ hence quasi-continuity can be considered as a special case of $\mathcal{B}r$-continuity.

The assumption that F is l-$\mathcal{B}r$-continuous (u-$\mathcal{B}r$-continuous) implies that X is a Baire space. The similar situation arises in the setting of l-$\mathcal{D}$-continuity (u-$\mathcal{D}$-continuity) where $\mathcal{D} = \{A \subset X : A \text{ is of second category}\}$ (i.e., F is l-$\mathcal{D}$-continuous (u-$\mathcal{D}$-continuous) at a point x if $x \in D(F^-(V))$ ($x \in D(F^+(V))$ for any open V with $V \cap F(x) \neq \emptyset$ ($V \subset F(x)$). Dealing with these cases the assumption that X is Baire will be omitted.

As for historical background, a function of two variables being quasi-continuous under the assumption that it is continuous in each variable was first mentioned by Volterra (see [3]). A generalization for the case $X = R_n$, $Y = R$ can be found in [17]. Another generalization for topological spaces as well as for multifunctions was studied in [39], [40], [45]. Further survey information concerning quasi-continuity and its applications can be found in [0]. Perhaps the first definition of $\mathcal{B}$-continuity of multifunction and its systematic study were introduced in [22], [23], [10]. The reader will be referred to further papers in the corresponding sections below.

3. The Baire Continuity and Other Continuity Types

It is evident that upper (lower) quasi-continuity implies u-$\mathcal{B}$-continuity (l-$\mathcal{B}$-continuity). The converse is not true as the following example shows.

Example 2 *Let $X = Y = [0,1]$ with the usual topology and $\mathcal{B} = \mathcal{B}r$. Define $F, G : X \to \mathcal{K}(Y)$ as*

$$F(x) = \begin{cases} \{0\} & \textit{if } x \textit{ is irrational} \\ [0,1] & \textit{if } x \textit{ is rational} \end{cases}$$

$$G(x) = \begin{cases} \{0\} & \textit{if } x \textit{ is rational} \\ [0,1] & \textit{if } x \textit{ is irrational.} \end{cases}$$

On the other hand, within the single valued multifunctions, the following theorem holds.

Theorem 3 *Let Y be regular. If $f : X \to Y$ is $\mathcal{B}$-continuous, then f is quasi-continuous.*

PROOF. Suppose that f is not quasi-continuous at p. Since Y is regular, there are open sets V, U with $p \in U$, $f(p) \in V$ such that $f^{-1}(Y \setminus cl(V)) \cap U$ is dense in U. f is $\mathcal{B}$-continuous at p, hence there is a set $B \in \mathcal{B}$ such that $B \subset U \cap f^{-1}(V)$. $B \in \mathcal{B}r \cup \mathcal{O}$, hence B is either open or $B = (G \setminus I) \cup J$ where G is an open set of second category and I, J are of first category. If B is open, then $f^{-1}(Y \setminus cl(V)) \cap U \cap B \neq \emptyset$ what is a contradiction with the inclusion $B \subset U \cap f^{-1}(V)$.

Let $B = (G \setminus I) \cup J$. Since G is of second category, $U \cap G \cap int(D(G)) \neq \emptyset$. Let $x \in U \cap G \cap int(D(G)) \cap f^{-1}(Y \setminus cl(V))$. Since f is $\mathcal{B}$-continuous at x, there is a set $B_1 \in \mathcal{B}$ such that $B_1 \subset U \cap G \cap int(D(G)) \cap f^{-1}(Y \setminus cl(V))$. That means B_1 is either open or of second category. Since $B_1 \subset int(D(G))$, B_1 is of second category in both cases. Thus $\emptyset \neq B_1 \setminus I \subset U \cap ((G \setminus I) \cup J) \cap f^{-1}(Y \setminus cl(V)) = U \cap B \cap f^{-1}(Y \setminus cl(V))$ what is a contradiction to the inclusion $B \subset U \cap f^{-1}(V)$.

The multifunction version of the previous theorem was proved in [22, Th.-2.5].

Theorem 4 *Let X be a Baire space and Y be a second countable regular one. A multifunction $F : X \to \mathcal{K}(Y)$ is $\mathcal{B}$-continuous if and only if it is quasi-continuous.*

Using the notion of a semi-open set [20] the quasi-continuity can be formulated as follows.

Theorem 5 *([38]). Let $\mathcal{Q}$ be a family of all semi-open subsets of X (i.e., $A \subset X$ is semi-open (or quasi-open) if $A \subset cl(int(A))$ or equivalently $A = N \cup G$ where G is open and N is nowhere dense such that $N \subset cl(G)$). A multifunction $F : X \to \mathcal{P}(Y)$ is upper (lower) quasi-continuous iff $F^+(V) \in \mathcal{Q}$ ($F^-(V) \in \mathcal{Q}$) for any open $V \subset Y$. That means, the family $\mathcal{Q}$ gives a characterization of the global lower quasi-continuity and upper quasi-continuity (the case $\mathcal{B} = \mathcal{O}$).*

In the case $\mathcal{B} = \mathcal{B}r$ we can obtain similar equivalence [34, Corollary 4].

