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ADVANCES IN GEOMETRIC INTEGRATION

Abstract

Over the last decade considerable progress has been made in developing extensions of the Lebesgue integral which provide the Gauss-Green theorem for noncontinuously differentiable vector fields and remain invariant under groups of transformations including diffeomorphisms. One particular extension, an averaging process defined by W. F. Pfeffer, accomplishes this in the setting of bounded sets of bounded variation—the most general class of sets for which the notions of "surface area" and "exterior normal" can be profitably defined. In this survey article we recover all of the notable geometric features behind Pfeffer's extension in the less forbidding (vs. bounded sets of bounded variation) setting of figures, i.e. finite unions of compact intervals.

Key Words: Henstock–Kurzweil Integral, gage, Gauss–Green Theorem, geometric integral

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1. Introduction

Let M be an m-dimensional compact oriented C^1 manifold with boundary, and let ω be a differentiable $(m-1)$-form on M. If ω is continuously differentiable, then using Stokes' theorem, we can recover $\int_{\partial M}\omega$ from $d\omega$ by means of ordinary integration:

$$\int_{\partial M}\omega = \int_M d\omega\,.$$

However, if ω is merely differentiable, the integral $\int_M d\omega$ need not exist (even in the Lebesgue sense). Yet, $\int_{\partial M}\omega$ is still uniquely determined by $d\omega$. Indeed, if η is a differential form on M with $d\eta = d\omega$, then $d(\eta-\omega)=0$, and it follows from Stokes' theorem that $\int_{\partial M}\eta = \int_{\partial M}\omega$. Thus a natural problem is to find an *averaging process* on M that enables us to calculate $\int_{\partial M}\omega$ from $d\omega$ for any ω which is differentiable, but not necessarily continuously.

The desired global averaging process can be obtained by standard means from a local *geometric integral*, i.e., an integral in R^m which

1. extends the Lebesgue integral;
2. integrates partial derivatives of differentiable functions so that the Gauss-Green theorem is satisfied;
3. is coordinate free, i.e. invariant with respect to a group containing all diffeomorphisms.

For dimension one this question has been resolved for some time. In this article we are concerned with a recent extension given by W. F. Pfeffer. Doubtless, the most useful version of the integral is obtained when as in [3] it is defined using the family BV of bounded sets of bounded variation; these sets afford a very general notion of "surface area" and "exterior normal". The family BV also has an important compactness property which has been used to solve variational problems of geometric measure theory. But the technical difficulties connected with the local behavior of BV sets tend to obscure the simple geometric ideas behind the integral.

For this reason, we here confine ourselves to a version defined by means of such simple geometric objects as figures (finite unions of compact intervals). After some preliminaries—familiarity with the Lebesgue theory as well as with Hausdorff measure is presumed—we start with a treatment of the one-dimensional HK-integral developed independently by R. Henstock [3] and J. Kurzweil [3] in the 1950's. While the HK-integral possesses all the properties of a geometric integral in R, its natural extension to R^m does not. The very fact that the (unrestricted) Fubini theorem holds [3] precludes property 2 of a geometric integral [3, section 11.1]. Consequently, two refinements are introduced: first the gage integral (sections 4. and 7.) which satisfies properties 1 and 2 of a geometric integral, and then the final version in section 8..

In the preparation of this survey I benefited greatly from several discussions with Washek Pfeffer.

2. Preliminaries

All functions are real valued and the set of all real numbers is denoted by R. Our ambient space is the m-fold Cartesian product of R, denoted by R^m; here m is a positive integer. For $x = (\xi_1, ..., \xi_m)$ and $y = (\eta_1, ..., \eta_m)$ in R^m, we let

$$x \cdot y = \sum_{j=1}^{m} \xi_j \eta_j \,, \qquad \|x\| = \sqrt{x \cdot x}\,, \qquad \text{and} \qquad |x| = \max\{|\xi_1|, ..., |\xi_m|\}\,.$$

In R^m we use exclusively the metric induced by the norm $|x|$. If $E \subset R^m$ then $\mathrm{cl}E$, E°, ∂E, and $d(E)$ denote, respectively, the closure, interior, boundary, and diameter of E. For $x \in R^m$ and $\varepsilon > 0$, we set $U(x, \varepsilon) = \{y \in R^m : |x - y| < \varepsilon\}$. The k-dimensional (outer) Hausdorff measure $\mathcal{H}^k$ in R^m is defined so that it agrees with the k-dimensional Lebesgue measure in $R^k \subset R^m$ for $k = 1, \ldots, m$; $\mathcal{H}^0$ is the counting measure in R^m. A set $E \subset R^m$ with $\mathcal{H}^k(E) = 0$ is called *$\mathcal{H}^k$-negligible.* We say two sets $C, D \subset R^m$ are *nonoverlapping* if $C \cap D$ is $\mathcal{H}^m$-negligible.

By an *interval* we always mean a compact nondegenerate subinterval of R^m, i.e. the product $K = \prod_{j=1}^{m}[a_j, b_j]$ where $a_j < b_j$ are real numbers for $j = 1, \ldots, m$. A *dyadic cube* is an interval of the form $[k_j 2^{-n}, (k_j + 1)2^{-n}]$ where $n, k_1, k_2, \cdots$ are integers. If C and D are dyadic cubes, then either $C \subset D$, $D \subset C$ or they are nonoverlapping, i.e. have disjoint interiors. Consequently, if $\mathcal{C}$ is a family of dyadic cubes it can be replaced with a nonoverlapping subfamily $\mathcal{K}$ such that $\bigcup \mathcal{K} = \bigcup \mathcal{C}$. A *figure* in R^m is a finite (posssibly empty) union of intervals in R^m.

3. Partitions

A *partition* in R^m is a collection (possibly empty)

$$P = \{(A_1, x_1), ..., (A_p, x_p)\}$$

where $A_1, \ldots, A_p$ are nonoverlapping sets in R^m and $x_i \in A_i$ for $i = 1, \ldots, p$. Throughout sections 3 to 7 the sets A_i will always be intervals; in section 8 they will be figures. The points $x_1, \ldots, x_p$ are called the *anchor points* of P, and the set of all anchor points (called the *anchor* of P) is denoted by an$P$; if $anP \subset E$, we say P is *anchored* in E. Given a function f on a set $E \subset R^m$ and a partition $P = \{(A_1, x_1), ..., (A_p, x_p)\}$ anchored in E, we define the Stieltjes sum

$$\sigma(f, P) = \Sigma f(x_i)\mathcal{H}^m(A_i).$$

If δ is a nonnegative function on a set $E \subset R^m$, then a partition $P = \{(A_1, x_1), ..., (A_p, x_p)\}$ anchored in E is called *δ-fine* whenever $d(A_i) < \delta(x_i)$ for $i = 1, \ldots, p$; if $\delta(x) = 0$, then x cannot occur as an anchor point of a δ-fine partition.

The set $\bigcup_{i=1}^{p} A_i$ is called the *body* of P, denoted by $\bigcup P$. We say that P is a *partition* ***in*** A whenever $\bigcup P \subset A$, and a *partition* ***of*** A whenever $\bigcup P = A$.

Lemma 3.1 *Let $A \subset R^m$ be an interval. For each positive function δ on A there is a δ-fine partition $P = \{(A_1, x_1), ..., (A_p, x_p)\}$ consisting of intervals. In particular, if $A = \prod_{i=1}^{m}[r_i, s_i]$ where r_i, s_i are integers, then we may assume P consists of dyadic cubes.*

PROOF. Suppose there is no δ-fine partition of A. Then there is a nested sequence $\{A_i\}$ of subintervals of A (with decreasing diameters) such that each A_i has no δ-fine partition. By the nested intervals theorem $\bigcap_{i=1}^{\infty} A_i = \{x\}$ for some $x \in A$. Now we have a contradiction: Certainly $d(A_i) < \delta(x)$ for some i and consequently$\{(A_i, x)\}$ is a δ-fine partition of A_i. This proves the first statement; the specialization to dyadic cubes is similar. □

If δ is not positive on all of A we have recourse to a result of E. J. Howard [3, Lemma 5].

Lemma 3.2 *Let $A = \prod_{i=1}^{m}[r_i, s_i]$ where r_i, s_i are integers. If $T \subset A$, δ is a positive function on $A - T$, and $\mathcal{C}$ is any collection of dyadic cubes contained in A with $T \subset (\bigcup \mathcal{C})^\circ$, then there is a finite nonoverlapping subcollection $\mathcal{D}$ of $\mathcal{C}$ and a δ-fine partition $\{(A_1, x_1), ..., (A_p, x_p)\}$ (consisting of dyadic cubes) in A such that*

$$A \ominus \bigcup P = \bigcup \mathcal{D}.$$

PROOF. We may assume that $\mathcal{C}$ is nonoverlapping. Set

$$\delta_+(x) = \begin{cases} \delta(x) & \text{if } x \in A - T \\ \min\{d(C) : C \in \mathcal{C},\ x \in C\} & \text{if } x \in T. \end{cases}$$

Let $\{(A_1, x_1), ..., (A_r, x_r)\}$ be a δ_+-fine partition of A consisting of dyadic cubes. Define $\mathcal{D} = \{C \in \mathcal{C} : A_i \subset C \text{ for some } i\}$. Now since $\mathcal{C}$ is nonoverlapping, $\mathcal{D}$ is finite. If A_i overlaps with $D \in \mathcal{D}$, then either $A_i \subset D$ or D is a proper subset of A_i by the property of dyadic cubes. The latter case is impossible since D contains an A_j that does not overlap with A_i. So for $i = 1, 2, \dots, r$ either $A_i \subset \bigcup \mathcal{D}$ or A_i overlaps with no $D \in \mathcal{D}$. Thus for some $J \subset \{1, 2, \dots, r\}$, $\bigcup \mathcal{D} = \bigcup_{i \in J} A_i$. Reordering, we obtain $\bigcup \mathcal{D} = \bigcup_{i=p+1}^{r} A_i$ for some p with $p \leq r$.

Define $P = \{(A_1, x_1), ..., (A_p, x_p)\}$. Then $anP \subset A - T$, as $T \subset (\bigcup \mathcal{C})^{\circ}$ and the definition of δ_+ implies $A_i \subset \bigcup \mathcal{D}$ if $x_i \in T$. □

If δ on [0,1] is given by $\delta : x \mapsto x$, then it is clear that no δ-fine partition of $[0, 1]$ exists. How Lemma 3.2 above addresses such a situation can be seen as follows. Suppose $A \subset R^m$ is an interval and δ is positive on $A - T$. If $T \subset A$ is $\mathcal{H}^m$-negligible, then given any positive number η we can find a dyadic covering $\mathcal{C}$ of T such that $\mathcal{H}^m(\bigcup \mathcal{C}) < \eta$. Applying the lemma above, there is a δ-fine partition P in A such that $\mathcal{H}^m(A \ominus \bigcup P) < \eta$. Our actual use will be similar but more demanding.

4. The Henstock-Kurzweil integral

Throughout this section A will be a subinterval of R. Also, for any partition $\{(A_1, x_1), ..., (A_p, x_p)\}$ the sets $A_1, \dots, A_p$ are intervals.

Definition 4.1 *A function f defined on A is HK-integrable in A if there is a real number I with the following property: given any $\varepsilon > 0$ there is a positive function δ on A such that*

$$|\sigma(f, P) - I| < \varepsilon$$

for each δ-fine partition P of A.

