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CHAOTIC BEHAVIOR OF NEWTON'S METHOD

A classical application of iterations of functions involves Newton's method. For a differentiable function f, one iterates the function $N_f(x) = x - f(x)/f'(x)$ usually with the idea that the initial point x_0 leads, under iteration of N_f, to a sequence converging to a zero of f. For f a complex polynomial, N_f is analytic on the Riemann sphere and typically there is a nonempty compact set $J(N_f)$, the Julia set of N_f, where $x_0 \in J(N_f)$ implies that the iterations of x_0 under N_f leads to a sequence that behaves chaotically, as described more precisely in Chapter 3 of [D]. In [SU] one finds an analogous result for f a real polynomial in which all of the more than 3 roots are simple, in which case the set of initial points leading to chaotic behavior is a null Cantor set. Moreover if we label the intervals of $\mathbb{R} \setminus f'^{-1}(\{0\})$ as $a_1, a_2, \ldots, a_k$, then for any sequence s with range $\{1, 2, \ldots, k\}$, there is an x_0 such that for each n the n^{th} iterate of N_f at x_0 belongs to $a_{s(n)}$.

One subject having to do with iterations of functions is that of attractors. A subset A of the range of a function f is an attractor, or ω-limit set, of f if there is a point x_0 such that A is the limit set of the sequence of iterates of x_0 under f. For f differentiable, we want each of the zeros of f to be an attractor for the function N_f representing the application of Newton's method on f. Yet even polynomial functions can have other kinds of attractors. For $p(x) = x^3 - 3x^2 - 144x$ and $I = [-5, 7]$ one can verify that I contains a zero of p as well as the x-coordinate of the only reflection point of p yet p' is never 0 in I. Also $N_p(N_p(x)) = x$ has only a finite number of solutions in I and, moreover, $N_p(-5) > 7$ and $N_p(7) < -5$. Thus by [G] there are points a and b with $-5 < a < b < 7$ such that $N_p(a) = b$ and $N_p(b) = a$, yielding a two point attractor. We seek to find out what further kinds of attractors Newton's

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method can yield. We then exhibit a C^∞ function f such that the ω-limit sets of N_f include every nonempty closed subset of Cantor's middle thirds set.

Before we look into the collections of attractors, or ω-limit sets, associated with Newton's method we must first look into the collections of ω-limit sets of certain classes of real functions. In particular, since derivatives need not be continuous, the iteration scheme of Newton's method need not be represented by a continuous function. Hence this paper deals with the iterative behavior of classes of functions broader than the class of continuous functions.

While studying the diversity of ω-limit sets we want to know when this diversity is limited. We see from Theorem 2.1 that continuous functions are limited with regard to the kinds of ω-limit sets they generate. We show that we can replace the hypothesis of continuity with a more general dense mapping property which we define in the second section. Any function with both the intermediate value property and with the dense mapping property can have only ω-limit sets possible for continuous functions. An example of such a function is N_f, where f is any C^1 function.

Each of the functions we consider has for its range a connected set of real numbers with the exception of functions representing Newton's method. For such cases if x is not a zero of a differentiable function f but is a zero of f', then applying Newton's method to x with respect to f yields the point at infinity. Hence typically, the range of a function representing Newton's method, N_f, is the circle which is the one-point compactification of the real line with the point at infinity. For this reason, the topological hypotheses of many of the theorems that follow must be general enough to include the possibility that either the line $\mathbb{R}$ or the circle $\mathbb{R} \cup \{\infty\}$ is the range of the function involved.

1. Preliminary Remarks

In this paper a function $f : \mathbb{R} \to \mathbb{R}$, is differentiable at x whenever f is continuous at x and

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \in [-\infty, +\infty].$$

For $|x_n| \to +\infty$, we say that $x_n \to \infty$, and we define $0 \cdot \infty = \infty \cdot 0 = \frac{0}{0} = 0$. Let

$$N_f(x) = \lim_{h \to 0} \left(x - \frac{hf(x)}{f(x+h) - f(x)} \right)$$

provided that this limit belongs to the space $\mathbb{R} \cup \{\infty\}$, which we consider as the unit circle S^1@. Note that for $f(x_0) = 0$, $N_f(x_0) = x_0$ and that for f differentiable, we have $N_f : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ with $N_f(x) = x - f(x)/f'(x)$.

Since derivatives need not be continuous, f being differentiable need not imply that N_f is continuous. Therefore, we need to look at classes of functions more general than the class of continuous functions.

Because our interest is limited to real or extended real valued functions of a real variable, the following definition of Darboux (there are others) will suit our purpose.

Definition 1.1 *Suppose X and Y are topological spaces. A function $f : X \to Y$ is Darboux ($f \in \mathcal{D}$) if f maps connected sets onto connected sets.*

Definition 1.2 *Suppose $I \subseteq \mathbb{R}$ is closed. A function $f : I \to \mathbb{R}$ is Baire 1 ($f \in \mathcal{B}_1$) if f is a pointwise limit of real continuous functions on I@.*

Theorem 1.3 *For connected $I \subseteq \mathbb{R}$, the following statements are equivalent:*

1. *The function $f : I \to \mathbb{R}$ is Baire 1,*
2. *If $E \subseteq \mathbb{R}$ is open, then $f^{-1}(E)$ is of Borel type F_σ,*
3. *If $\emptyset \neq K \subseteq I$ is closed, then there is an $x \in K$ with $f|K$ continuous at x.*

PROOF. See [N].