Theorem 6 *Let X be a Baire space and let $\mathcal{B}s = \{A \subset X : A = (G \setminus I) \cup J$ where G is an open set, I, J are of first category and $J \subset cl(G)$ $\}$. If $F : X \to \mathcal{K}(M)$, then the following conditions are equivalent*

(i) F is u-$\mathcal{B}r$-continuous (l-$\mathcal{B}r$-continuous),

(ii) F is u-$\mathcal{B}s$-continuous (l-$\mathcal{B}s$-continuous),

(iii) $F^+(V) \in \mathcal{B}s$ $(F^-(V) \in \mathcal{B}s)$ *whenever* V *is open.*

In connection with the family of semi-open sets as well as the family $\mathcal{B}s$ we can consider other ones which have been discussed e.g. in [38], [5].

Let $\mathcal{P}s = \{A \subset X : A$ is pseudo-open i.e., $A = G \cup N$ where G is open and N is of first category$\}$.

$\mathcal{S} = \{A \subset X : A$ is simply-open i.e., $A = G \cup N$ where G is open and N is nowhere dense$\}$.

We can see immediately that l-$\mathcal{P}s$, l-$\mathcal{S}$, l-$\mathcal{Q}$ (u-$\mathcal{P}s$, u-$\mathcal{S}$, u-$\mathcal{Q}$) continuities in the sense of Definition 1 are equivalent. Owing to this fact, the pseudo and simple-continuity are defined as follows in literature.

Definition 7 *A multifunction* $F : X \to \mathcal{P}(Y)$ *is said to be lower (upper) pseudo-continuous (lower (upper) simply-continuous) if*

$$F^-(V) \in \mathcal{P}s \ (F^+(V) \in \mathcal{P}s) \ (F^-(V) \in \mathcal{S} \ (F^+(V) \in \mathcal{S}))$$

for any open $V \subset Y$.

Besides the basic relation between quasi, simple, and pseudo-continuity (given by $\mathcal{Q} \subset \mathcal{S} \subset \mathcal{P}$) we would like to mention in this section a decomposition theorem. Further results will be included in the next sections.

Theorem 8 *([22, Th. 1.1], [38]). Let* Y *be second countable and* X *Baire.* $F : X \to \mathcal{P}(Y)$ $(F : X \to \mathcal{K}(Y))$ *is lower (upper) quasi-continuous if and only if it is lower (upper) pseudo-continuous and l-$\mathcal{D}$-continuous (u-$\mathcal{D}$-continuous).*

The proof follows from [22, Th. 1.1], [30, Th. 6] and Theorem 5. Consequently, for u-$\mathcal{D}$-continuous (l-$\mathcal{D}$-continuous) multifunction, the following equivalences hold [30, Corollaries 13, 14].

Theorem 9 *Let* M *be a separable metric space and* X *be a Baire one. Let* $F : X \to \mathcal{P}(Y)$ $(F : X \to \mathcal{K}(Y))$ *be u-$\mathcal{D}$-continuous (l-$\mathcal{D}$-continuous). Then the following conditions are equivalent*

(i) F *is upper quasi-continuous (lower quasi-continuous),*

(ii) F *is upper simply-continuous (lower simply-continuous),*

(iii) F *is upper pseudo-continuous (lower pseudo-continuous).*

Another type of generalized continuity which will be discussed in our paper is cliquishness. As for the cliquishness of functions the reader is referred e.g. to [4], [12], [46]. The cliquish multifunctions were studied in [32], [33], [8].

Definition 10 *([32]). A multifunction $F : X \to \mathcal{P}(M)$ is said to be $\mathcal{B}$-cliquish at a point $p \in X$ if for any $\epsilon > 0$ and any neighborhood U of p there is a set $B \in \mathcal{B}$ such that $\bigcap_{x \in B} S_\epsilon(F(x)) \neq \emptyset$. F is cliquish if it is cliquish at any point.*

For a function $f : X \to M$ and $\mathcal{B} = \mathcal{O}$ Definition 10 is equivalent to that of original notion of cliquishness.

Definition 11 *A function f is cliquish at p if for any $\epsilon > 0$ and any neighborhood U of p there is a non-empty open set $B \subset U$ such that $d(f(x_1), f(x_2)) < \epsilon$ for any $x_1, x_2 \in B$ where d is a metric for M.*

The main result concerning the cliquishness of functions can be found in [12, Th.-1].

Theorem 12 *Let X be a Baire space. A function $f : X \to M$ is $\mathcal{O}$-cliquish if and only if the set of continuity points of f is residual.*

More general results were proved in [33, Corollaries 1, 3].

Theorem 13 *([33, Corollary 1]). Let X be a Baire space. For a function $f : X \to M$ (M-separable) the following conditions are equivalent*

(i) f is $\mathcal{B}$-cliquish,

(ii) the set of $\mathcal{B}$-continuity points of f is residual,

(iii) the set of $\mathcal{B}$-continuity points of f is dense.