Uniqueness of the number I above follows from the existence of δ-fine partitions (Lemma 3.1); it is called the integral of f on A and designated by $(HK)\int_A f$. The family of all HK-integrable functions on A, denoted by $\mathcal{H}K(A)$, is a linear space and the map $f \mapsto (HK)\int_A f$ is a nonnegative linear functional on $\mathcal{H}K(A)$. The proof of the following test for integrability is left to the reader.

Lemma 4.2 *(Cauchy Criterion) A function f is HK-integrable in A if and only if for each $\varepsilon > 0$ there is a positive δ on A such that*

$$\left|\sigma(f, P) - \sigma(f, Q)\right| < \varepsilon$$

for all δ-fine partitions P and Q of A.

Using the easily established fact that any δ-fine partition *in* A can be enlarged to a δ-fine partition *of* A and the Cauchy Criterion, the next lemma follows.

Lemma 4.3 *Let f be HK-integrable in A, then f is HK-integrable in each subinterval $B \subset A$.*

It is certainly desirable that the HK-integral of a function f be unaffected when the values of f are changed on a $\mathcal{H}^1$-negligible set; the lemma below will be useful in establishing this and other facts about the HK-integral.

Lemma 4.4 *Let $E \subset R^m$ be $\mathcal{H}^m$-negligible and let $\varepsilon > 0$. There is a function α defined on all subsets of R^m which satisfies the following conditions:*

1. *$0 \leq \alpha(B) \leq \varepsilon$ for each $B \subset R^m$;*
2. *$\alpha(B \cup C) = \alpha(B) + \alpha(C)$ for each pair of nonoverlapping Lebesgue measurable sets $B, C \subset R^m$;*
3. *given $x \in E$ and an integer $n \geq 1$, there is a $\delta > 0$ such that $\alpha(B) \geq n\mathcal{H}^m(B)$ for each $B \subset U(x, \delta)$.*

PROOF. Find a decreasing sequence $\{U_k\}$ of open sets containing E so that $\mathcal{H}^m(U_k) < \varepsilon 2^{-k}$ for $k = 1, 2, \ldots$, and set

$$\alpha(B) = \sum_{k=1}^{\infty} \mathcal{H}^m(B \cap U_k)$$

for each $B \subset R^m$. Clearly, the function α satisfies the first two conditions. Given $x \in E$ and a positive integer n, find $\delta > 0$ so that $U(x, \delta) \subset U_n$. Now if $B \subset U(x, \delta)$, then

$$\alpha(B) = n\mathcal{H}^m(B) + \sum_{k=n+1}^{\infty} \mathcal{H}^m(B \cap U_k) \geq n\mathcal{H}^m(B).$$

□

Proposition 4.5 *If f is zero $\mathcal{H}^1$-almost everywhere on A, then $f \in \mathcal{HK}(A)$ and $(HK)\int_A f = 0$.*

PROOF. Suppose f is zero outside a $\mathcal{H}^1$-negligible set $E \subset A$. Let $\varepsilon > 0$ be given and choose α according to Lemma 4.4 for ε and E. For each $x \in E$ there is a δ_x such that $\alpha(B) \geq |f(x)|\mathcal{H}^1(B)$ whenever B is an interval such that $d(B) < \delta_x$ and $x \in B$. Set

$$\delta(x) = \begin{cases} \delta_x & \text{if } x \in E \\ 1 & \text{otherwise.} \end{cases}$$

If P is a δ-fine partition of A, then

$$|\sigma(f,P)-0|<\alpha(A)<\varepsilon.$$

Corollary 4.6 *Let f and g be functions defined on A such that $f(x)=g(x)$ for $\mathcal{H}^1$-almost all $x\in A$. Then f belongs to $\mathcal{HK}(A)$ if and only if g does, in which case*

$$(HK)\int_A f=(HK)\int_A g.$$

We now state and prove an important technical tool known as *Henstock's lemma.* It gives the connection between the *Riemann sum characterization* of the integral found in Definition 4.1 and the *indefinite integral characterization* found in Proposition 4.9. The latter formulation is the proper setting for a study of the relation between integral and derivative—one ought to look at the proofs of Theorems 4.13 and 4.15.

Lemma 4.7 *Let f be HK-integrable in A. Given any $\varepsilon>0$, there is a positive function δ on A such that*

$$\sum_{i=1}^{p}\left|f(x_i)\mathcal{H}^1(A_i)-(HK)\int_{A_i}f\right|<\varepsilon$$

for each δ-fine partition $\{(A_1,x_1),...,(A_p,x_p)\}$ in A.

PROOF. Let δ be a positive function on A such that

$$\left|\sigma(f,P)-(HK)\int_A f\right|<\varepsilon/3$$

for each δ-fine partition P of A. Choose any δ-fine partition $\{(A_1,x_1),...,(A_p,x_p)\}$ of A and reorder it so that $f(x_i)\mathcal{H}^1(A_i)-(HK)\int_{A_i}f$ is nonnegative for $i=1,\cdots,k$ and negative for $i=k+1,\cdots,p$ where k is an integer with $0\le k\le p$. Let P_i be a δ-fine partition of A_i so that $|\sigma(f,P_i)-(HK)\int_{A_i}f|<\varepsilon/3p$. If $Q=\bigcup_{i=k+1}^{p}P_i$, then $P=\{(A_1,x_1),\cdots,(A_k,x_k)\}\cup Q$ is a δ-fine partition of A and

$$\begin{aligned}
\frac{\varepsilon}{3} &> \left|\sigma(f,P)-(HK)\int_A f\right| \\
&\ge \sum_{i=1}^{k}\left[f(x_i)\mathcal{H}^1(A_i)-(HK)\int_{A_i}f\right]-\left|\sum_{i=k+1}^{p}\left[\sigma(f,P_i)-(HK)\int_{A_i}f\right]\right| \\
&\ge \sum_{i=1}^{k}\left|f(x_i)\mathcal{H}^1(A_i)-(HK)\int_{A_i}f\right|-\frac{\varepsilon(p-k)}{3p}.
\end{aligned}$$

Similarly, one can establish that

$$\frac{\varepsilon}{3} > \sum_{i=k+1}^{p} \left| f(x_i)\mathcal{H}^1(A_i) - (HK)\int_{A_i} f \right| - k\frac{\varepsilon}{3p}.$$

Adding these two inequalities yields

$$\sum_{i=1}^{p} \left| f(x_i)\mathcal{H}^1(A_i) - (HK)\int_{A_i} f \right| < \varepsilon$$

for each δ-fine partition $\{(A_1, x_1), ..., (A_p, x_p)\}$ of A.

Since any δ-fine partition $\{(A_1, x_1), ..., (A_p, x_p)\}$ *in* A can be enlarged to a δ-fine partition *of* A, we are done. □

Proposition 4.8 *Let B and C be nonoverlapping intervals whose union is an interval A, and let f be a function on A. If f is HK-integrable in B and C, it is HK-integrable in A and*

$$(HK)\int_A f = (HK)\int_B f + (HK)\int_C f.$$

Proof. Suppose $D \subset A$ is an interval and $P = \{(A_1, x_1), ..., (A_p, x_p)\}$ is any partition of A. Then we can define a partition

$$P_D = \{(A_i \cap D, x_i) \; : \; x_i \in D \text{ and } A_i \cap D^\circ \neq \emptyset\}$$

in D. Further, given any positive function δ_D on an interval $D \subset A$, we can find a positive function $\delta'_D \leq \delta_D$ on D with the property that $x \in D^\circ$ implies $U(x, \delta'_D(x)) \subset D^\circ$.

Choose $\varepsilon > 0$ and set $I = (HK)\int_B f + (HK)\int_C f$. Find a positive function δ_B on B such that

$$\left| \sigma(f, Q) - (HK)\int_B f \right| < \varepsilon$$

whenever Q is a δ_B-fine partition of B. Similarly, choose such a δ_C on C and define

$$\delta(x) = \begin{cases} \delta'_B(x) & \text{if } x \in B - C \\ \delta'_C(x) & \text{if } x \in C - B \\ \min\{\delta'_B(x), \delta'_C(x)\} & \text{if } x \in B \cap C\,. \end{cases}$$

By our choice of δ, we have that P_B is a δ_B-fine partition *of* B whenever P is a δ-fine partition of A. Of the course the same holds for P_C whence

$$\begin{aligned}|\sigma(f,P)-I| &= |\sigma(f,P_B\cup P_C)-I| \\ &\leq \left|\sigma(f,P_B)-(HK)\int_B f\right|+\left|\sigma(f,P_C)-(HK)\int_C f\right| < 2\varepsilon.\end{aligned}$$

Thus the proposition is proved. □

We say F is an *additive* function (of intervals) in a set $S \subset R$ if

$$F(B\cup C)=F(B)+F(C)$$

whenever B and C are nonoverlapping subintervals of S. It follows from Lemma 4.3 and Proposition 4.8 that if f is HK-integrable in an interval A, then $B \mapsto (HK)\int_B f$ is an additive function in A; we call it the *indefinite integral* of f in A.

Proposition 4.9 *A function f defined on A is HK-integrable in A if and only if we can find an additive function F in A satisfying the following condition: given $\varepsilon > 0$, there is a positive function δ on A such that*

$$\sum_{i=1}^{p}\left|f(x_i)\mathcal{H}^1(A_i)-F(A_i)\right|<\varepsilon$$

for each δ-fine partition $\{(A_1,x_1),\dots,(A_p,x_p)\}$ in A.

If F is an additive function defined on the (nondegenerate) subintervals of $A=[a,b]$, we define the *associated point function*

$$G(x)=\begin{cases} F([a,x]) & \text{if } a<x\leq b \\ 0 & \text{if } x=a.\end{cases}$$

Conversely, if G is a point function on $[a,b]$, then we can recover an additive function, called the *associated interval function*, by setting $F([c,d])=G(d)-G(c)$ for each $[c,d]\subset[a,b]$.

Proposition 4.10 *Let f be an HK-integrable function in an interval $[a,b]$, set $G(a)=0$ and $G(x)=(HK)\int_a^x f$ for each $x\in(a,b]$. Then G is continuous in $[a,b]$.*

PROOF. Choose an $\varepsilon>0$ and an $x\in[a,b]$. By Proposition 4.9 there is a $\delta>0$ such that

$$\left|f(x)(y-x)-[G(y)-G(x)]\right|<\varepsilon$$

for each $y \in [a,b]$ with $|y-x| < \delta$. Making δ smaller so that $f(x)(y-x) \leq \varepsilon$, we obtain $|G(y) - G(x)| < 2\varepsilon$ whenever $|y-x| < \delta$. □

Let F be the indefinite integral of $f \in \mathcal{HK}(A)$. Since A is compact, its associated point function G is uniformly continuous and F inherits this as:

Corollary 4.11 *If F is the indefinite integral of $f \in \mathcal{HK}(A)$, then F is continuous in the following sense: given $\varepsilon > 0$ there is an $\eta > 0$ such that $|F(B)| < \varepsilon$ whenever $B \subset A$ is an interval with $\mathcal{H}^1(B) < \eta$.*

We say a sequence $\{B_i\}$ of intervals *shrinks* to a point x if $d(B_i) \to 0$ as $i \to \infty$ and $x \in B_i$ for each i. An additive function F is *differentiable* at x if for each sequence (B_i) shrinking to x,

$$\lim_{i\to\infty} \frac{F(B_i)}{\mathcal{H}^1(B_i)}$$

exists; in this case all such limits have the same value called the derivative of F at x and written $F'(x)$. It is not difficult to show that F is differentiable at x if and only if the associated point function is differentiable at x in which case the derivatives are equal.