An analogous result holds if we replace I by any closed subset of $\mathbb{R}$.

We know that real functions that are derivatives are Darboux, Baire 1functions. Also, if a function f is everywhere differentiable with real nonzero derivative, then $N_f \in \mathcal{DB}_1$ (see Theorem 3.2 of [B]).

Suppose $f : X \to Y$ with $X \subseteq Y$. Typically in our examples X will be a connected subset of $\mathbb{R}$ and Y will be either $\mathbb{R}$ or $\mathbb{R} \cup \{\infty\}$. For $x_0 \in X$, we define $f^0(x_0) = x_0$, $f^1(x_0) = f(x_0)$, and, if for any positive integer n, $f^n(x_0) \in X$, then $f^{n+1}(x_0) = f(f^n(x_0))$.

Definition 1.4 *Suppose $f : X \to Y$ with $X \subseteq Y$ and suppose $f^n(x_0) \in X$ for all $n \geq 0$. Then the orbit of x_0 under f is the sequence $\{f^n(x_0)\}_{n=0}^{\infty}$.*

For a differentiable function f, there may be an x_0 and an n such that

$$f(N_f^n(x_0)) \neq 0 \text{ but } f'(N_f^n(x_0)) = 0.$$

Then $N_f^{n+1}(x_0) = \infty$ and there is no orbit of x_0 under N_f, that is to say, Newton's method ends after $n+1$ iterations.

The following definition is central to this work.

Definition 1.5 *Suppose Y is a topological space, $X \subseteq Y$, with $f : X \to Y$, and suppose further that for some $x_0 \in X$, the orbit of x_0 under f exists. The omega-limit set (ω-limit set) of f at x_0, which we denote by $\omega(x_0, f)$, is the limit set of the orbit of x_0 under f, that is to say,*

$$\omega(x_0, f) = \bigcap_{i=0}^{\infty} cl\,\{ f^n(x_0) : n \geq i \}.$$

For f as above, we define $\Lambda(f) = \{ \omega(x, f) : x \in X \}$.

2. The Dense Mapping Property

Certain properties of functions may impose restrictions on the sorts of ω-limit sets such functions can yield.

Theorem 2.1 (Agronsky, Bruckner, Ceder, and Pearson [ABCP]) *If the function $f : [0,1] \to [0,1]$ is continuous and $x_0 \in [0,1]$, then $\omega(x_0, f)$ is nonempty and is either nowhere dense and closed or is a finite union of nonsingleton closed intervals. Conversely, if $\emptyset \neq K \subseteq [0,1]$ is either nowhere dense and closed or is a finite union of nonsingleton closed intervals, then there is a continuous function $f : [0,1] \to [0,1]$ and an $x_0 \in [0,1]$ with $K = \omega(x_0, f)$.*

The converse statement is the more subtle one; see [BS] for a shorter proof.

If D is dense in an interval I then, for f continuous, $f(D)$ is dense in $f(I)$. This dense mapping property is what restricts the sorts of ω-limit sets continuous functions can have.

Definition 2.2 *Suppose X and Y are topological spaces. A function $f : X \to Y$ has the dense mapping property if for any $D \subseteq X$ with $cl\,D$ connected, $f(cl\,D) \subseteq cl\,f(D)$.*

Lemma 2.3 *Suppose X and Y are topological spaces, $f : X \to Y$ is Darboux and has the dense mapping property, and $cl\,D$ is connected in X. Then $f(cl\,D)$, $cl\,f(D)$, and $cl\,f(cl\,D)$ are each connected.*

PROOF. $f(\mathrm{cl}\,D) \subseteq \mathrm{cl}\,f(D) \subseteq \mathrm{cl}\,f(\mathrm{cl}\,D)$. □

Lemma 2.4 *For X a topological space, suppose $f : X \to X$ is Darboux and has the dense mapping property. Then for each nonnegative integer n, f^n is Darboux and has the dense mapping property.*

PROOF. Since f^0 is the identity function on X, the basis step is done. Suppose f^{n-1} is Darboux and has the dense mapping property. Clearly f^n is Darboux. Suppose $\mathrm{cl}\,D$ is connected in X. By Lemma 2.3 $\mathrm{cl}\,f^{n-1}(D)$ is connected. Thus we have $f^n(\mathrm{cl}\,D) = f(f^{n-1}(\mathrm{cl}\,D)) \subseteq f(\mathrm{cl}\,f^{n-1}(D)) \subseteq \mathrm{cl}\,f(f^{n-1}(D)) = \mathrm{cl}\,f^n(D)$. □

Theorem 2.5 *Suppose Y is T_1, with no isolated points, locally connected and such that any closed connected set is either a singleton or the closure of its interior. If $X \subseteq Y$, if $f : X \to Y$ is Darboux and has the dense mapping property, and if the orbit of x_0 under f exists, then $\omega(x_0, f)$ is either nowhere dense or is a finite union of nonsingleton connected closed sets.*

PROOF. Suppose there is a nonempty open set $G \subseteq \omega(x_0, f)$. For $x \in G$, let K be the component of $\omega(x_0, f)$ containing x. Let $E = \operatorname{int} K$, which is nonempty since Y is locally connected. Since Y is T_1 with no isolated points, any nonempty open set contains an infinite number of points. Moreover, K contains an infinite number of points in the orbit of x_0 under f.