Theorem 14 *([33, Corollary 3]). Let X be a Baire space. A function $f : X \to M$ (M-separable) is $\mathcal{B}r$-cliquish if and only if f has the Baire property i.e., $f^{-1}(V)$ has the Baire property for any open $V \subset Y$.*

Further results concerning the cliquishness of multifunctions and their selections will be discussed in section 6.

Some generalized continuity notions such as α-continuity, somewhat continuity, almost continuity, almost quasi-continuity are also related to Baire continuity but they are not included in this work. The reader is referred to the corresponding papers [13], [15], [39], [41], [42], [43], [48].

4. The Sets of Upper and Lower Semi-Continuity Points

In this section we will recall the notion of semi-continuity of multifuntions and the structure of the set of semi-continuity points will be discussed.

Definition 15 *[19, p. 32], [0]. A multifunction $F : X \to \mathcal{P}(Y)$ is lower (upper) semi-continuous at $x \in X$ (briefly l.s.c. (u.s.c.)) if $x \in int(F^-(V))$ ($x \in int(F^+(V))$) for any open $V \subset Y$ such that $V \cap F^-(V) \neq \emptyset$ ($V \subset F^+(V)$). The set of all points in which F is lower (upper) semi-continuous will be denoted by $C_l(F)$ ($C_u(F)$). Further, put $D_l(F) = X \setminus C_l(F)$ ($D_u(F) = X \setminus C_u(F)$).*

A fundamental result concerning the set of continuity points of quasi-continuous functions is due to Levine [20]. A generalization for multifunctions can be found in [11], [16]. Perhaps, the most general results were proved in [22], [23].

Theorem 16 *([22, Th. 2.1 and 2.2]). Let Y be a second countable regular space. If $F : X \to \mathcal{P}(Y)$ ($F : X \to \mathcal{K}(Y)$) is u-$\mathcal{B}r$-continuous (l-$\mathcal{B}r$-continuous), then $C_l(F)$ ($C_u(F)$) is residual.*

Theorem 17 *([23, Th. 1 and 2]). If $F : X \to \mathcal{K}(M)$ is u-$\mathcal{B}$-continuous (l-$\mathcal{B}$-continuous), then $D_l(F)$ ($D_u(F)$) is of first category.*

Theorem 18 *([30, Th. 6, 7]). Let Y be a second countable topological space. A multifunction $F : X \to \mathcal{P}(Y)$ ($F : X \to \mathcal{K}(Y)$) is lower (upper) pseudo-continuous if and only if $D_l(F)$ ($D_u(F)$) is of first category.*

From the previous two theorems follows:

Theorem 19 *Let M be a separable metric space. If $F : X \to \mathcal{K}(M)$ is u-$\mathcal{B}$-continuous (l-$\mathcal{B}$-continuous), then F is lower (upper) pseudo-continuous i.e., $F^-(V) = G \cup I$ ($F^+(V) = G \cup I$) for any open $V \subset Y$ where G is open and I is of first category.*

In connection with the structure $D_l(F)$ ($D_u(F)$) of a lower (upper) quasi-continuous multifunction we mention the results of Ewert and Lipinski [6].

Theorem 20 *([6, Th. 15, 16]). Let Y be a second countable space. If $F : X \to \mathcal{P}(Y)$ is lower quasi-continuous ($F : X \to \mathcal{K}(Y)$ is upper quasi-continuous), then $D_l(F)$ ($D_u(F)$) is of first category.*

As we can see from Example 2 an analogous theorem does not hold for l-$\mathcal{B}$-continuity (u-$\mathcal{B}$-continuity).

Further results concerning $D_l(F)$ and $D_u(F)$ will be discussed in section 8 that is devoted to the limits of multifunctions.

5. The Baire Property

In this section we will deal with the Baire property of multifunctions which also has a deep connection with the Baire continuity. First we recall a definition.

Definition 21 *A multifunction $F : X \to \mathcal{P}(Y)$ has the Baire property if $F^{-}(V)$ has the Baire property for any open $V \subset Y$.*

This well-known result in this direction says that a function f has the Baire property if and only if there is a residual set A such that f/A is continuous. Using the Baire continuity concept we can obtain another characterization.

Theorem 22 *([22-1, Th. 3.3]). Let X be a Baire space and M be a separable metric. Then for a multifunction $F : X \to \mathcal{K}(M)$ the following conditions are equivalent.*

(i) F has the Baire property,

(ii) F is l-$\mathcal{B}r$-continuous except for a set of first category,

(iii) F is u-$\mathcal{B}r$-continuous except for a set of first category,

(iv) F is $\mathcal{B}r$-continuous except for a set of first category.

The next characterization is based on the closed connection between the Baire property and the upper semi-continuity and the lower quasi-continuity in the following sense:

Theorem 23 *([22, Corollary 2 of Th. 5.6]). Let X be a Baire space and M be metric compact. A multifunction $F : X \to \mathcal{K}(M)$ has the Baire property if and only if there is a lower quasi-continuous and u.s.c. multifunction $G : X \to \mathcal{K}(M)$ such that the set $\{x \in X : F(x) \neq G(x)\}$ is of first category.*

Using selection methods and dense representation we can obtain another equivalent set of conditions for the Baire property. These will be discussed in section 6. Here we recall only one case which seems to be the most natural.