Proposition 4.12 *Let $f \in \mathcal{HK}(A)$ and let F be the indefinite integral of f. Then $F'(x) = f(x)$ for almost all $x \in A$.*

PROOF. Let E_n be the set of points $x \in A$ with the property that there is a sequence $\{B_i\}$ shrinking to x and satisfying

$$\left| F(B_i) - f(x)\mathcal{H}^1(B_i) \right| > \mathcal{H}^1(B_i)/n \tag{1}$$

for $i = 1, 2, \cdots$. The set $\bigcup_n E_n$ is precisely the set of points where F is either not differentiable or $F'(x) \neq f(x)$.

Our task is to show that $\mathcal{H}^1(E_n) = 0$ for each positive integer n. To this end, let $\varepsilon > 0$ be given. By the integrability of f there is a positive function δ on A such that

$$\sum_{i=1}^{p} \left| f(x_i)\mathcal{H}^1(A_i) - F(A_i) \right| < \varepsilon/n \tag{2}$$

for each δ-fine partition $\{(A_1, x_1), ..., (A_p, x_p)\}$ in A.

For $x \in E_n$ let $\mathcal{E}_x$ denote the collection of intervals satisfying inequality 1 with diameters less than $\delta(x)$. Then $\bigcup_{x\in E_n} \mathcal{E}_x$ covers E_n in the sense of Vitali and by the Vitali covering theorem there is a disjoint collection of intervals $\mathcal{K}$ for which $\mathcal{H}^1(E_n - \bigcup \mathcal{K}) = 0$.

Let $\mathcal{F} \subset \mathcal{K}$ be finite; for each $C \in \mathcal{F}$ there is at least one $x_C \in E_n$ which makes $\{(C, x_C)\}_{C\in\mathcal{F}}$ a δ-fine partition. Inequalities 1 and 2 imply

$$\mathcal{H}^1(\bigcup \mathcal{F}) \leq n \sum_{C\in\mathcal{F}} \left| f(x_C)\mathcal{H}^1(C) - F(C) \right| < \varepsilon$$

and consequently $\mathcal{H}^1(E_n) \leq \mathcal{H}^1(E_n - \bigcup \mathcal{K}) + \mathcal{H}^1(\bigcup \mathcal{K}) \leq \varepsilon$. As ε is arbitrary, $\mathcal{H}^1(E_n) = 0$. □

Theorem 4.13 *(The fundamental theorem of calculus) If F is differentiable in an interval $A = [a, b]$, then $F' \in \mathcal{HK}([a, b])$ and*

$$(HK)\int_a^b F' = F(b) - F(a)\,.$$

PROOF. Let G be the interval function associated with F, i.e. $G([c, d]) = F(d) - F(c)$. Let $\varepsilon > 0$ be given. Since F is differentiable in A, there is a positive number δ_x such that $|F'(x)\mathcal{H}^1(B) - G(B)| < \varepsilon\mathcal{H}^1(B)$ whenever $d(B) < \delta_x$ and $x \in B$. Letting $\delta(x) = \delta_x$ for each $x \in A$, we obtain

$$\sum_{i=1}^{p} \left| f(x_i)\mathcal{H}^1(A_i) - G(A_i) \right| < \varepsilon\mathcal{H}^1(A)$$

for each δ-fine partition $\{(A_1, x_1), ..., (A_p, x_p)\}$ in A. Now apply Proposition 4.9.

We turn our attention to the relationship between the family $L^1(A)$ of Lebesgue integrable functions on A and $\mathcal{HK}(A)$. If $f \in L^1(A)$, its Lebesgue integral will be written $(L)\int_A f$ or sometimes $(L)\int_A f\,d\mathcal{H}$ for clarity.

Proposition 4.14 *If $f \in \mathcal{HK}(A)$, then f is measurable.*

PROOF. Notice $G(x) = (HK)\int_a^x f$ is measurable by Proposition 4.10 and

$$f(x) = \lim_{n\to\infty} \frac{G(x + \frac{1}{n}) - G(x)}{1/n}$$

almost everywhere in $[a, b]$ by Proposition 4.12. □

The proof of the next theorem highlights the interplay between differentiation and locally fine partitions—the classical Riemann integral uses uniformly fine partitions.

Theorem 4.15 *$L^1(A) \subset \mathcal{HK}(A)$ and*

$$(HK)\int_A f = (L)\int_A f$$

for each $f \in L^1(A)$.

PROOF. Let $f \in L^1(A)$ and $\varepsilon > 0$. Define the additive function $F : B \mapsto (L)\int_B f$. Then F is differentiable with derivative $F'(x) = f(x)$ at each x outside some $\mathcal{H}^1$-negligible set E. Thus at each $x \in A - E$, there is a positive number δ_x such that

$$|f(x)\mathcal{H}^1(B) - F(B)| < \varepsilon\mathcal{H}^1(B)$$

whenever B is an interval with $d(B) < \delta_x$ and $x \in B$.

Now there is an open set U containing E such that $|F(U)| < \varepsilon$ and for each $x \in E$ there is a γ_x such that $B \subset U$ whenever $d(B) < \gamma_x$ and $x \in B$.

Set

$$\delta(x) = \begin{cases} \delta_x & \text{if } x \in A - E \\ \gamma_x & \text{if } x \in E. \end{cases}$$

Invoking Corollary4.6, we assume for convenience that f is zero on E. If $P = \{(A_1, x_1), ..., (A_p, x_p)\}$ is δ-fine, then

$$\begin{aligned} \sum_{i=1}^{p} \left| f(x_i)\mathcal{H}^1(A_i) - F(A_i) \right| &= \sum_{x_i \in E} \left| F(A_i) \right| + \sum_{x_i \notin E} \left| f(x_i)\mathcal{H}^1(A_i) - F(A_i) \right| \\ &< 2|F(U)| + \varepsilon\mathcal{H}^1(A) < \varepsilon(2 + \mathcal{H}^1(A)). \end{aligned}$$

□

Theorem 4.16 *$f \in L^1(A)$ if and only if $f, |f| \in \mathcal{HK}(A)$.*

PROOF. Suppose $f, |f| \in \mathcal{HK}(A)$. Set $g_n = \min(|f|, n)$, then since g_n is bounded and measurable, $g_n \in L^1(A) \subset \mathcal{HK}(A)$. Upon applying the monotone convergence theorem we obtain:

$$(L)\int_A |f| = \lim(L)\int_A g_n = \lim(HK)\int g_n \le (HK)\int |f| < +\infty$$

which implies that $|f| \in L^1(A)$. The other implication follows from Theorem 4.15.

A function f defined on $E \subset R^m$ is *almost differentiable* at $x \in E^\circ$ if

$$\limsup_{y \to x} \frac{|f(y) - f(x)|}{|y - x|} < \infty.$$

If $E \subset A^\circ$ and f (defined on A) is almost differentiable at all $x \in E$, then f is differentiable almost everywhere in E by Stepanoff's theorem [3, Theorem 6.6.8]. We could now easily give an analog of Theorem 4.13 for a function

F almost differentiable in A but an apparent change in the definition of the HK-integral affords the even better result realized in Theorem 4.18 below.

Let δ be a nonnegative function on A. The null set of δ is the set

$$\mathcal{Z}_\delta = \{x \in A : \delta(x) = 0\}.$$

Should $\mathcal{Z}_\delta$ be countable, then δ is called a *gage*.

Definition 4.17 *A function f defined on A is gage integrable if we can find an additive continuous function F (the indefinite integral) in A satisfying the following condition: given $\varepsilon > 0$, there is a gage δ on A such that*

$$\sum_{i=1}^{p} \left| f(x_i)\mathcal{H}^1(A_i) - F(A_i) \right| < \varepsilon$$

for each δ-fine partition $\{(A_1, x_1), ..., (A_p, x_p)\}$ in A.

In the case that f is gage integrable in A with indefinite integral F, the integral of f on A is $(g)\int_A f = F(A)$. The family of all gage integrable functions defined on A is denoted by $\mathcal{G}(A)$. Uniqueness of the indefinite integral will be resolved shortly.

Theorem 4.18 *(The fundamental theorem of calculus: version two)*

Let $T \subset R$ be a countable set. If F is a continuous function almost differentiable at each $x \in [a, b] - T$, then F' is gage integrable and

$$(g)\int_a^b F' = F(b) - F(a).$$

PROOF. We let F stand for both the point function and its associated interval function. By Stepanoff's theorem there is a $\mathcal{H}^1$-negligible set E such that F is differentiable outside E. Thus for each $x \in A - E$ there is a $\gamma_x > 0$ such that

$$|F(B) - F'(x)\mathcal{H}^1(B)| < \varepsilon\mathcal{H}^1(B)$$

whenever $d(B) < \gamma_x$ and $x \in B$.

Let α be chosen for E and ε according to lemma 4.4. For $x \in A - E \cup T$ there is a $c_x > 0$ and a $\beta_x > 0$ such that

$$|F(B)| < c_x\mathcal{H}^1(B) \quad \text{and} \quad \alpha(B) > c_x\mathcal{H}^1(B)$$

whenever $d(B) < \beta_x$ and $x \in B$.

Set

$$\delta(x) = \begin{cases} \gamma_x & \text{if } x \in A - E \\ \beta_x & \text{if } x \in E - T \\ 0 & \text{otherwise} \end{cases}$$

and observe that δ is a gage. To make the estimates simpler, we employ an analog of Corollary 4.6 and assume $F'(x) = 0$ whenever $x \in E$. Suppose $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ is a δ-fine partition in A, then

$$\begin{aligned} \sum_{i=1}^{p} |f(x_i)\mathcal{H}^1(A_i) - F(A_i)| &= \sum_{x_i \in E} |F(A_i)| + \sum_{x_i \notin E} |f(x_i)\mathcal{H}^1(A_i) - F(A_i)| \\ &< \sum_{x_i \in E} \alpha(A_i) + \sum_{x_i \notin E} \varepsilon \mathcal{H}^1(A_i) \\ &< \alpha(A) + \varepsilon\mathcal{H}^1(A) \le \varepsilon(1 + \mathcal{H}^1(A)). \end{aligned}$$

□

It is immediate that $\mathcal{HK}(A) \subset \mathcal{G}(A)$ and $(HK)\int_A f = (g)\int_A f$ when $f \in \mathcal{HK}(A)$. We show $\mathcal{HK}(A) = \mathcal{G}(A)$ and coincidentally resolve uniqueness of the indefinite gage integral.