Suppose there is a point p which is isolated in $\omega(x_0, f)$. For each nonnegative integer n, let $x_n = f^n(x_0)$. There is a nonnegative integer j with $x_j \in E$. Also, there is an open neighborhood $A \ni p$ with $A \cap \omega(x_0, f) = \{p\}$ as well as a $k > 0$ with $x_{j+k} \in A$. Thus $f^k(K) \cap A \ni x_{j+k}$. Since K is the closure of its interior E, $K = \bigcap_{i=1}^{\infty} \operatorname{cl}(E \cap \{x_n : n \geq i\})$. Using Lemma 2.4, we have

$$\begin{aligned} f^k(K) &= f^k\left(\bigcap_{i=0}^{\infty} \operatorname{cl}(E \cap \{x_n : n \geq i\})\right) \subseteq \bigcap_{i=0}^{\infty} f^k(\operatorname{cl}(E \cap \{x_n : n \geq i\})) \\ &\subseteq \bigcap_{i=0}^{\infty} \operatorname{cl} f^k(E \cap \{x_n : n \geq i\}) \subseteq \bigcap_{i=0}^{\infty} \operatorname{cl} f^k(\{x_n : n \geq i\}) \\ &= \bigcap_{i=0}^{\infty} \operatorname{cl}\{x_n : n \geq i+k\} = \omega(x_0, f). \end{aligned}$$

Thus $x_{j+k} \in \omega(x_0, f) \cap A$ and so $x_{j+k} = p$. Therefore the orbit is cyclic and $\omega(x_0, f)$ is a finite union of points and hence nowhere dense. This contradiction shows that $\omega(x_0, f)$ contains no isolated points.

Let l be the least positive integer such that for some $x_m \in K$, $x_{m+l} \in K$. Pick j with x_j, $x_{j+l} \in K$. As in the previous paragraph, $f^l(K) \subseteq \omega(x_0, f)$. Since by Lemma 2.4 $\operatorname{cl} f^l(K)$ is connected, and K is a component of $\omega(x_0, f)$, and moreover $x_{j+l} \in \operatorname{cl} f^l(K) \cap K$, we have $\operatorname{cl} f^l(K) \subseteq K$. Thus for $n \geq j$, $x_n \in K$ iff l divides $n-j$. If $K \setminus \operatorname{cl} f^l(K) \neq \emptyset$, then, since K is the closure of its interior E, open $E \setminus \operatorname{cl} f^l(K) \neq \emptyset$, contradicting the fact that for $n \geq j+l$, $x_n \in K$ only if $x_n \in \operatorname{cl} f^l(K)$. Thus $K = \operatorname{cl} f^l(K)$, and $\omega(x_0, f) = \bigcup_{n=0}^{l} \operatorname{cl} f^n(K)$, a finite union of connected closed sets. By the previous paragraph, we see that none of these connected closed sets can be singletons. □

Let us state a clear implication of the previous paragraph to which we will later refer.

Corollary 2.6 *Suppose $f : X \to Y$ satisfies the hypotheses of Theorem 2.5 and K is a component of $\omega(x_0, f)$ with nonempty interior. Then for any nonnegative integer n, $cl\, f^n(K)$ is also a component of $\omega(x_0, f)$.*

Next we shall identify some classes of functions that satisfy the hypotheses of Theorem 2.5.

Lemma 2.7 *Let U denote the interior of the set of points where $f : X \to Y$ is continuous. If for any connected set $S \subseteq X$, $f(S) \subseteq cl\, f(U \cap S)$, then f has the dense mapping property*

PROOF. Let $D \subseteq X$ with $\mathrm{cl}\, D$ connected and let f satisfy the hypothesis of Lemma 2.7. Then $f(\mathrm{cl}\, D) \subseteq \mathrm{cl}\, f(U \cap \mathrm{cl}\, D)$. Since U is open and f is continuous on $U \cap \mathrm{cl}\, D$, $f(U \cap \mathrm{cl}\, D) \subseteq \mathrm{cl}\, f(U \cap D)$ and hence $\mathrm{cl}\, f(U \cap \mathrm{cl}\, D) \subseteq \mathrm{cl}\, f(U \cap D)$. Linking together these relations yields $f(\mathrm{cl}\, D) \subseteq \mathrm{cl}\, f(U \cap \mathrm{cl}\, D) \subseteq \mathrm{cl}\, f(U \cap D) \subseteq \mathrm{cl}\, f(D)$. □

We don't know if there is a Darboux, Baire 1function $f : [0,1] \to \mathbb{R}$ such that $\Lambda(f)$ contains, up to homeomorphism, every nonempty compact set. As we shall see and make precise, if such a function exists it cannot be a nice Darboux, Baire 1function.

Definition 2.8 *Suppose $I \subseteq \mathbb{R}$ is connected. A function $f : I \to \mathbb{R}$ is Baire*1 ($f \in \mathcal{B}_1^*$) if whenever $\emptyset \neq K \subseteq I$ is closed, there is an $x \in K$ with $f|K$ continuous on a K-neighborhood of x.*

If in addition $f \in \mathcal{D}$, then $f \in \mathcal{DB}_1^*$, a proper subset of $\mathcal{DB}_1$ that quite adequately corresponds to the nice, sketchable Darboux, Baire 1functions.