Theorem 24 *([14, Th. 5.6]). Let M be a separable metric space and X be a topological one. Let F be a multifunction with complete values. Then F has the Baire property if and only if there is a sequence $\{f_n\}_{n=1}^{\infty}$ of selections of F having the Baire property such that $F(x) = cl(\bigcup_{n=1}^{\infty} \{f_n(x)\})$ for any $x \in X$.*

6. Selection Theorems

The definition of Baire continuity was motivated by the idea of existence of quasi-continuous selection [22]. As we will see below, u-$\mathcal{B}$-continuity which is more general then upper quasi-continuity implies the existence of quasi-continuous selection. Consequently, within functions, $\mathcal{B}$-continuity is equivalent to quasi-continuity as we proved in Theorem 3. This section will also be devoted to cliquish and pseudo-continuous selections [29]. A dense representation of multifunctions and selections having the Baire property will be given, too.

Perhaps the first result in this direction was proved in [22, Th.-5.3].

Theorem 25 *Let M be a metric compact space and X be a Baire space. If $F : X \to \mathcal{K}(M)$ is u-$\mathcal{B}r$-continuous, then there is a quasi-continuous selection of F.*

A generalization of the previous theorem can be found in [33, Corollary-5].

Theorem 26 *Let M be separable metric and X be a Baire space. If $F : X \to \mathcal{K}(M)$ is u-$\mathcal{B}$-continuous then it has a quasi-continuous selection.*

Considering a multifunction $F : R \to \mathcal{P}(R)$ (R - the real line with the usual topology) defined as: $F(x) = \{\frac{1}{x}\}$ for $x \neq 0$ and $F(0) = R$, we see that the compactness of values of F is essential.

A dense representation of a multifunction $F : X \to \mathcal{P}(Y)$ is any sequence $\{f_n\}_{n=1}^{\infty}$ of selections of F such that $F(x) = cl(\bigcup_{n=1}^{\infty}\{f_n(x)\})$ for any $x \in X$. A dense representation $\{f_n\}_{n=1}^{\infty}$ is called quasi-continuous if f_n is quasi-continuous for any n.

The questions concerning a quasi-continuous dense representation were studied in [22] where quasi-continuity in the Hausdorff metric on $\mathcal{K}(M)$ was considered. The situation is different from that described in Definition 1.

Recall that the Hausdorff metric h_d on $\mathcal{K}(M)$ is defined as

$$h_d(A, B) = max\{sup_{x\in B}d(x, A), sup_{x\in A}d(x, B)\}$$

where d is metric for M.

Now we consider h_d-quasi-continuity of a multifunction $F : X \to \mathcal{K}(M)$ in such a way that we consider F as a quasi-continuous single-valued mapping into $(\mathcal{K}(M), h_d)$.

It is easy to see that if $F : X \to \mathcal{K}(M)$ is h_d-quasi-continuous, then it is lower and upper quasi-continuous and the converse is not true [41, 1.2.7].

Theorem 27 *Let X be a Baire space and M be a separable metric one. If $F : X \to \mathcal{K}(M)$ is h_d-quasi-continuous, then it has a quasi-continuous dense representation.*

PROOF. Considering F as a multifunction into $\mathcal{K}(M^o)$ the proof follows from [22, Th.-5.6].

It is evident that an upper quasi-continuous multifunction need not have a quasi-continuous dense representation. It is sufficient to consider multifunction $F : [0,1] \to [0,1]$ defined as $F(x) = \{0\}$ if $x \neq 1$ and $F(1) = [0,1]$. As for a lower quasi-continuous multifunction we have no information about existence of its quasi-continuous dense representation.

Another interesting question concerning selection theorems is the existence of cliquish selection. Next theorem concerns a sufficient and necessary condition for a $\mathcal{B}$-cliquish selection.

Theorem 28 *([33, Th. 1, Corollary 1]). Let X be a Baire space and M be a separable metric one. Then for $F : X \to \mathcal{K}(M)$ the following conditions are equivalent*

(i) F is $\mathcal{B}$-cliquish,

(ii) F has a selection which is $\mathcal{B}$-continuous at any $x \in S$ where S is residual,

(iii) F has a selection which is $\mathcal{B}$-continuous at any $x \in T$ where T is dense.

The first part of the following Corollary follows from Theorem 22 and the second one from Theorems 12, 18.

Corollary 29 *Under the same conditions on X, M and F as in Theorem 28 the following conditions hold*

(i) F is $\mathcal{B}r$-cliquish iff F has a selection having the Baire property,

(ii) F is $\mathcal{O}$-cliquish iff F has a pseudo-continuous selection.

For the u-$\mathcal{D}$-continuous multifunctions we can find a sufficient and necessary condition for existence of quasi-continuous selection.

Theorem 30 *([33, Th. 2]). Let X be a Baire space and M be a separable metric one. Let $F : X \to \mathcal{K}(M)$ be u-$\mathcal{D}$-continuous. Then F has a quasi-continuous selection if and only if F is $\mathcal{B}$-cliquish.*

The well-known fact that a function being continuous on a dense set is continuous on a residual set can be formulated in multifunction setting as follows.