Proposition 4.19 *If f is gage integrable in A, then f is HK-integrable in A.*

PROOF. Suppose that f is gage integrable with indefinite integral F. Let $\varepsilon > 0$ be given, then there is a gage δ on A such that

$$\sum_{i=1}^{p} \left| f(x_i)\mathcal{H}^1(A_i) - F(A_i) \right| < \varepsilon$$

for each δ-fine partition $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ in A. Since F is continuous and $\mathcal{Z}_\delta$ is countable, there is a positive function β defined on $\mathcal{Z}_\delta$ such that $\sum_{i=1}^{p} |F(A_i)| < \varepsilon$ whenever $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ is a β-fine partition anchored in $\mathcal{Z}_\delta$; making β smaller—but keeping it positive—we may assume that $\sum_{i=1}^{p} |f(x_i)|\mathcal{H}^1(A_i) < \varepsilon$ as well. Define a positive function on A

$$\Delta(x) = \begin{cases} \delta(x) & \text{if } x \notin \mathcal{Z}_\delta \\ \beta(x) & \text{otherwise.} \end{cases}$$

For any Δ-fine partition $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ in A, we obtain

$$\sum_{i=1}^{p} \left| f(x_i)\mathcal{H}^1(A_i) - F(A_i) \right| < 3\varepsilon$$

courtesy of the triangle inequality and the above estimates. □

We round out this section with a change of variables theorem. Let $E \subset R^m$ be a figure. A map $\Phi : E \to R^n$ is called *Lipschitz* if there is a positive number c such that $|\Phi(y) - \Phi(x)| \leq c|y - x|$ for all $x, y \in E$; the least such c exists and is called the *Lipschitz constant* of Φ and denoted by $\mathrm{Lip}(\Phi)$. An injective Lipschitz map $\Phi : E \to R^n$ with Lipschitz inverse is called a *lipeomorpism.* If $E \subset R$ is an interval and $\Phi : E \to R$ is a lipeomorphism, then Φ is a monotone function on E. By standard real variable theory such a Φ is differentiable almost everywhere in E. For consistency with later notation we define

$$\det \Phi(x) = \begin{cases} \Phi'(x) & \text{if } \Phi \text{ is differentiable at } x \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 4.20 *Let A, B be intervals and let $\Phi : A \to B$ be a lipeomorphism. If f is HK-integrable in B, then $f \circ \Phi \cdot |\det \Phi|$ is HK-integrable in A and*

$$\int_A f \circ \Phi \cdot |\det \Phi| = \int_B f$$

PROOF. We give a proof which generalizes well. Choose $\varepsilon > 0$, by [3, Theorem 7.2.4] there is a set $N \subset A$ with $\mathcal{H}^1(N) = 0$ and a positive function Δ on A such that

$$|f \circ \Phi(x)| \cdot \Big| |\det \Phi(x)| \mathcal{H}^1(C) - \mathcal{H}^1(\Phi(C)) \Big| < \varepsilon \mathcal{H}^1(C)$$

for each $x \in A - N$ and each interval $C \subset A$ with $x \in C$ and $d(C) < \Delta(x)$. According to Corollary 4.6 we may assume that f is zero on $\Phi(N)$, which set is $\mathcal{H}^1$-negligible since Φ is Lipschitz.

Let F be the indefinite integral of f in B and choose δ_B according to Proposition 4.9 so that

$$\sum_{i=1}^{p} \Big| f(y_i)\mathcal{H}^1(B_i) - F(B_i) \Big| < \varepsilon$$

for each δ-fine partition $\{(B_1, y_1), ..., (B_p, y_p)\}$ in B. Define

$$\delta_A = \min\{\frac{\delta_B \circ \Phi}{\mathrm{Lip}(\Phi)}, \Delta\}$$

and let $\{(A_1, x_1), ..., (A_p, x_p)\}$ be any δ_A-fine partition in A, then

$$\{(\phi(A_1), \Phi(x_1)), \cdots, (\phi(A_p), \Phi(x_p))\}$$

is a δ_B-fine partition in B and

$$\begin{aligned}
&\sum_{i=1}^{p}\left|f(\Phi(x_i))\cdot|\det\Phi(x_i)|\mathcal{H}^1(A_i)-F(\Phi(A_i))\right| \\
\leq\ &\sum_{i=1}^{p}|f(\Phi(x_i))|\cdot\left||\det\Phi(x_i)|\mathcal{H}^1(A_i)-\mathcal{H}^1(\Phi(A_i))\right| \\
+\ &\sum_{i=1}^{p}\left|f(\Phi(x_i))\mathcal{H}^1(\Phi(A_i))-F(\Phi(A_i))\right| \\
\leq\ &\varepsilon(\mathcal{H}^1(A)+1)
\end{aligned}$$

An application of Proposition 4.9 shows that $f\circ\Phi\cdot|\det\Phi|$ is HK-integrable in A with indefinite integral $F\circ\Phi$. □

5. Continuous additive functions

On an interval $A\subset R$ the difference between $L^1(A)$ and $\mathcal{HK}(A)$ can be likened to that between absolutely and conditionally convergent series.

Example 5.1 *Let $A=[0,1]$ and for $i=1,2,\cdots$ define the function $f_i: x\mapsto (-1)^i 2^i/i$ on the interval $A_i=[1/2^i,1/2^{i-1}]$. Observe that*

$$F: B\mapsto\sum_{i=1}^{\infty}\int_{A_i\cap B} f_i,$$

is an additive function on the subintervals of A which is differentiable in the interior of A_i for each i. Furthermore, the conditional convergence of $\sum_{i=1}^{\infty}(-1)^i/i$ implies that F is continuous in the sense of Corollary 4.11 and so by Theorem 4.18, $F'\in\mathcal{HK}(A)$. On the other hand, since $\sum_{i=1}^{\infty}(-1)^i/i$ is not absolutely convergent, $|F'|\notin L^1(A)$ and consequently $F'\notin L^1(A)$. □

The previous example shows the extension $L^1(A)\subset\mathcal{HK}(A)$ is proper and that an indefinite HK-integral F is not (in general) absolutely continuous. Evidently, if B is a finite union of nonoverlapping intervals, then in order to guarantee that $|F(B)|$ is small we must regulate not only $\mathcal{H}^1(B)$, but also the number of components of B; consider $B=\bigcup_{i=k}^{n}A_{2i}$ for various positive integers $k<n$ in Example 5.1.

Recall that a figure $B\subset R^m$ is a finite (possibly empty) union of intervals in R^m. We define the *perimeter* of B as $\|B\|=\mathcal{H}^{m-1}(\partial B)$; if $m=1$, then $\|B\|=\mathcal{H}^0(\partial B)$—twice the number of components of B. We now reinterpret our observations about continuity using the perimeter.

Definition 5.2 *F is an additive function (of figures) in a set $E \subset R^m$ if*

$$F(B \cup C) = F(B) + F(C)$$

whenever B, $C \subset E$ are nonoverlapping figures. (Note each additive function of intervals has a unique extension to an additive function of figures.)

An additive function F (of figures) in a bounded set $E \subset R^m$ is continuous if given $\varepsilon > 0$ there is an $\eta > 0$ such that $|F(C)| < \varepsilon$ for each figure $C \subset E$ with $\|C\| < 1/\varepsilon$ and $\mathcal{H}^m(C) < \eta$.

We now relax the restriction of the previous section that A is always a subinterval of R. Let $\mathcal{F}$ be the collection of all figures in R^m and set

$$\mathcal{F}_n = \{A \in \mathcal{F} : A \subset [-n, n]^n \text{ and } \|A\| \leq n\}.$$

If we define the metric

$$\rho(A, B) = \mathcal{H}^m((A - B) \cup (B - A))$$

on $\mathcal{F}$, then an additive function on the family $\mathcal{F}$ continuous in the sense of Definition 5.2 is ρ-continuous on $\mathcal{F}_n$ for each n. However, such a function will not generally be ρ-continuous on $\mathcal{F}$ since this is equivalent to absolute continuity. The correct topology on $\mathcal{F}$ compatible with Definition 5.2 is the largest topology τ on $\mathcal{F}$ for which each of the embeddings $(\mathcal{F}_n, \rho) \hookrightarrow \mathcal{F}$ is continuous—it is a nonmetrizable topology induced by a uniformity and convergence has the characterization:

$$A_n \to A \Leftrightarrow (\lim_n \rho(A_n, A) = 0 \text{ and } \sup_n \|A_n\| < \infty) \tag{3}$$

We next consider the completion of $(\mathcal{F}, \rho)$. This first lemma follows from [3, Theorem 5.1]

Lemma 5.3 *There is a positive constant κ_m with the following property: if $E \subset R^m$ and $\mathcal{H}^{m-1}(E) < a$, then for each $\eta > 0$ we can find a sequence of cubes $\{C_n\}$ with diameters less than η and such that $E \subset \bigcup_n C_n^\circ$ and $\sum_n \|C_n\| < \kappa_m a$.*

Proposition 5.4 *Let $E \subset R^m$ be a bounded set with $\mathcal{H}^{m-1}(\partial E) < a$. Then there is a positive constant κ_m (depending only on the dimension m) and a sequence $\{A_n\}$ of figures such that $A_n \subset E^\circ$, $\|A_n\| < \kappa_m a$ for $n = 1, 2, \cdots$, and $\lim \mathcal{H}^m(E - A_n) = 0$.*

PROOF. The compactness of ∂E and Lemma 5.3 imply there are cubes $C_1 \ldots, C_k$ each with diameter less than $1/n$ and such that $\partial E \subset \bigcup_{i=1}^k C_i^\circ$

and $\sum_{i=1}^{k} \|C_n\| < \kappa_m a$. Observe that $A_n = cl(E - \bigcup_{i=1}^{k} C_i)$ is a subfigure of E° with

$$\mathcal{H}^m(E - A_n) \leq \sum_{i=1}^{k} \mathcal{H}^m(C_i) \leq \frac{1}{n} \cdot \frac{1}{2m} \sum_{i=1}^{k} \|C_i\| < \frac{1}{n} \cdot \frac{\kappa_m a}{2m},$$

and $\|A_n\| < \kappa_m a$ □

Define the *solids* $\mathcal{S} = \{E \subset R^m : E$ bounded and $\mathcal{H}^{m-1}(\partial E) < \infty\}$. Proposition 5.4 then says that each solid E is the limit of a sequence of figures; this is convergence in the sense given in (3) above.

We extend Definition 5.2 to solids. F is an *additive* function (of solids) in a set $E \subset R^m$ if $F(B \cup C) = F(B) + F(C)$ whenever $B, C \subset E$ are nonoverlapping solids. An additive function F (of solids) in a bounded set $E \subset R^m$ is *continuous* if given $\varepsilon > 0$ there is an $\eta > 0$ such that $|F(C)| < \varepsilon$ for each solid $C \subset E$ with $\mathcal{H}^{m-1}(\partial C) < 1/\varepsilon$ and $\mathcal{H}^m(C) < \eta$.

Remark 5.5 *Definition 5.2 employed the perimeter in the definition of continuity. The proper definition of the perimeter of Lebesgue measurable sets should be based upon the measure theoretic boundary [3, Sections 5.1,5.8]; while for a figure B this is the same as the topological boundary ∂B—this is not true for solids. Although it is preferable to continue the use of perimeter in defining continuity, we yield to convenience; for our present purposes it is not critical.*

For figures A and B, define a new figure $A \ominus B = cl(A - B)$ and observe that $\|A \ominus B\| \leq \|A\| + \|B\|$. If F is an additive continuous function defined on the family of all subfigures of E and $\{A_i\}$, $\{B_j\}$ are two (not necessarily distinct) sequences of subfigures of E converging to E, then the estimate

$$|F(A_i) - F(B_j)| = |F(A_i \ominus B_j) - F(B_j \ominus A_i)| \leq |F(A_i \ominus B_j)| + |F(B_j \ominus A_i)|$$

implies that $|F(A_i) - F(B_j)| \to 0$ as $i, j \to \infty$. It is now routine to verify:

Proposition 5.6 *Each additive continuous function F defined on the family of all subfigures of a solid E has a unique extension to an additive continuous function on the family of all subsolids of E.*

In fact, the completion of $(\mathcal{F}, \tau)$ is much larger than $(\mathcal{S}, \tau)$; it is the geometrically rich space of bounded Caccioppoli sets—bounded sets of bounded variation—which we denote BV. Although we will not use these sets here, some comments are appropriate. First, the obvious analog of Proposition 5.6 holds for these sets. Indeed, by [3, Theorem 1.24] each bounded Caccioppoli set can be approximated in the sense of (3) by special solids, namely C^∞ manifolds. Consequently, every bounded Caccioppoli set can be approximated by figures. That the completion of $(\mathcal{F},\tau)$ is no larger than (BV,τ) follows from [3, Theorem 1.9].