Theorem 2.9 (O'Malley) *Let $f : [0,1] \to \mathbb{R}$ be Darboux, Baire* 1@. For U denoting the interior of the set of points where f is continuous, $f([0,1]) \setminus f(U)$ is nowhere dense.*

PROOF. See Theorem 2 of [O].

Theorem 2.10 *If $f : [0,1] \to [0,1]$ is Darboux, Baire* 1and $x_0 \in [0,1]$, then $\omega(x_0, f)$is nonempty and is either nowhere dense and closed or is a finite union of nonsingleton closed intervals. Conversely, if $\emptyset \neq K \subseteq [0,1]$ is either nowhere dense and closed or is a finite union of nonsingleton closed intervals, then there is a Darboux, Baire* 1function $f : [0,1] \to [0,1]$ and an $x_0 \in [0,1]$ with $K = \omega(x_0, f)$.*

PROOF. Clearly Theorem 2.9 generalizes to Darboux, Baire* 1functions having a subinterval I of $[0,1]$ as the domain. If $f : [0,1] \to [0,1] \in \mathcal{DB}_1^*$ then $f|I \in \mathcal{DB}_1^*$ so f satisfies the hypothesis of Lemma 2.7@. This lemma, together with Theorem 2.5, yields the first implication. The second, converse, implication follows from Theorem 2.1 since continuous functions are Darboux, Baire* 1@. □

It is natural to ask if the above characterization on ω-limit sets applies to Newton's method. The remainder of this section gives a partial result.

The following theorem is an attempt to scrape together various conditions on f allowing N_f to be Darboux.

Theorem 2.11 *Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable on an open set D dense in $\mathbb{R}$ such that for any endpoint p of any component A of D*

1. *either $\limsup_{A \ni x \to p} |f'(x)| = \infty$ or $f(x) \to 0$ as $A \ni x \to p$, and*
2. *either there is an infinite one-sided derivative at p on A or $f(p) = 0$,*
3. *and finally for any limit point p of $\mathbb{R} \setminus D$, $f(p) = 0$.*

Then $N_f : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ is Darboux.

Before giving the proof of Theorem 2.11 we need a lemma which states that with some conditions, a line which is secant to the graph of a differentiable function can be rotated about its zero so that it becomes tangent to this graph.

Lemma 2.12 *Suppose, for $a < b$, $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b) and moreover that either $f < 0$ on (a, b) or $f > 0$ on (a, b). Further suppose that either $f(a) \neq 0$ or $f(b) \neq 0$ and, for $l : \mathbb{R} \to \mathbb{R}$ being the linear function with $l(a) = f(a)$ and $l(b) = f(b)$, that either $f < l$ on (a, b) or $f > l$ on (a, b). Let z be the zero of l ($z = \infty$ for l constant). Then if $z \in \mathbb{R} \cup \{\infty\} \setminus [a, b]$ there is a $c \in (a, b)$ with $N_f(c) = z$. However, if $z = a$ ($z = b$) then f has either a right(left)-sided derivative at z or there is a $c \in (a, b)$ with $N_f(c) = z$.*

PROOF. If l is constant, then the lemma follows from Rolle's theorem. So now suppose $z \in \mathbb{R} \setminus [a, b]$, and without loss of generality let $z < a$. Similarly we will let both $f > l$ and $f > 0$ on (a, b). Now f is a compact arc so there is an $M > 0$ with $f \subseteq \{ (x, y) : (x - z)^2 + y^2 < M^2 \}$. For $0 \leq \theta \leq \frac{\pi}{2}$, let $l_\theta = \{ (z + r\cos\theta, r\sin\theta) : 0 \leq r \leq M \}$. Let $d(\theta)$ be the distance between f and l_θ. Note that d is continuous, that $d(\frac{\pi}{2}) > 0$ and, because there is a $\varphi \in (0, \frac{\pi}{2})$ with $l_\varphi(b) = l(b) = f(b)$, that $d(\varphi) = 0$. Hence there is a $t \in (0, \frac{\pi}{2})$ with $d(t) = 0$ and $d(\theta) > 0$ for all $\theta > t$. By our choice of t, $l_t(x) \geq f(x)$ for all x in the domain of l_t, thus there is a $c \in (a, b)$ with $l_t(c) = f(c)$. Therefore $f'(c) = l'(c)$ and $N_f(c) = z$.

The last case to consider is for $z \in \{a, b\}$, so we will let $z = a$. For $x \in (a, b)$, continuous $(f(x) - f(a))/(x - a) = f(x)/(x - a)$ is either a monotone function of x or there are points a_1 and b_1 with $a < a_1 < b_1 < b$ such that

$f(a_1)/(a_1-a) = f(b_1)/(b_1-a) \neq f(x)/(x-a)$ for any $x \in (a_1, b_1)$. For the latter case, the first paragraph of this proof applies showing that there is a $c \in (a_1, b_1)$ with $N_f(c) = z = a$. For the former case, as $a < x \to a$, $f(x)/(x-a)$ converges to a (possibly infinite) right-sided derivative of f at a@. □PROOF. PROOF OF THEOREM 2.11 Let (a, b) be any component of $D \setminus f^{-1}(\{0\})$. Note that $f < 0$ on (a, b) or $f > 0$ on (a, b) and that f is differentiable on (a, b). Consider E with $(a, b) \subseteq E \subseteq [a, b]$: a $(b) \in E$ iff f has a right(left)-sided derivative at a (b). Suppose $G \subseteq E$ is connected.