Theorem 31 *Let X be a Baire space and M be a separable metric one. If $F : X \to \mathcal{K}(M)$ is l-$\mathcal{B}$-continuous on a dense set, then it has a selection which is $\mathcal{B}$-continuous on a residual set. Consequently, if F is lower quasi-continuous on a dense set (F has the Baire property), then F has a selection which is continuous on a residual set (which has the Baire property).*

Proof follows from [33, Corollary 4] and Theorem 12.

Note that a multifunction being l-$\mathcal{B}$-continuous on a dense set need not have the Baire property. It is sufficient to consider $F : R \to \mathcal{K}(R)$ defined as $F(x) = \{0\}$ if x is rational, $F(x) = [0,1]$ if $x \in R \setminus A$ and $F(x) = [-1,0]$ if $x \in A$, where A is a subset of irrational numbers with $D(A) = D(R \setminus A) = R$. From this point of view Theorem 31 seems to be rather general.

7. Inclusion Relations

In this section we will deal with inclusion relations between Baire continuous multifunctions and quasi-continuous (semi-continuous) ones. In a certain sense (as for u-$\mathcal{B}r$-continuity) we find a nice multivalued selection which is equal to a given multifunction on a residual set. Similar results connected with multivalued selections can be found in [47].

By $F \subset G$ we denote the fact that $F(x) \subset G(x)$ for any $x \in X$.

Theorem 32 *Let X be Baire and M be metric compact. If $F : X \to \mathcal{K}(M)$, then the following conditions are equivalent*

(i) F is l-$\mathcal{B}r$-continuous (u-$\mathcal{B}r$-continuous),

(ii) there is a lower quasi-continuous and u.s.c. (lower and upper quasi-continuous) multifunction $G : X \to \mathcal{K}(M)$ such that $G = F$ except for a set of first category and $F \subset G$ ($G \subset F$).

Proof follows from Theorems 1, 2, 5 of [34].

Theorem 33 *Let X be a Baire space and M be separable metric. If $F : X \to \mathcal{K}(M)$ is u-$\mathcal{B}r$-continuous, then there is a sequence $\{f_n\}_{n=1}^{\infty}$ of quasi-continuous selections of F such that $cl(\bigcup_{n=1}^{\infty} \{f_n(x)\}) = F(x)$ except for a set of first category.*

PROOF. Considering F as a multifunction into $\mathcal{K}(M^o)$ the proof follows from [34, Th. 4].

8. Limits of Multifunctions

J. Ewert [9] has investigated the set $C_u(F)$ ($C_l(F)$) of multifunction which is a limit of sequence of lower (upper) quasi-continuous compact valued multifunctions. The similar questions were studied in [21] for limits of c-quasi-continuous multifunctions. In both papers, convergence in the Hausdorff metric was considered. As we shall see the results can be generalized in two directions. The lower quasi-continuity will be replaced by l-$\mathcal{B}$-continuity and convergence in Hausdorff metric can be replaced by upper Kuratowski limit [27], [35].

Note that the upper (lower) Kuratowski limit [18, p.241] of a given sequence of sets $\{A_n\}_{n=1}^{\infty}$ is defined as the set of all points x such that every neighborhood U of x the set $\{n : A_n \cap U \neq \emptyset\}$ ($\{n : A_n \cap U = \emptyset\}$) is infinite (finite). The upper (lower) Kuratowski limit of $\{A_n\}_{n=1}^{\infty}$ will be denoted by $Ls(A_n)$ ($Li(A_n)$). We say that a multifunction $F : X \to \mathcal{P}(Y)$ is upper (lower) Kuratowski limit of a sequence of multifunctions $F_n : X \to \mathcal{P}(Y)$ if $F(x) = Ls(F_n(x))$ ($F(x) = Li(F_n(x))$) for any $x \in X$. We write $F = LsF_n$ ($F = LiF_n$).

The main result proved in [9] says that if $F_n, F : X \to \mathcal{K}(M)$ (X- topological space, M-separable metric) are multifunctions such that F_n are lower (upper) quasi-continuous and $F = LsF_n = LiF_n$, then $D_u(F)$ ($D_l(F)$) is of first category. There are example in [9] that $D_l(F)$ ($D_u(F)$) need not be of first category.

As for a sequence of l-$\mathcal{B}r$-continuous multifunctions one can find a generalization of Ewert result in [27].

Theorem 34 *([27, Th. 1 and Remark 1c]). Let X be a T_1-Baire topological space, Y be a compact metric one. If $F = LsF_n$, where $F_n : X \to \mathcal{P}(Y)$ are l-$\mathcal{B}r$-continuous, then $D_u(F)$ is of first category.*

As the following example shows an upper Kuratowski limit of a sequence of upper quasi-continuous multifunctions need not be l.s.c. at any point.