The next lemma lays the foundation for a *uniform* characterization of additive continuous functions in a figure A; a result which is crucial in proving the multiplier theorem (Theorem 9.2).

Lemma 5.7 *If F is an additive continuous function defined on the family of all subfigures of a figure A, then*

$$\lim_{\|B\|\to+\infty}\frac{F(B)}{\|B\|}=0 \qquad \textit{and} \qquad \sup_{\|B\|<c}|F(B)|<+\infty$$

for each $c>0$.

PROOF: Let $0=(0,\dots,0)$ be the origin of R^m, and find an $r>0$ with $A\subset U(0,r)$. If $a=2^m r^{m-1}$, then the $\mathcal{H}^{m-1}$ measure of each face of $U(0,r)$ equals $a/2$. Given $c\geq 1$, choose a positive $\varepsilon<1/[c(1+2a)]$ and find an $\eta>0$ so that $|F(B)|<\varepsilon$ for each figure $B\subset A$ with $\|B\|<1/\varepsilon$ and $\mathcal{H}^m(B)<\eta$. Select an integer $p>(2r)^m/\eta$ and for $i=1,\dots,p$, let

$$A_i=\left[-r+(i-1)\frac{2r}{p},-r+i\frac{2r}{p}\right)\times[-r,r]^{m-1}.$$

If C is a subfigure of A and $\|C\|<c$, then

$$\mathcal{H}^m(C\odot A_i)\leq\mathcal{H}^m(A_i)<\eta \qquad \textit{and} \qquad \|C\odot A_i\|\leq\|C\|<c\leq\frac{1}{\varepsilon}$$

for $i=1,\dots,p$. Thus $|F(C)|\leq\sum_{i=1}^p|F(C\odot A_i)|<p\varepsilon$, and the second claim is proved.

To prove the first claim, choose a figure $C\subset A$ so that $\|C\|>\max\{p,1/\varepsilon\}$, and for each $t\in(-r,r)$, let

$$C_-(t)=C\odot([-r,t)\times[-r,r]^{m-1}) \quad \textit{and} \quad C_+(t)=C\odot([t,r]\times[-r,r]^{m-1}).$$

Then C is the disjoint union of figures $C_\pm(t)$, and it is easy to see that

$$\|C_-(t)\|+\|C_+(t)\|\leq\|C\|+a.$$

Notice that $t\mapsto\|C_-(t)\|$ is an increasing function on $(-r,r)$, which increases from 0 to $\|C\|$. Since

$$\lim_{t\to\tau+}\|C_-(t)\|-\lim_{t\to\tau-}\|C_-(t)\|\leq a$$

for each $\tau \in (-r, r)$, there is a $\theta \in (-r, r)$ such that $\|C\|/2 < \|C_-(\theta)\| \leq \|C\|/2 + a$. We conclude that $\|C_\pm(\theta)\| \leq \|C\|/2 + a$. Next find an integer $n \geq 1$ with

$$\frac{\|C\|}{2^n} < \frac{1}{\varepsilon} - 2a \leq \frac{\|C\|}{2^{n-1}},$$

and proceeding inductively, construct nonoverlapping figures $C_1, \ldots, C_{2^n}$ whose union is C and such that

$$\|C_k\| \leq \frac{\|C\|}{2^n} + \sum_{j=0}^{n-1} \frac{a}{2^j} < \frac{\|C\|}{2^n} + 2a < \frac{1}{\varepsilon}.$$

Note that the inequality $1 < 1/\varepsilon - 2a$ yields $2^{n-1} < \|C\|$. For $i = 1, \ldots, p$ and $k = 1, \ldots, n$, we have

$$\mathcal{H}^m(A_i \odot C_k) \leq \mathcal{H}^m(A_i) < \eta \quad \textit{and} \quad \|A_i \odot C_k\| \leq \|C_k\| < 1/\varepsilon.$$

By construction, the collection

$$\{A_i \odot C_k : i = 1, \ldots, p;\ k = 1, \ldots, 2^n\}$$

contains at most $2^n + p - 1$ nonempty figures whose union is C. Therefore

$$|F(C)| \leq \sum_{i=1}^{p} \sum_{k=1}^{2^n} |F(A_i \odot C_k)| < \varepsilon(2^n + p - 1) < \varepsilon(2\|C\| + p),$$

and hence

$$\frac{|F(C)|}{\|C\|} < \varepsilon\left(2 + \frac{p}{\|C\|}\right) < 3\varepsilon$$

□

Lemma 5.8 *(Uniformity Lemma) An additive function F defined on the family of all subfigures of a figure A is continuous if and only if the following condition is satisfied: given $\varepsilon > 0$, there is a $\theta > 0$ such that*

$$|F(B)| < \theta \mathcal{H}^m(B) + \varepsilon(\|B\| + 1)$$

for each subfigure B of A.

PROOF. As the converse is obvious, assume that F is continuous and choose an $\varepsilon > 0$. According to Lemma 5.7, there are positive numbers b and c such that $|F(B)| < \varepsilon\|B\|$ and $|F(C)| < b$ whenever B, C are subfigures of A such that $\|B\| \geq c$ and $\|C\| < c$. We can find an $\eta > 0$ so that $|F(C)| < \varepsilon$ for each subfigure C of A for which $\|C\| < c$ and $\mathcal{H}^m(C) < \eta$. Now if $\|C\| < c$ and $\mathcal{H}^m(C) \geq \eta$, then $|F(C)| < b \leq (b/\eta)\mathcal{H}^m(C)$. Letting $\theta = b/\eta$, the previous alternatives yield the desired inequality. □

Example 5.9 *Let $A \subset R^m$ be a figure and let $v : A \to R^m$ be a continuous vector field. If $B \subset A$ is a figure, then n_B is the usual exterior normal, defined $\mathcal{H}^{m-1}$-almost everywhere on ∂B. We let*

$$F(B) = (L) \int_{\partial B} v \cdot n_B \, d\mathcal{H}^{m-1},$$

for each subfigure $B \subset A$. Clearly F is additive, we show that it is continuous. Choose an $\varepsilon > 0$ and find a continuously differentiable vector field $w : R^m \to R^m$ so that $\|v(x) - w(x)\| < \varepsilon$ for each $x \in A$. Let θ be a positive upper bound for $|div\, w|$ on A. The Schwarz inequality and standard divergence theorem applied to w over B yield

$$\begin{aligned} \left| (L) \int_{\partial B} v \cdot n_B \, d\mathcal{H}^{m-1} \right| &\leq (L) \int_B |div\, w| \, d\mathcal{H}^m + (L) \int_{\partial B} \|v - w\| \, d\mathcal{H}^{m-1} \\ &\leq \theta \mathcal{H}^m(B) + \varepsilon \|B\|. \end{aligned}$$

□

6. Gages and calibers

We call $T \subset R^m$ *thin* if it has σ-finite $\mathcal{H}^{m-1}$-measure. Let δ be a nonnegative function on a set $E \subset R^m$. As before, the null set of δ is the set

$$\mathcal{Z}_\delta = \{x \in E : \delta(x) = 0\}.$$

If $\mathcal{Z}_\delta$ is thin, then δ is called a *gage*; note that a countable set has σ-finite $\mathcal{H}^0$-measure so this is consistent with our earlier usage.

By a *caliber* we mean a sequence $\eta = \{\eta_j\}$ of positive numbers. For a given $\varepsilon > 0$ and a caliber η, we say B is (ε, η)-small if B can be written as a union of figures $B_1, \cdots, B_k$ such that $\mathcal{H}^m(B_i) < \eta_i$ and $\|B_i\| < 1/\varepsilon$ for $i = 1, \cdots, k$. A partition P is a *partition of A mod (ε, η)* if $A \ominus \bigcup P$ is (ε, η)-small.

Proposition 6.1 *Let $A \subset R^m$ be a figure and let δ be a positive function on $A - T$, where T is a thin set. Then there is a positive constant λ, depending only on the dimension m, with the following property: for each positive $\varepsilon < \lambda$ and each caliber η there is a δ-fine partition $\{(A_1, x_1), ..., (A_r, x_r)\}$ of A mod (ε, η) such that $A_1, \ldots, A_r$ are dyadic cubes.*

PROOF. Let $\lambda = 1/(2\kappa_m)$ where κ_m is the constant of Lemma 5.3, and select a positive $\varepsilon < \lambda$. We first suppose A is an interval with integer endpoints. As T has σ-finite $\mathcal{H}^{m-1}$-measure, we can write $T = \bigcup_{j=1}^{\infty} T_j$ where $\mathcal{H}^{m-1}(T_i) < 2$.

By Lemma 5.3 there are countable collections $\mathcal{C}_j$ of dyadic cubes with diameters less than η_j/κ_m and satisfying

$$T_j \subset (\bigcup_{C\in\mathcal{C}_j} C)^\circ \quad \text{and} \quad \sum_{C\in\mathcal{C}_j} \|C\| \le 2\kappa_m.$$

Further, eliminating redundancies, we require that $\mathcal{C}_{j_1}\cap\mathcal{C}_{j_2}=\emptyset$ whenever $j_1\neq j_2$. Set $\mathcal{C}=\bigcup_j \mathcal{C}_j$ and apply Lemma 3.2 to find a partition $P=\{(A_1,x_1),\ldots,(A_p,x_p)\}$ in A such that $A\ominus\bigcup P=\bigcup\mathcal{D}$ with $\mathcal{D}\subset\mathcal{C}$. To complete this first part, define the (nonoverlapping) figures $D_j=\bigcup(\mathcal{D}\cap\mathcal{C}_j)$ and notice that there are only finitely many D_j which are nonempty since $\mathcal{D}$ is finite. Observing that

$$\mathcal{H}^m(D_j)\le\mathcal{H}^m(\bigcup\mathcal{C}_j)\le\frac{\eta_j}{\kappa_m}\sum_{C\in\mathcal{C}_j}[d(C)]^{m-1}\le\frac{\eta_j}{2m\kappa_m}\sum_{C\in\mathcal{C}_j}\|C\|\le\eta_j,$$

it is easy to check that $A\ominus\bigcup P=\bigcup_j D_j$ is an (ε,η)-small figure.

For a general interval A, enclose it in a larger interval K with integer endpoints and define a new function on K

$$\Delta(x)=\begin{cases}\min\{\delta(x),dist(x,\partial A)\} & \text{if } x\in A\\ dist(x,\partial A) & \text{if } x\in K-A.\end{cases}$$

By the first part there is a Δ-fine partition $Q=\{(A_1,x_1),\ldots,(A_p,x_p)\}$ of K mod (ε,η). Notice that if B is (ε,η)-small, then $B\cap A$ is (ε,η)-small—this is true since A is an interval. Thus $P=\{(A_i,x_i):x_i\in A\}$ is a δ-fine partition of A mod (ε,η).