Let S be the square $\{(x, y) : x, y \in G\}$. Consider $F : S \to \mathbb{R} \cup \{\infty\}$, where

$$F(x,y) = \begin{cases} N_f(x) & \text{if } f(x) = 0 \text{ or } x = y \\ x - f(x)\frac{y-x}{f(y)-f(x)} & \text{if } f(x) \neq 0 \text{ and } x \neq y. \end{cases}$$

Claim: $F(S)$ is pathwise connected. For any (x_0, y_0), $(x_1, y_1) \in S$ with $x_0 \leq x_1$, we will construct a continuous map $\{(t, (x_t, y_t)) : 0 \leq t \leq 1\}$ that maps $[0, 1]$ into S so that F is continuous on its range $\{(x_t, y_t) : 0 \leq t \leq 1\}$. Note that F is continuous with respect to y, and, except when $f(x) = 0 = f(y)$ and $x \neq y$, F is continuous with respect to x.

For $0 \leq t \leq 13$, let $x_t = x_0$. Furthermore, if $x_0 = a$ and $y_0 = b$, pick a $c \in E$ with $f(c) \neq 0$, then let $y_t = (1-3t)y_0 + 3tc$. Else let $y_t = y_0$.

For $\frac{1}{3} \leq t \leq \frac{2}{3}$, let $x_t = (2-3t)x_{\frac{1}{3}} + (-1+3t)x_1$ and $y_t = y_{\frac{1}{3}}$.

For $\frac{2}{3} \leq t \leq 1$, let $x_t = x_1$ and $y_t = (3-3t)y_{\frac{2}{3}} + (-2+3t)y_1$.

Clearly $N_f(G) \subseteq F(S)$. Lemma 2.12 shows that $F(S) \subseteq N_f(G)$ so $N_f(G) = F(S)$ is connected, and hence $N_f(E)$ is also connected.

To show that $N_f([a, b])$ is connected, it suffices to show that $N_f(a) \in \operatorname{cl} N_f(E)$ for $a \notin E$. (The case for $N_f(b)$ is analogous.) For $f(a) = 0$ and $f(x) \to 0$ as $E \ni x \to a$, Lemma 2.12 shows that either f has a right-sided derivative at a (in which case $a \in E$) or there is a $c \in E$ with $N_f(c) = a = N_f(a) \in N_f(E)$. For $f(a) = 0$ and $f(x) \not\to 0$ as $E \ni x \to a$, $N_f(a) = a$ and by hypothesis $\limsup_{E \ni x \to a} |f'(x)| = \infty$, so it's clear, even for f unbounded about a, that for any $\varepsilon > 0$, one can find an $x \in E$ with $N_f(x) - a < \varepsilon$, so $N_f(a) = a \in \operatorname{cl} N_f(E)$. For $f(a) \neq 0$, f has by hypothesis a right-sided (infinite) derivative at a so $N_f(a) = a \in E$.

To complete the proof, note that $N_f(a) = a$ and $N_f(b) = b$. Hence, using the final hypothesis, $N_f|(\mathbb{R} \setminus D) \cup f^{-1}(\{0\})$ is continuous since it is the identity function. Let I be any nonempty bounded interval and suppose $N_f(I) \subseteq U \cup V$ where U and V are disjoint open sets. Pick $x \in I$ with, say, $N_f(x) \in U$. Let $s = \sup\{t : t \in I, t \geq x, N_f(t) \in U\}$. From the above comments and from what we've already shown, $N_f(s) \in U$. Also, for $\alpha = \inf\{t : t \in I, t \leq x, N_f(t) \in U\}$, $N_f(\alpha) \in U$. Now suppose $V \neq \emptyset$, that is to say there is a $y \in I$ with $N_f(y) \in V$. We can assume that $y > x$. For

$\beta = \inf\{ s < t \leq y : N_f(t) \in V \}$ we have $N_f(\beta) \in V$. Thus $s < \beta$ and for $s < t < \beta$, $N_f(t) \notin U \cup V$, a contradiction. Therefore $V = \emptyset$ and $N_f(I)$ is connected. Hence N_f is Darboux. □

We now have the machinery to prove a concrete result concerning ω-limit sets stemming from Newton's method.

Theorem 2.13 *Suppose $f : \mathbb{R} \to \mathbb{R}$ is C^1. Then for any x_0 such that the orbit of x_0 under N_f exists, $\omega(x_0, N_f)$ is either nowhere dense or is a finite union of nonsingleton connected closed sets.*

Proof. $N_f(x)$ is continuous whenever $f'(x) \neq 0$ and for $x \in \operatorname{int} f'^{-1}(\{0\})$. For $f'(x) = 0$, $N_f(x) \in \{x, \infty\}$. Thus for

$$U = \operatorname{int}\{ x \in \mathbb{R} : N_f \text{ is continuous at } x \},$$

$N_f(\mathbb{R} \setminus U)$ is nowhere dense. By Theorem 2.11, for any connected set S, $N_f(S)$ is connected and so $N_f(S) \subseteq \operatorname{cl} N_f(U \cap S)$. The conclusion now follows from Lemma 2.7 and Theorem 2.5. □

3. Collections of Nowhere Dense ω-limit Sets

The following lemma is the key to some nice examples of functions with comprehensive collections of nowhere dense ω-limit sets.