Example 35 *There is a sequence of quasi-continuous functions $f_n : [0,1] \to [0,1]$ such that $C_l(F) = \emptyset$ where $F = Lsf_n$. The interval [0,1] is considered with the usual topology. Moreover, every selection of F is discontinuous at any point.*

PROOF. Let $A = \{a_k\}_{k=1}^{\infty}$, $B = \{b_k\}_{k=1}^{\infty}$ be dense disjoint subsets of (0,1). Let $f_n : [0,1] \to [0,1]$ be a continuous function such that $f_n(a_k) = 1$ and $f_n(b_k) = 0$ for $k \leq n$. Then $F(x) = Lsf_n(x)$ is non-empty compact subset of [0,1] for any $x \in [0,1]$. Since $F(x) = \{1\}$ for $x \in A$ and $F(x) = \{0\}$ for $x \in B$, $C_l(F) = \emptyset$.

Despite the fact that the behavior of the multifunction F from Example 35 is very "bad" there is a multivalued selection of F which is "nice" as we will see from the following theorem.

Theorem 36 *Let X be a T_1-Baire topological space, M be a compact metric one. Let a multifunction $F : X \to \mathcal{K}(M)$ be an upper Kuratowski limit of a sequence $\{F_n : X \to \mathcal{P}(M)\}_{n=1}^{\infty}$ of u-$\mathcal{B}r$-continuous multifunctions. Then there is a multifunction $G : X \to \mathcal{K}(M)$ such that $G \subset F$ and the sets $\{x \in X : F(x) \neq G(x)\}$ and $D_u(G)$ are of first category.*

PROOF. Define $cl(F_n) : X \to \mathcal{K}(M)$ as $cl(F_n)(x) = cl(F_n(x))$ for any x and any n. It is easy to prove that $cl(F_n)$ is u-$\mathcal{B}r$-continuous and $Lscl(F_n) = F$. By Theorem 33, for any $n = 1, -2, \ldots$ there is a lower quasi-continuous multifunction $G_n : X \to \mathcal{K}(M)$ such that $G_n \subset cl(F_n)$ and $A_n = \{x \in X : G_n(x) = cl(F_n(x))\}$ is residual (more precisely $G_n(x) = cl(\bigcup_{i=1}^{\infty}\{f_n^i(x)\})$ where $\{f_n^i\}_{i=1}^{\infty}$ is from Theorem 33). It is clear that for any $x \in X$ $Ls(G_n(x))$ is a non-empty closed subsets of $F(x)$. Since $F(x)$ is compact for any $x \in X$, a multifunction $G = LsG_n$ is compact valued and by Theorem 34, $D_u(G)$ is of first category. Moreover, $G(x) = F(x)$ for any $x \in \bigcap_{n=1}^{\infty} A_n$.

The next theorem is perhaps the most general result concerning the existence of selection with the Baire property.

Theorem 37 *Let X be Baire and M compact metric. Let $F : X \to \mathcal{K}(M)$ be an upper Kuratowski limit of a sequence $\{F_n : X \to \mathcal{K}(M)\}_{n=1}^{\infty}$ of $\mathcal{B}r$-cliquish multifunctions. Then F has a selection having the Baire property.*

PROOF. By Corollary 29, F_n has a selection f_n having the Baire property. Since M is compact, a multifunction $G = Lsf_n \subset F$ is non-empty and compact valued. It is easy to prove that G has the Baire property. By Theorem 31, G has a selection having the Baire property.

9. Separate and Joint Continuity

One of the nicest results concerning the quasi-continuity is the Kempisty's theorem [17] and its generalization. Under some general conditions on the spaces X, Y, M the quasi-continuity of x-sections and y-sections implies the quasi-continuity of $f : X \times Y \to M$. Recall, if F is a multifunction defined on the product space $X \times Y$, we call an x-section (y-section) for given $x \in X$ ($y \in Y$) the multifunction $F_x : Y \to M$ ($F_y : X \to M$) defined as $F_x(y) = F(x, y)$ ($F_y(x) = F(x, y)$). Another result in this direction was proved by

Neubrunn [40] dealing with the quasi-continuity of multifunctions. Roughly speaking the upper (lower) quasi-continuity of F_x and both upper and lower quasi-continuity of F_y implies the upper (lower) quasi-continuity of F. There is the example in [40] that the lower (upper) quasi-continuity of F_y cannot be omitted. For example, the upper quasi-continuity of F_x and F_y for any $x \in X$, $y \in Y$ does not imply the upper quasi-continuity of F. As we shall see below it implies the existence of quasi-continuous selection of F.

Another direction connected with the separate properties is the problem of finding the assumptions on spaces X, Y, M and the sections f_x, f_y such that $f : X \times Y \to M$ has at least one point of joint continuity. A recently published result [46, Th. 2] is very close to that of [12, Th.-3], assuming the cliquishness of f_x and the quasi-continuity of f_y. Corollary 40 below generalizes both these results. The notion of order upper (lower) quasi-continuity of real functions raises another questions concerning separate and joint quasi-continuity [41 3].

A π-base ([44, p.-56], [46]) for a space $(Y, \mathcal{T})$ is a subset $\mathcal{H}$ of $\mathcal{T} \setminus \{\emptyset\}$ such that every non-empty set U of $\mathcal{T}$ contains a non-empty set G of $\mathcal{H}$.