For the case where $A=\bigcup_{k=1}^n B_k$, a finite union of intervals, apply the foregoing to the intervals B_k, calibers $\{\eta_{ni-k+1}\}_{i=1}^\infty$, and $\varepsilon/2^k$. □

Lemma 6.2 *Let F be a continuous additive function in a figure $A\subset R^m$. Given $\varepsilon>0$, there is a caliber η such that $|F(B)|<\varepsilon$ whenever $B\subset A$ is an (ε,η)-small figure.*

PROOF. By the continuity of F, for each $\varepsilon/2^i$ (i a positive integer) there is a positive number η_i such that $|F(B)|<\varepsilon/2^i$ whenever B is a figure with $\mathcal{H}^m(B)<\eta_i$ and $\|B\|<1/\varepsilon$. Let $\eta=\{\eta_i\}$. □

Uniqueness of the gage integral in dimension one followed from its equivalence to the HK-integral. We now give a direct proof which generalizes to higher dimensions.

Proposition 6.3 *The indefinite gage integral is unique.*

PROOF. Let F_1, F_2 be two continuous additive functions which satisfy Definition 4.17 with respect to a function f defined on an interval A. Let $\varepsilon > 0$ be given and choose a gage δ so that $\sum_{i=1}^{p} |f(x_i)\mathcal{H}^m(A_i) - F_j(A_i)| < \varepsilon$ whenever $Q = \{(A_1, x_1), ..., (A_p, x_p)\}$ is a δ-fine partition in A; $j = 1, 2$. Pick a caliber η according to Lemma 6.2 so that $|F_j(C)| < \varepsilon$ whenever C is an (ε, η)-small figure; $j = 1, 2$. Choose any interval $B \subset A$ and apply Proposition 6.1 to δ, B, and η to obtain a partition $P = \{(B_1, y_1), ..., (B_q, y_q)\}$ of B mod (ε, η). We have the inequality

$$
\begin{aligned}
&|F_1(B) - F_2(B)| \\
&\quad \le |F_1(B) - \sigma(f, P)| + |F_2(B) - \sigma(f, P)| \\
&\quad \le \sum_{i=1}^{q} \left| f(y_i)\mathcal{H}^m(B_i) - F_1(B_i) \right| + \sum_{i=1}^{q} \left| f(y_i)\mathcal{H}^m(B_i) - F_2(B_i) \right| \\
&\qquad + |F_1(B \ominus \bigcup P)| + |F_2(B \ominus \bigcup P)| \\
&\quad \le 4\varepsilon.
\end{aligned}
$$

□

7. The Gauss-Green theorem

One of the main features of the HK-integral in dimension one is the generality of the *Gauss-Green theorem* it affords, i.e. *the fundamental theorem of calculus.* As the obvious generalization of the Henstock Kurzweil integral to R^m no longer provides this, we next state an infinitesimal version of the Gauss-Green theorem and use it to outline an alternate proof of Theorem 4.18. In this setting, it becomes clear how the mode of approximation—the allowable partition sets in the Stieltjes sum—must be altered in order to retain the generality of Theorem 4.18.

Lemma 7.1 *Let v be a continuous vector field on a figure $A \subset R^m$ that is differentiable at $x \in A^\circ$. Given $\varepsilon > 0$, there is a number $\delta_x > 0$ such that*

$$\left| div\, v(x)\mathcal{H}^m(B) - (L)\int_{\partial B} v \cdot n_B \, d\mathcal{H}^{m-1} \right| < \varepsilon d(B)\|B\|$$

for each figure $B \subset A \cap U(x, \delta_x)$ for which $x \in B$.

PROOF. Choose an $\varepsilon > 0$, and let $w(y) = v(x) + D_x v(y - x)$ for each $y \in R^m$; here $D_x v$ is the differential of v at x. Then $div\, w(y) = div\, v(x)$ for all $y \in R^m$, and there is a $\delta_x > 0$ such that $|v(y) - w(y)| < \varepsilon|y - x|$ for all $y \in A$ with $|y - x| < \delta_x$. Now if B is as in the statement of the lemma, then

$$\begin{aligned}
&\left|div\, v(x)\mathcal{H}^m(B) - (L)\int_{\partial B} v \cdot n_B d\mathcal{H}^{m-1}\right| \\
= &\left|(L)\int_B div\, w(y)\mathcal{H}^m - (L)\int_{\partial B} v \cdot n_B d\mathcal{H}^{m-1}\right| \\
= &\left|(L)\int_{\partial B} (w - v) \cdot n_B d\mathcal{H}^{m-1}\right| < \varepsilon d(B)\|B\|
\end{aligned}$$

by the usual Gauss-Green theorem for the linear vector field w. □

Theorem (The fundamental theorem of calculus) *If F is differentiable in an interval $A = [a, b]$, then $F' \in \mathcal{HK}([a, b])$ and*

$$(HK)\int_a^b F' = F(b) - F(a).$$

PROOF. Set $v = F$, then $F' = div\, v$. Using Lemma 7.1, given $\varepsilon > 0$ we can find a δ such that if $P = \{(A_1, x_1), ..., (A_p, x_p)\}$ in A is δ-fine, we obtain

$$\sum_{i=1}^{p}\left|div\, v(x_i)\mathcal{H}^1(A_i) - (L)\int_{\partial A_i} v \cdot n_{A_i}\, d\mathcal{H}^0\right| \;\le\; \sum_{i=1}^{p} \varepsilon d(A_i)\|A_i\| \le 2\varepsilon\mathcal{H}^1(A),$$

since $\|B\| = 2$ and $d(B) = \mathcal{H}^1(B)$ for each interval $B \subset A \subset R$. Thus the indefinite HK-integral of F' is the function $B \mapsto (L)\int_{\partial B} v \cdot n_B\, d\mathcal{H}^0 = F(d) - F(c)$ where $B = [c, d]$. □

As the relationships $\|B\| = 2$ and $d(B) = \mathcal{H}^1(B)$ in the previous proof do not hold for intervals $B \subset R^m$ when $m \ge 2$, we must somehow restrict the intervals entering into a partition. The previous proof suggests this should be done by means of a number

$$r(B) = \begin{cases} \frac{\mathcal{H}^m(B)}{d(B)\|B\|} & \text{if } B \ne \emptyset \\ 0 & \text{otherwise} \end{cases}$$

associated to each figure B, and called the *regularity* of B. We simply say a partition $P = \{(A_1, x_1), ..., (A_p, x_p)\}$ is *ε-regular* if $r(A_i) > \varepsilon$ for each interval A_i.

Definition 7.2 *Let $A \subset R^m$ be a figure. A function f defined on A is gage integrable if we can find an additive continuous (Definition 5.2) function F in*

A satisfying the following condition: given $\varepsilon > 0$, there is a gage δ on A such that

$$\sum_{i=1}^{p}\left|f(x_i)\mathcal{H}^m(A_i) - F(A_i)\right| < \varepsilon$$

for each δ-fine ε-regular partition $\{(A_1, x_1), ..., (A_p, x_p)\}$ in A.

As before, if f is gage integrable in A with indefinite integral F (unique by Proposition 6.3), the integral of f on A is $(g)\int_A f = F(A)$ and the family of all gage integrable functions defined on A is denoted by $\mathcal{G}(A)$. Observe that for any interval $B \subset R$, we have $r(B) = 1/2$ and consequently the present definition of the gage integral coincides with that given previously in section 4.

Remark 7.3 *Sometimes the shape $s(B) = \mathcal{H}^m(B)/[d(B)]^m$ of a figure $B \subset R^m$ is used in place of the regularity (cf. [3, Theorem 11.4.9]). Since*

$$[2mr(B)]^m \leq s(B) \leq 2mr(B)$$

for any interval B, this variation is of no consequence. We note, however, that there is a significant difference between the shape and regularity when figures are involved (see [3, Remark 12.1.7]).

Theorem 7.4 *(Gauss-Green theorem) Let $T \subset R^m$ be a thin set, and let v be a continuous vector field on a figure A that is almost differentiable at each $x \in A^\circ - T$. If f is a function on A such that $f(x) = \operatorname{div} v(x)$ for every $x \in A^\circ - T$ at which v is differentiable, then $f \in \mathcal{G}(A)$ and*

$$(g)\int_A f = (L)\int_{\partial A} v \cdot n_A \, d\mathcal{H}^{m-1}.$$

PROOF. For each figure $B \subset A$, let $F(B) = (L)\int_{\partial B} v \cdot n_B \, d\mathcal{H}^{m-1}$. Then the additive continuous (by Example 5.9) function F is the natural candidate for the indefinite integral of f.

By the Stepanoff theorem there is a $\mathcal{H}^m$-negligible set $E \subset A^\circ - T$ such that v is differentiable at each $x \in A^\circ - (E \cup T)$. By Corollary 4.6 we may assume that f is zero at each point of E. Choose $\varepsilon > 0$ and let α be associated with ε and E according to Lemma 4.4. For each $x \in E$ there are positive numbers c_x and δ_x such that

$$\|v(y) - v(x)\| \leq c_x|y - x| \quad \text{and} \quad \alpha(B) \geq \frac{c_x}{\varepsilon}\mathcal{H}^m(B)$$

for every $y \in A \cap U(x, \delta_x)$ and every $B \subset A \cap U(x, \delta_x)$. As a consequence, for each $x \in E$,

$$\begin{aligned}
|F(B)| &= |(L)\int_{\partial B} v(y) \cdot n_B(y)\, d\mathcal{H}^{m-1}(y)| \\
&= |(L)\int_{\partial B} [v(y) - v(x)] \cdot n_B(y)\, d\mathcal{H}^{m-1}(y)| \\
&\leq |(L)\int_{\partial B} c_x|y - x|\, d\mathcal{H}^{m-1}(y)| \\
&\leq c_x d(B)\|B\| \leq \frac{c_x}{\varepsilon}\mathcal{H}^m(B) \leq \alpha(B)
\end{aligned}$$

whenever $B \subset A \cap U(x, \delta_x)$ is a figure with $r(B) > \varepsilon$.

If $x \in A^\circ - (E \cup T)$ then Lemma 7.1 yields a $\delta_x > 0$ such that

$$|f(x)\mathcal{H}^m(B) - F(B)| < \varepsilon^2 d(B)\|B\| < \varepsilon\mathcal{H}^m(B)$$

for every ε-regular figure such that $B \subset A \cap U(x, \delta_x)$ and $x \in B$.

Define a gage on A

$$\delta(x) = \begin{cases} \delta_x & \text{if } x \in A^\circ - T \\ 0 & \text{otherwise.} \end{cases}$$

If $P = \{(A_1, x_1), ..., (A_p, x_p)\}$ is a δ-fine ε-regular partition in A, then

$$\begin{aligned}
\sum_{i=1}^{p} |f(x_i)\mathcal{H}^m(A_i) - F(A_i)| &= \sum_{x_i \in E} |F(A_i)| + \sum_{x_i \notin E} |f(x_i)\mathcal{H}^m(A_i) - F(A_i)| \\
&< \sum_{x_i \in E} \alpha(A_i) + \sum_{x_i \notin E} \varepsilon\mathcal{H}^m(A_i) \\
&< \alpha(A) + \varepsilon\mathcal{H}^m(A) \leq \varepsilon(1 + \mathcal{H}^m(A)).
\end{aligned}$$

□

8. A geometric integral

Recall that by a (local) geometric integral we understand an integral in R^m which

1. extends the Lebesgue integral;

2. integrates partial derivatives of differentiable functions so that the Gauss-Green theorem is satisfied;

3. is coordinate free, i.e. invariant with respect to a group containing all diffeomorphisms.