Theorem 3.1 *Suppose Y is a locally compact metric space, $X \subseteq Y$, and X contains the range of $\{p_n\}_{n=0}^{\infty}$. If $f : X \to Y$ is such that for each n, every neighborhood of p_n contains a neighborhood B_n with f continuous on B_n and with $p_{n+1} \in \operatorname{int} f(B_n)$, then $\bigcap_{n=0}^{\infty} \operatorname{cl}\{ p_k : k \geq n \}$ is an ω-limit set of f.*

Proof. Let d be the metric of X, and let $A_n = \{ x \in X : d(x, p_n) \leq \frac{1}{n+1} \}$ for each nonnegative integer n. Let $B_{1,0} \subseteq A_0$ be a compact neighborhood with f continuous on $B_{1,0}$ and with $p_1 \in \operatorname{int} f(B_{1,0})$. Recursively, for each $n \geq 1$, let $B_{1,n} \subseteq A_n \cap f(B_{1,n-1})$ be a compact neighborhood with f continuous on $B_{1,n}$ and with $p_{n+1} \in \operatorname{int} f(B_{1,n})$. Further, for each $m \geq 2$, let $B_{m,n} \subseteq B_{m-1,n} \cap f^{-1}(B_{m-1,n+1})$ be a compact neighborhood for each $n \geq 0$. For any positive integers i and j with $i < j$, $B_{i,0} \supseteq B_{j,0}$ and so there is an $x_0 \in \bigcap_{m=1}^{\infty} B_{m,0}$. Notice that for each n, $f^n(x_0) \subseteq A_n$ and so x is a limit point of the sequence $\{p_n\}_{n=0}^{\infty}$ if and only if x is a limit point of the orbit of x_0 under f. Thus $\bigcap_{n=0}^{\infty} \operatorname{cl}\{ p_k : k \leq n \} = \omega(x_0, f)$. □

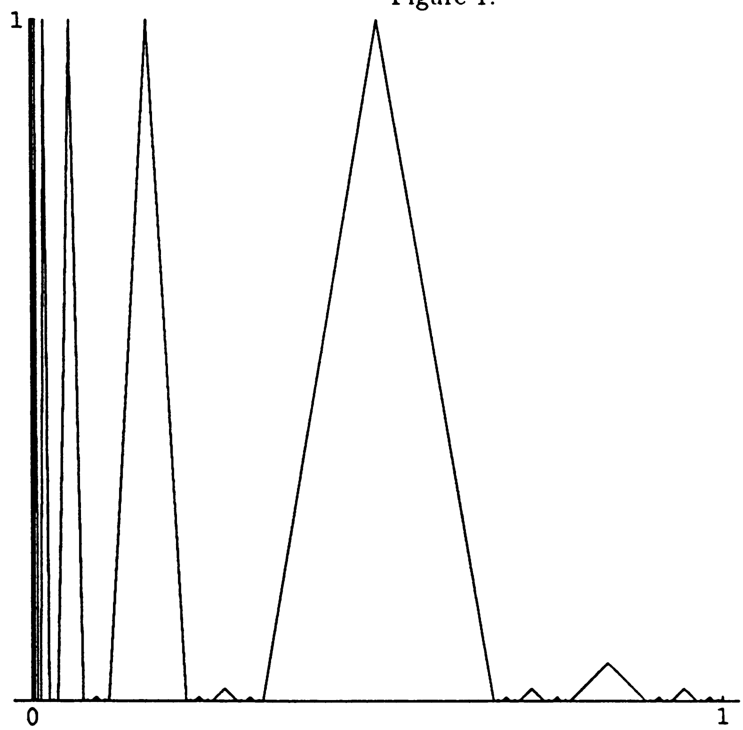
Example 1 Let $C =$ Cantor's middle thirds set, and for $x \in [0,1]$ let $d(x, C)$ be the distance between x and C. Define a function $g : [0,1] \to [0,1]$ such that for $x \in [0,1] \setminus \bigcup_{n=1}^{\infty} [3^{-n}, 2 \cdot 3^{-n}]$, $g(x) = d(x, C)$ and for each positive integer n, $g|[3^{-n}, 2 \cdot 3^{-n}]$ is a spike of height 1, as we see in figure 1.

Note that $g \in \mathcal{DB}_1^*$; in fact g is continuous everywhere except at the point 0, where g takes on every value in its range on every neighborhood of 0.

We can use Theorem 3.1 to show that any compact set F with $0 \in F \subseteq C$ is an ω-limit set of g. Consider a sequence $\{p_n\}_{n=0}^{\infty}$ with range contained in C, with limit set F, and such that for each k, $p_{2k} = 0$. Since every neighborhood of 0 contains a continuous spike with range $[0,1]$ hence containing the range of $\{p_n\}_{n=0}^{\infty}$, and since any neighborhood of any point in C contains a continuous spike with range containing 0, $\bigcap_{n=0}^{\infty} \mathrm{cl}\,\{p_k : k \leq n\} \subseteq \Lambda(g)$.