Definition 38 *Let d be a metric for M. A multifunction $F : X \to \mathcal{K}(M)$ is said to be h_d-cliquish at a point $x \in X$ (where h_d is the Hausdorff metric on $\mathcal{K}(M)$ induced by d) if for any $\epsilon > 0$ and any neighborhood U of x there is a non-empty open set $B \subset U$ such that $h_d(F(a), F(b)) < \epsilon$ for any $a, b \in B$. F is h_d-cliquish if it is so at any $x \in X$. By other words h_d-cliquishness of F is understood as the original cliquishness of F considered as a function into $(\mathcal{K}(M), h_d)$ (see Definition 11).*

Theorem 39 *([MA 11, Th. 3]). Let X be a Baire space and Y be locally of π-countable type (i.e., each open non-empty subset of Y contains an open non-empty subset having a countable π-base). Let $F : X \times Y \to \mathcal{K}(M)$. If F_x is h_d-cliquish for any $x \notin S$ ($S \subset X$ of first category) and F_y is u-$\mathcal{B}$-continuous for any $y \in Y$, then F is $\mathcal{O}$-cliquish.*

The following consequence of Theorem 39 is a generalization of [46, Th.-2] as well as [12, Th.-3].

Corollary 40 *Under the same conditions on X, Y, M as in Theorem 39, the $\mathcal{O}$-cliquishness of x-sections (except for a set of first category) and the quasi-continuity of y-sections implies the $\mathcal{O}$-cliquishness of a function $f : X \times Y \to M$. Moreover, if $X \times Y$ is Baire, f is continuous on a residual set, by Theorem 12.*

The main result of [32] concerns the existence of a quasi-continuous selection.

Theorem 41 *([32, Th. 4]). Let X be a Baire space and Y be Baire locally of π-countable type. Let M be a separable metric space. If $F : X \times Y \to \mathcal{K}(M)$ is a multifunction such that F_x is upper quasi-continuous for any $x \notin S$ (S is of first category) and F_y is u-$\mathcal{B}$-continuous for any $y \in Y$, then F has a quasi-continuous selection.*

Note that u-$\mathcal{B}$-continuity of F_y in Theorem 41 cannot be replaced by l-$\mathcal{B}$-continuity (see [MA-11, Example]). Further, the multifunction F from Theorem 41 need not have the Baire property. Consider $F : R \times R \to \mathcal{K}(R)$ defined as $F(p) = \{0\}$ if $p \in A$ and $F(p) = [0,1]$ if $p \in R \setminus A$ where $A \subset R \times R$ is a set of second category such that any section of A contains at most two points (see [44, Theorem 15.5]). Moreover, A does not have the Baire property [44, Theorem 15.4]. It is clear that F_x and F_y are upper quasi-continuous for any x,y.

Using Theorems 34 and 41 we have:

Theorem 42 *Let X be a Baire space and Y be Baire locally of π-countable type such that $X \times Y$ is T_1-Baire. Let M be compact metric and let $F = LsF_n$ where $F_n : X \times Y \to \mathcal{K}(M), n = 1,2,...$ Then upper quasi-continuity of x-sections (except for a set of first category) and u-$\mathcal{B}$-continuity of y-sections of F_n implies the existence of a multifunction $G : X \times Y \to \mathcal{K}(M)$ such that $G \subset F$ and $D_u(G)$ is of first category. (Note $G = Lsf_n$ where f_n are quasi-continuous selections of F_n, by Theorem 41.)*

Theorem 43 *Let X, Y, M, F_n, F be from Theorem 42. Then h_d-cliquish- ness of x-sections (except for a set of first category) and u-$\mathcal{B}$-continuity of y-sections of F_n implies the existence of a selection of F having the Baire property.*

PROOF. By Theorem 39, F_n is $\mathcal{O}$-cliquish, $n = 1,2,...$ Colloraly 29 implies the existence of a selection f_n of F_n having the Baire property. It is easy to prove that a multifunction $G = Lsf_n \subset F$ has the Baire property. Finally, there is a selection of G having the Baire property, by Theorem 31.

10. Generalized Derivatives

This section is devoted to an application of multifunction approach to differentiation of the real functions based on the concept of path system which was introduced in [2]. The main results include the semi-Borel and projective classification of the multifunction of all path derived numbers, its measurable properties as well as the search for a Borel selection of the mentioned multifunction.

The path system differentiation gives a unified method of the study of a number of generalized derivatives. A collection $E = \{E(x) : x \in R\}$ (R- real line) is a system of paths if each set $E(x) \subset R$ has x as a point of accumulation. It can be considered as a multifunction $E : R \to \mathcal{P}(R), x \mapsto E(x)$. By $E_f : R \to \mathcal{K}(R^*)$ (R^* - the extended real line with the topology of two-point compactification) we denote the following non-empty and compact valued multifunction of all E-derived numbers of a given function $f : R \to R$ defined as $E_f(x) = \{y \in R^* : \exists$ a sequence $\{x_n\}_{n=1}^{\infty}$ in $E(x)\setminus\{x\}$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} \frac{f(x_n)-f(x)}{x_n - x} = y\}$. The upper and lower E-derivatives of f at x are defined as $\overline{f}'_E = sup\ E_f(x)$ and $\underline{f}'_E = inf\ E_f(x)$.