Z. Buczolich has shown in [3] that the gage integral of the previous section is not rotation invariant. As indicated by the title of this section, we now define an integral which meets all the criteria of a geometric integral.

An *$\mathcal{F}$-partition* in E is a collection $P = \{(A_1, x_1), ..., (A_p, x_p)\}$ where $A_1, \ldots, A_p$ are nonoverlapping subfigures of E and $x_i \in A_i$ for $i = 1, \ldots, p$. Given $\varepsilon > 0$ and a gage δ on E, we say that P is *ε-regular* or *δ-fine* if $r(A_i) > \varepsilon$ for $i = 1, \ldots, p$ or $d(A_i) < \delta(x_i)$ for $i = 1, \ldots, p$, respectively.

Definition 8.1 *Let A be a figure and let f be a function defined on A. We say that f is $\mathcal{F}$-integrable (or simply integrable) in A if there is an additive continuous function F defined on the family of all subfigures of A having the following property: given $\varepsilon > 0$, we can find a gage δ on A so that*

$$\sum_{i=1}^{p} |f(x_i)\mathcal{H}^m(A_i) - F(A_i)| < \varepsilon$$

for each δ-fine ε-regular $\mathcal{F}$-partition $\{(A_1, x_1), ..., (A_p, x_p)\}$ in A.

The function F is called the *indefinite integral* of f in A and the number $\int_A f = F(A)$ the *integral* of f on A. The family of integrable functions is denoted by $\mathcal{F}(A)$. It is clear that $\mathcal{F}(A) \subset \mathcal{G}(A)$ and $\int_A f = (g)\int_A f$ whenever both are defined; thus uniqueness of the indefinite integral is already resolved.

We state the next two theorems for completeness. The proofs are the same as for their counterparts Theorem 4.15 and Theorem 7.4.

Theorem 8.2 *$L^1(A) \subset \mathcal{F}(A)$ and*

$$\int_A f = (L)\int_A f$$

for each $f \in L^1(A)$.

Theorem 8.3 *(Gauss-Green theorem) Let $T \subset R^m$ be a thin set, and let v be a continuous vector field on a figure A that is almost differentiable at each $x \in A^\circ - T$. If f is a function on A such that $f(x) = div\, v(x)$ for every $x \in A^\circ - T$ at which v is differentiable, then $f \in \mathcal{F}(A)$ and*

$$\int_A f = (L)\int_{\partial A} v \cdot n_A \, d\mathcal{H}^{m-1}.$$

The proof given for coordinate invariance of the HK-integral (Theorem 4.20) depends upon the fact that intervals (in R) are *stable* under lipeomorphisms, i.e. the lipeomorphic image of an interval is again an interval. Consequently, partitions are mapped to partitions by a lipeomorphism. Except in R, an $\mathcal{F}$-partition is certainly not mapped to an $\mathcal{F}$-partition, but by using the approximation result in Proposition 5.4 we can give almost the same proof for the invariance theorem below.

Let $E \subset R^m$ be a figure. If $\Phi : E \to R^m$ is Lipschitz, $\det \Phi$ denotes, as before, the determinant of the differential $D\Phi$ of Φ. By the Rademacher and Kirszbraun theorems [3, Corollary 10.4.8 and Theorem 10.3.3], the function $\det \Phi$ is defined almost everywhere in E° and hence in E. The following change of variables theorem has been established in [3].

Theorem 8.4 *Let $\Phi : A \to B$ be a lipeomorphism from a figure A onto a figure B, and let f be an integrable function in B. Then $f \circ \Phi \cdot |\det \Phi|$ is integrable in A and*

$$\int_A f \circ \Phi \cdot |\det \Phi| = \int_B f\,.$$

PROOF. There are positive constants α and β such that

$$\alpha|x - x'| \leq |\Phi(x) - \Phi(x')| \leq \beta|x - x'|$$

for all $x, x' \in A$. If C is a subfigure of A, then it follows from [3, Lemma 1.8] that $\Phi(C)$ belongs to $\mathcal{S}$ and satisfies the inequalities

$$\mathcal{H}^m[\Phi(C)] \geq \alpha^m \mathcal{H}^m(C) \quad \text{and} \quad \mathcal{H}^{m-1}[\partial\Phi(C)] \leq \beta^{m-1}\|C\|.$$

By Proposition 5.6, the indefinite integral of f in B has a unique additive continuous extension F to the family of all subsolids of B. In view of the above inequalities, the map $G : C \mapsto F[\Phi(C)]$ is an additive continuous function defined on all subfigures of A. We show the function G is the indefinite integral of $f \circ \Phi|\det \Phi|$ in A. Choose $\varepsilon > 0$, by [3, Theorem 7.2.4] there is a set $N \subset A$ with $\mathcal{H}^1(N) = 0$ and a positive function Δ on A such that

$$|f \circ \Phi(x)| \cdot \Big| |\det \Phi(x)| \mathcal{H}^m(C) - \mathcal{H}^m(\Phi(C)) \Big| < \varepsilon \mathcal{H}^m(C)$$

for each $x \in A - N$ and each figure $C \subset A$ with $x \in C$, $r(C) > \varepsilon$ and $d(C) < \Delta(x)$. By an easy analog of Corollary 4.6 we may assume that f is zero on $\Phi(N)$, which set is $\mathcal{H}^m$-negligible since Φ is Lipschitz.

Since f is integrable in B, there is a gage δ_B on B such that

$$\sum_{i=1}^{p} \Big| f(y_i)\mathcal{H}^m(B_i) - F(B_i) \Big| < \varepsilon$$

for each δ_B-fine ε-regular partition $\{(B_1, y_1), ..., (B_p, y_p)\}$ in B. With no loss of generality, we may assume that $\delta_B(x) = 0$ for each $x \in \partial B$. Let $\varepsilon' = (\beta/\alpha)^m \kappa_m \varepsilon$ where κ_m is the constant of Proposition 5.4 and define a gage δ_A on A by setting $\delta_A = \min\{\delta_B \circ \Phi/\beta, \Delta\}$. Choose a δ_A-fine ε'-regular partition $\{(A_1, x_1), ..., (A_p, x_p)\}$ in A. For $i = 1, \dots, p$, let $K_i = \Phi(A_i)$ and $y_i = \Phi(x_i)$, and observe that $d(K_i) < \delta_B(y_i)$ and

$$\frac{\mathcal{H}^m(K_i)}{d(K_i)\mathcal{H}^{m-1}(\partial K_i)} \geq (\frac{\alpha}{\beta})^m r(A_i) > \kappa_m \varepsilon.$$

It follows from Proposition 5.4 and Proposition 5.6 that each K_i contains a figure B_i such that

$$r(B_i) > \varepsilon,\ |f(y_i)| \cdot |\mathcal{H}^m(K_i) - \mathcal{H}^m(B_i)| < \varepsilon/p,\ |G(A_i) - F(B_i)| < \varepsilon/p.$$

As the figure B_i may not contain the point y_i, an additional adjustment is necessary. Fix an integer i with $1 \leq i \leq p$, and observe that $y_i \in B^\circ$ by the choice of δ_B. Thus we can select nonoverlapping cubes $C_1, \dots, C_{2^m}$ contained in B whose common vertex is the point y_i. If $\{j_1, \dots, j_k\}$ is the set of all indexes j for which $x_i \in A_j$, then $1 \leq k \leq 2^m$ and we let

$$B'_{j_s} = cl\left[(B_{j_s} \cup C_s) - \bigcup_{r \neq s} C_r\right]$$

for $s = 1, \dots, k$. Since $y_i \in K_i$, the cubes $C_1, \dots, C_{2^m}$ can be chosen so small that $d(B'_i) < \delta_B(y_i)$ and the above inequalities hold when B_i is replaced by B'_i. Thus

$$\begin{aligned}
& \sum_{i=1}^{p}\left|f(\Phi(x_i)) \cdot |\det \Phi(x_i)|\mathcal{H}^m(A_i) - G(A_i)\right| \\
\leq\ & \sum_{i=1}^{p}|f(\Phi(x_i))| \cdot \left||\det \Phi(x_i)|\mathcal{H}^m(A_i) - \mathcal{H}^m(K_i)\right| \\
+\ & \sum_{i=1}^{p}|f(y_i)| \cdot \left|\mathcal{H}^m(K_i) - \mathcal{H}^m(B'_i)\right| \\
+\ & \sum_{i=1}^{p}\left|f(y_i)\mathcal{H}^m(B'_i) - F(B'_i)\right| + \sum_{i=1}^{p}\left|F(B'_i) - G(A_i)\right| \\
<\ & \sum_{i=1}^{p}\varepsilon\mathcal{H}^m(A_i) + p\frac{\varepsilon}{p} + \varepsilon + p\frac{\varepsilon}{p} \leq \varepsilon[\mathcal{H}^m(A) + 3],
\end{aligned}$$

since $\{(B'_1, y_1), \ldots, (B'_p, y_p)\}$ is a δ_B-fine ε-regular partition in B. $\square$

For dimensions greater than one Buczolich's example [3] and Theorem 8.4 imply $\mathcal{F}(A)$ is properly contained in $\mathcal{G}(A)$. Surprisingly, even in dimension one the containment is proper [3, Example 12.3.5].

9. Multipliers

A *multiplier* for a family $\mathcal{E}$ of functions on a set E is a function g on E such that $fg \in \mathcal{E}$ (pointwise product) whenever $f \in \mathcal{E}$. For the family $L^1([0,1])$ of Lebesgue integrable functions defined on $[0,1]$, bounded measurable functions are multipliers. Recall the notation of Example 5.1 and let $g = \sum_{i=1}^{\infty} \chi_{A_{2i}}$, then it is clear that $gF' \notin \mathcal{HK}([0,1])$. Repeating the argument in Example 5.1 (using the Gauss-Green theorem of section 8. in place of Theorem 4.18), it is clear that $F' \in \mathcal{F}([0,1]) \subset \mathcal{HK}([0,1])$. Consequently, bounded measurable functions are not generally multipliers for $\mathcal{F}([0,1])$. Recently, B. Bongiorno and V. Skvortsov [3] showed that functions of bounded variation are multipliers for $\mathcal{F}(A)$ when A is a subfigure of R, but it is not known at this time whether the result extends to subfigures of R^m.

In this section we show that the Lipschitz functions on a figure A are multipliers for the family $\mathcal{F}(A)$ in any dimension (see [3] for this in the context of BV sets). This is accomplished by constructing the indefinite integral of fg from the indefinite integral of f. We first outline the ideas behind this construction.