Now suppose K is any nonempty nowhere dense compact real set and that $[a,b]$ is the least interval containing K. Then there is a homeomorphism $h : [0,1] \to [a,b]$ with $h(F) = K$. Thus $\Lambda(g)$ contains, up to homeomorphism (on $[0,1]$), every nonempty nowhere dense compact real set.

Figure 1:

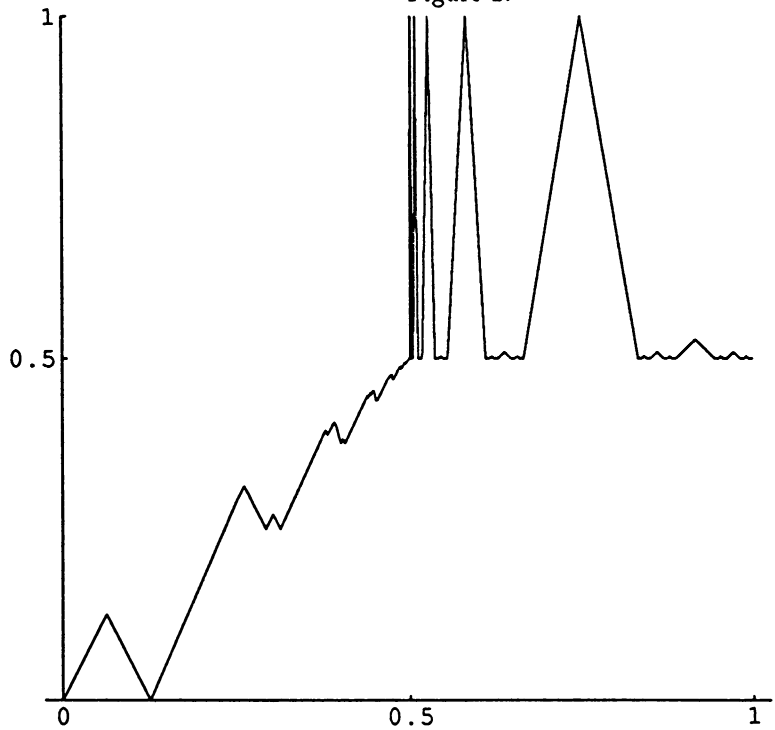


Example 2 From Theorem 2.1 we see that for each n, there is a continuous function $g_n : \left[\frac{1}{2} - \frac{1}{2^n}, \frac{1}{2} - \frac{1}{2^n} + \frac{1}{2^{n+2}}\right] \to \left[\frac{1}{2} - \frac{1}{2^n}, \frac{1}{2} - \frac{1}{2^n} + \frac{1}{2^{n+2}}\right]$ with an ω-limit set consisting of n nonsingleton closed components. If g is the function

of Example 1 then $g_0 = \frac{1}{2}g(2x-1) + \frac{1}{2}$ is topologically conjugate to g but with domain and range $\left[\frac{1}{2}, 1\right]$.

Interpolating these functions yields the function $f : [0,1] \to [0,1]$ which we see in Figure 2. Note that $f \in \mathcal{DB}_1^*$ and is continuous everywhere except at the point $\frac{1}{2}$. If K is an ω-limit set for some continuous function with domain and range being the unit interval, then $\Lambda(f)$ contains a set homeomorphic to K.

Figure 2:



We haven't yet found a function continuous everywhere on the unit interval which otherwise has the properties of the function f of Example 2 with respect to ω-limit sets.

The following proposition introduces the function of Example 3 at the end of this section.

Proposition 3.2 *Let C = Cantor's middle thirds set. Define a function $f : \mathbb{R} \to \mathbb{R}$ such that for (a,b) any component of $[0,1] \setminus C$ and for $x \in (a,b)$,*

$$f(x) = \exp\left(-\frac{1}{b-a}\right)\exp\left(\frac{b-a}{a-x}\right)\exp\left(\frac{b-a}{x-b}\right).$$

Let $f(x) = 0$ on $(-\infty, 0] \cup C \cup [1,\infty)$. Then $f \in C^\infty$ and for each nonnegative integer n and for each $x \in C$, $f^{(n)}(x) = 0$.

PROOF. Let $\{(a_k, b_k)\}_{k=1}^{\infty}$ be an ordering of the components of $[0,1] \setminus C$ such that $b_j - a_j \le b_i - a_i$ if $i < j$. For each k, let, for $x \in (a_k, b_k)$, $x = (1-t)a_k + tb_k$ whenever $0 < t < 1$. Then, for $x \in (a_k, b_k)$,

$$f(x) = \exp\left(-\frac{1}{b_k - a_k}\right)\exp\left(-\frac{1}{t(1-t)}\right), \qquad \frac{dt}{dx} = \frac{1}{b_k - a_k},$$

and for each positive integer n there is a rational function $R_n(t)$ defined on $(0,1)$ with n^{th} derivative of $f(x)$ being

$$f^{(n)}(x) = \left(\frac{1}{b_k - a_k}\right)^n \exp\left(\frac{1}{a_k - b_k}\right) R_n(t)\exp\left(\frac{1}{t(t-1)}\right). \tag{1}$$