The Borel and semi-Borel classification of E_f were given in [24] and [26], generalizing the results from [1]. We will mention only the main results (for more detailed information see [24], [25], [26] containing also the classification of approximate and qualitative path derivatives).

Definition 44 *A multifunction $F : R \to \mathcal{P}(R^*)$ is said to be of lower (upper) class α, if $F^-(G)$ ($F^+(G)$) is in the Borel additive class α for any open $G \subset R^*$. A function $f : R \to R^*$ is said to be of lower (upper) class α, if $f^{-1}((a, \infty >)$ ($f^{-1}(< -\infty, a))$) is in the Borel additive class α.*

Theorem 45 *([31, Th.-5]). Let f be a function of class α. If $Gr(E)$ =(graph of E)= $\bigcup_{n=1}^{\infty} A_n \times B_n$, A_n is in the additive Borel class α, $B_n \subset R$, then E_f is a multifunction of upper class $\alpha + 1$ and $\overline{f}'_E$ ($\underline{f}'_E$) is a function of upper (lower) class $\alpha + 1$.*

Theorem 46 *([31, Th.-6]). If f is Baire 1 and $Gr(E)$ is an F_σ-set, then E_f is a multifunction of upper class 2 and $\overline{f}'_E$ ($\underline{f}'_E$) is a function of upper (lower) class 2.*

Theorem 47 *([31, Th.-7]). Let E be a l.s.c. multifunction and f be a continuous function. Then E_f is a multifunction of upper class 1 and $\overline{f}'_E$ ($\underline{f}'_E$) is a function of upper (lower) class 1. Consequently, $D_u(E_f)$ is of first category, by [21, Lemma 8].*

In the paper [1] one can find the Laczkovich's example of a function f of class 2 and a l.s.c. system of paths E such that $\overline{f}'_E$ is not Borel. Hence the multifunction E_f is not Borel measurable. As we can see from the following theorem E_f has a "nice" selection.

Theorem 48 *([28, Th. 4 and Corollary 6]). Let E be a closed valued system of paths of lower class α ($\alpha \geq 1$) and f be a function of class β ($\beta \geq 0$). Then*

(i) *there is a multifunction* $S : R \to \mathcal{K}(R^*)$ *of upper class* $\alpha + \beta + 1$ *such that* $S \subset E_f$,

(ii) E_f *has a selection of class* $\alpha + \beta + 2$,

(iii) *if* f *has* E*-derivative (i.e.,* $\overline{f}'_E = \underline{f}'_E$*), then it is of class* $\alpha + \beta + 1$.

As for the measurability of E_f the main results can be found in [31]. Given a family $\mathcal{M}$ of subset of R, we say that a multifunction $F : R \to \mathcal{P}(R^*)$ is $\mathcal{M}$-measurable if $F^-(G) \in \mathcal{M}$ for any G open in R^*.

Theorem 49 *[31, Th. 22]). Let* E *have closed values. If* f *and* E *are* $\mathcal{M}$*-measurable where* $\mathcal{M}$ *is a* σ*-algebra closed with respect to operation* $\mathcal{A}$ *(see [KU-1, p.-4]), then* $\overline{f}'_E$, $\underline{f}'_E$, E_f *are* $\mathcal{M}$*-measurable.*

Corollary 50 *Let* E *have closed values. If* f *and* E *are Lebesgue measurable (have the Baire property), then* $\overline{f}'_E$, $\underline{f}'_E$, E_f *are Lebesgue measurable (have the Baire property).*

Theorem 51 *([31, Corollary 18]). If* f *is Borel measurable and* $Gr(E)$ *is a Borel set, then* $E_f^-(K)$ *is an analytic set for any closed* $K \subset R^*$.

Theorem 52 *([31, Corollary 24]). Let* E *be Lebesgue measurable with closed values. If* $Gr(E)$ *belongs to a* σ*-algebra generated by* $\{A \times B : A, B$ *are Lebesgue measurable*$\}$*, then* $\overline{f}'_E$, $\underline{f}'_E$, E_f *are Lebesgue measurable.*

Without quoting we note that the paper [31] also contains the projective classification of E_f and the extreme E-derivatives.

We will close this section by a solution of Mišík's problem concerning the Lebesgue measurability of primitive function [37]. An extended real number a is the E-approximate derived number of a function $f : R \to R$ at a point p, if the outer upper density of $\{x \in E(p) \setminus \{p\} : \frac{f(x)-f(p)}{x-p} \in G\}$ is positive at p for any open $G \subset R^*$ such that $a \in G$. The upper (lower) extreme E-approximate derivative is defined as the supremum (infimum) of the set of all E-approximate derived numbers, respectively.

Theorem 53 *([26, Corollary 5.4]). Let* E *a one-sided system of paths such that* $E(x)$ *is Lebesgue measurable for any* x*. If a function* f *has one of the extreme* E*-approximate derivatives finite except for a set of Lebesgue measure zero, then* f *is Lebesgue measurable.*

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