For $C \subset R^{m+1}$ and $t \in R$, let $C^t = \{x \in R^m : (x,t) \in C\}$. Suppose that a Lipschitz function g on a figure A maps A into the unit interval $I = [0,1]$, and observe that for each figure $B \subset A$, the subgraph $\Sigma_B = \{(x,t) \in B \times I : t \leq g(x)\}$ of g is a solid. Choose an $f \in \mathcal{F}(A)$, and let $(f \otimes 1)(x,t) = f(x)$ for every $(x,t) \in A \times I$. Assuming that $f \otimes 1$ belongs to $\mathcal{F}(A \times I)$ and applying formally Fubini's theorem, we obtain

$$\begin{aligned}\int_B fg\, d\mathcal{H}^m &= \int_B \left[f(x) \int_0^{g(x)} d\mathcal{H}^1(t) \right] d\mathcal{H}^m(x) = \int_{\Sigma_B} f \otimes 1\, d\mathcal{H}^{m+1} \\ &= \int_I \left[\int_{(\Sigma_B)^t} f(x)\, d\mathcal{H}^m(x) \right] d\mathcal{H}^1(t).\end{aligned}$$

Although the Fubini theorem does not hold for this integral we show that the function

$$G : B \mapsto \int_I \left[\int_{(\Sigma_B)^t} f(x)\, d\mathcal{H}^m(x) \right] d\mathcal{H}^1(t)$$

is still the indefinite integral of fg.

Given an additive function F on the family of all subfigures of a figure A, let

$$\hat{F}(C) = \int_R F(C^t)\, d\mathcal{H}^1(t)$$

for each subfigure $C \subset A \times R$. Note that $t \mapsto F(C^t)$ is a step function and therefore the integral is well defined.

Lemma 9.1 *If F is an additive continuous function on the family of all subfigures of a figure A, then $\hat{F}$ is an additive continuous function on the family of all subfigures of $A \times [0,1]$.*

PROOF. As the additivity of $\hat{F}$ is clear, choose an $\varepsilon > 0$ and find a $\theta > 0$ associated with F and ε according to the Uniformity Lemma (Lemma 5.8). If $C \subset A \times [0,1]$ is a figure, an appeal to the Fubini theorem yields

$$\int_0^1 \|C^t\|\, d\mathcal{H}^1(t) \le \|C\|.$$

The estimate

$$\begin{aligned} |\hat{F}(C)| \le \int_0^1 |F(C^t)|\, d\mathcal{H}^1(t) < \theta \int_0^1 \mathcal{H}^m(C^t)\, d\mathcal{H}^1(t) + \varepsilon \int_0^1 (1 + \|C^t\|)\, d\mathcal{H}^1(t) \\ \le \theta\, \mathcal{H}^{m+1}(C) + \varepsilon(1 + \|C\|), \end{aligned}$$

implies the continuity of $\hat{F}$. □

Theorem 9.2 *Let g be a Lipschitz function on a figure A. If f belongs to $\mathcal{F}(A)$, then so does fg.*

PROOF. Avoiding a triviality, suppose that $\mathcal{H}^m(A) > 0$. Since g is bounded and $\mathcal{F}(A)$ is a linear space containing the constant functions, we may assume that g maps A into the unit interval $I = [0,1]$. For each figure $B \subset A$, let

$$\Sigma_B = \{(x,t) \in B \times I : t \le g(x)\} \quad \text{and} \quad \Gamma_B = \{(x,t) \in B \times I : t = g(x)\}.$$

Set $c = \max\{1, \mathrm{Lip}(g)\}$, and observe that $\mathcal{H}^m(\Gamma_B) \le c^m \mathcal{H}^m(B)$ [3, Lemma 1.8]. Estimating the perimeter of the base, sides and top of Σ_B, we obtain

$$\mathcal{H}^m(\partial \Sigma_B) \le \mathcal{H}^m(B) + \|B\| + \mathcal{H}^m(\Gamma_B) \le (1 + c^m)\mathcal{H}^m(B) + \|B\| < +\infty$$

thus Σ_B is a solid. Select an $f \in \mathcal{F}(A)$ and denote by F the indefinite integral of f in A. Let $\hat{F}$ be as in Lemma 9.1 and using Proposition 5.6 extend $\hat{F}$ to an additive continuous function on the family of all subsolids of $A \times [0,1]$, still denoted by $\hat{F}$. Then $G : B \mapsto \hat{F}(\Sigma_B)$ is a additive continuous function on the subfigures of A. Using a technique similar to that in [3, Theorem 6.8], we show that G is the indefinite integral of fg.

To this end, choose an $\varepsilon > 0$ and find an $\eta > 0$ so that $|\hat{F}(C)| < \varepsilon$ for each solid $C \subset A \times I$ with $\|C\| < (1+c^m+c/\varepsilon)\mathcal{H}^m(A)$ and $\mathcal{H}^{m+1}(C) < \eta[c\mathcal{H}^m(A)]$. There is a gage δ on A such that

$$\sum_{i=1}^{p} |f(x_i)\mathcal{H}^m(A_i) - F(A_i)| < \varepsilon$$

for each δ-fine ε-regular $\mathcal{F}$-partition $\{(A_1, x_1), ..., (A_p, x_p)\}$ in A. With no loss of generality, we may assume that $\delta \leq \eta$. Let $\{(A_1, x_1), ..., (A_p, x_p)\}$ be a δ-fine ε-regular $\mathcal{F}$-partition in A, and let $J_i = [0, g(x_i)]$ for $i = 1, \ldots, p$. We obtain

$$\begin{aligned}
\sum_{i=1}^{p} |f(x_i)g(x_i)\mathcal{H}^m(A_i) - G(A_i)| &\leq \sum_{i=1}^{p} g(x_i)|f(x_i)\mathcal{H}^m(A_i) - F(A_i)| \\
+\sum_{i=1}^{p} |F(A_i)\mathcal{H}^1(J_i) - \hat{F}(\Sigma_{A_i})| &< \varepsilon + \sum_{i=1}^{p} |\hat{F}(A_i \times J_i) - \hat{F}(\Sigma_{A_i})| .
\end{aligned}$$

If $S = \sum_{i=1}^{p} |\hat{F}(A_i \times J_i) - \hat{F}(\Sigma_{A_i})|$, then after a suitable reordering, we find an integer k with $0 \leq k \leq p$ such that

$$\begin{aligned}
S &= \left|\sum_{i=1}^{k}\left[\hat{F}(A_i \times J_i) - \hat{F}(\Sigma_{A_i})\right]\right| + \left|\sum_{i=k+1}^{p}\left[\hat{F}(A_i \times J_i) - \hat{F}(\Sigma_{A_i})\right]\right| \\
&= \left|\hat{F}\left[\bigcup_{i=1}^{k}(A_i \times J_i)\right] - \hat{F}\left[\bigcup_{i=1}^{k}\Sigma_{A_i}\right]\right| + \\
&\qquad + \left|\hat{F}\left[\bigcup_{i=k+1}^{p}(A_i \times J_i)\right] - \hat{F}\left[\bigcup_{i=k+1}^{p}\Sigma_{A_i}\right]\right| \\
&\leq \left|\hat{F}\left[\bigcup_{i=1}^{k}(A_i \times J_i - \Sigma_{A_i})\right]\right| + \left|\hat{F}\left[\bigcup_{i=1}^{k}(\Sigma_{A_i} - A_i \times J_i)\right]\right| \\
&\qquad + \left|\hat{F}\left[\bigcup_{i=k+1}^{p}(A_i \times J_i - \Sigma_{A_i})\right]\right| + \left|\hat{F}\left[\bigcup_{i=k+1}^{p}(\Sigma_{A_i} - A_i \times J_i)\right]\right| .
\end{aligned}$$

We let $C = \bigcup_{i=1}^{k}(A_i \times J_i - \Sigma_{A_i})$, and estimate $|\hat{F}(C)|$ by observing that

$$\begin{aligned}\mathcal{H}^{m+1}(A_i \times J_i - \Sigma_{A_i}) &\leq c\,d(A_i)\mathcal{H}^m(A_i) < \eta[c\mathcal{H}^m(A_i)]\\ \|A_i \times J_i - \Sigma_{A_i}\| &\leq \mathcal{H}^m(A_i) + \mathcal{H}^m(\Gamma_{A_i}) + c\,d(A_i)\|A_i\|\\ &< \left(1 + c^m + \frac{c}{\varepsilon}\right)\mathcal{H}^m(A_i)\end{aligned}$$

for $i = 1, \ldots, k$. Indeed, these estimates imply that

$$\mathcal{H}^{m+1}(C) < \eta[c\mathcal{H}^m(A)] \qquad \text{and} \qquad \|C\| < \left(1 + c^m + \frac{c}{\varepsilon}\right)\mathcal{H}^m(A),$$

and consequently $|\hat{F}(C)| < \varepsilon$. Completely analogous verifications show that $S < 4\varepsilon$, and the theorem is proved. □

We now give an important application of Theorem 9.2. Let H be a function defined on the subfigures of some figure $A \subset R^m$. Given a partition $P = \{(A_1, x_1), ..., (A_p, x_p)\}$ and a function f on a set $E \subset R^m$ containing $\{x_1, \ldots, x_p\}$, we define the Stieltjes sum

$$\sigma(f, P, H) = \Sigma f(x_i)H(A_i).$$

Definition 9.3 *Let A be a figure and let f be a function defined on A. We say that f is H-integrable in A if there is an additive continuous function F defined on the family of all subfigures of A having the following property: given $\varepsilon > 0$, we can find a gage δ on A so that*

$$\sum_{i=1}^{p} |f(x_i)H(A_i) - F(A_i)| < \varepsilon$$

for each δ-fine ε-regular $\mathcal{F}$-partition $\{(A_1, x_1), ..., (A_p, x_p)\}$ in A.

The function H is called the *integrator* and the function F the *indefinite H-integral* of f in A. The number $\int_A f\,dH = F(A)$ is the H-*integral* of f on A. If $H = \mathcal{H}^m$, then $\int_A f\,dH = \int_A f$.

Proposition 9.4 *Let A be a figure in R^m, H be a function defined on the subfigures of A, and g be H-integrable in A. If $G(B) = \int_B g\,dH$ for each subfigure $B \subset A$, then f is G-integrable if and only if fg is H-integrable, in which event*

$$\int_A f\,dG = \int_A fg\,dH.$$

PROOF. Choose an $\varepsilon > 0$ and for $n = 1, 2, \ldots$, find positive functions δ_n on A such that

$$\sum_{i=1}^{p} \left| g(x_i)H(A_i) - G(A_i) \right| < \frac{\varepsilon}{n2^n}$$

for every ε-regular δ_n-fine partition $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ in A. If $E_n = \{x \in A : n-1 \leq |f(x)| < n\}$ for $n = 1, 2, \ldots$, then A is the disjoint union of the E_n's. For each $x \in A$ let $\delta(x) = \delta_n(x)$ whenever $x \in E_n$. If $Q = \{(B_1, y_1), \ldots, (B_q, y_q)\}$ is an ε-regular δ-fine partition in A, then

$$\begin{aligned}
\left| \sigma(fg, Q, H) - \sigma(f, Q, G) \right| &\leq \sum_{j=1}^{q} |f(y_i)| \cdot \left| g(y_i)H(B_j) - G(B_j) \right| \\
&\leq \sum_{n=1}^{\infty} \sum_{y_i \in E_n} |f(y_i)| \cdot \left| g(y_i)H(B_j) - G(B_j) \right| \\
&< \sum_{n=1}^{\infty} n \frac{\varepsilon}{n2^n},
\end{aligned}$$

and the proposition is proved. □

Any improvement on the multiplier result in Theorem 9.2 would directly yield an improvement in:

Corollary 9.5 *Let $G(B) = \int_B g \, d\mathcal{H}^m$, $F(B) = \int_B f \, d\mathcal{H}^m$ for each subfigure B of a figure A where g is Lipschitz and $f \in \mathcal{F}(A)$. Then f is G-integrable, g is F-integrable, and*

$$\int_A f \, dG = \int_A fg = \int_A g \, dF$$

□

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