For $x \in C$, the basis step is to note that $f^{(0)}(x) = 0$. Suppose, inductively, that $f^{(n)}(x) = 0$ for $x \in C$. For $h = x - a_k = t(b_k - a_k)$, the right-sided $(n+1)^{\text{th}}$ derivative of f at a_k equals

$$\begin{aligned}
&\lim_{h \to 0+} \frac{f^{(n)}(a_k + h) - f^{(n)}(a_k)}{h} &=& \\
\lim_{t \to 0+} \frac{1}{t}\left(\frac{1}{(b_k - a_k)}\right)^{n+1} &\exp\left(\frac{1}{a_k - b_k}\right) R_n(t)\exp\left(-\frac{1}{t}\right)\exp\left(\frac{1}{t-1}\right) &=& \quad (2) \\
\left(\frac{1}{b_k - a_k}\right)^{n+1} &\exp\left(\frac{1}{a_k - b_k}\right)\exp(-1)\lim_{t \to 0+}\frac{R_n(t)}{t}\exp\left(-\frac{1}{t}\right) &=& \quad 0.
\end{aligned}$$

Similarly, the left-sided $(n+1)^{\text{th}}$ derivative of f at b_k is 0. Consider $\{f_i\}_{i=0}^{\infty}$ where

$$f_i(x) = \begin{cases} f(x) & \text{if } x \in \bigcup_{j=1}^{i}(a_j, b_j) \\ 0 & \text{elsewhere.} \end{cases}$$

From (2) we see that $f_i^{(n+1)}$ exists on $\mathbb{R}$ for each i, and moreover, that for each k, $f^{(n+1)}|(a_k, b_k)$ obtains its maximum absolute value for some $t_{n+1} \in$

(0, 1) which does not depend on k. From (1),

$$\begin{aligned} f^{(n+1)}(t_{n+1}) &= \\ \left(\frac{1}{b_k - a_k}\right)^{n+1} \exp\left(\frac{1}{a_k - b_k}\right) R_{n+1}(t_{n+1}) \exp\left(\frac{1}{t_{n+1}(t_{n+1}-1)}\right) &\to 0 \quad (3) \\ \text{as } b_k - a_k &\to 0 \quad (4) \end{aligned}$$

Thus if we define

$$g(x) = \begin{cases} f_i^{(n+1)}(x) & \text{if } x \in (a_i, b_i) \text{ for each } i \\ 0 & \text{if } x \in (-\infty, 0) \cup C \cup (0, +\infty) \end{cases}$$

then by (3), $\left\| f_i^{(n+1)} - g \right\|_\infty \to 0$ as $i \to \infty$. Hence by Theorem 7.17 of [R], $g = \lim_{i\to\infty} f_i^{(n)\prime} = \left(\lim_{i\to\infty} f_i^{(n)}\right)' = f^{(n)\prime} = f^{(n+1)}$. Thus $f^{(n+1)} = g$ exists and $f^{(n+1)}(x) = 0$ for all $x \in C$, thereby completing the inductive step and the proof. □

Example 3 Let f be the C^∞ function of Proposition 3.2. If point p belongs to Cantor's middle thirds set C, then every neighborhood of p contains a component (a, b) of $[0, 1] \setminus C$. Looking at Newton's method geometrically with Theorem 2.11 in mind, we see that $N_f((a, b))$ contains all of $\mathbb{R} \cup \{\infty\}$ except for a closed interval contained in (a, b). Thus any given neighborhood of any given point of C contains a component (a, b) of $[0, 1] \setminus C$ with N_f continuous on (a, b) and with $C \subseteq \operatorname{int} N_f((a, b))$.

We can now apply Theorem 3.1 as we did for Example 1 but now we no longer require that for the sequence $\{a_n\}_{n=0}^\infty$, $a_n = 0$ whenever n is even. Let K be any nonempty nowhere dense compact real set and let $I = [\inf K, \sup K]$. Then there is an $F \subseteq C$ and a homeomorphism $h : I \to [0, 1]$ such that $h(K) = F$ and $F \in \Lambda(N_f)$ because $\Lambda(N_f)$ contains every nonempty closed subset of C.

Moreover, suppose that F_1 and F_2 are nonempty closed subsets of C and suppose $\varepsilon > 0$. There are points x_1 and y_1 in the unit interval with orbits $\{N_f^n(x_1)\}_{n=0}^\infty$ and $\{N_f^n(y_1)\}_{n=0}^\infty$ such that $F_1 = \omega(x_1, N_f)$ and $F_2 = \omega(y_1, N_f)$. Let (a_i, b_i) be a component of $[0, 1] \setminus C$ with $b_i - a_i \le \varepsilon$. Reflecting on the second sentence of this example, we see that there is an $x_0 \in (a_i, b_i)$ and a $y_0 \in (a_i, b_i)$ with $N_f(x_0) = x_1$ and $N_f(y_0) = y_1$. Hence $F_1 = \omega(x_0, N_f)$ and $F_2 = \omega(y_0, N_f)$.

So we see that Newton's method can have chaotic behavior with sensitive dependence on the initial point of iteration even when applied to a C^∞ function.

4. Countable Collections of ω-limit Sets

Analyzing the collections of ω-limit sets that are possible for some function of a given class becomes more difficult when we broaden our consideration to include nonnowhere dense ω-limit sets.

One limitation is worth mentioning. If $f : X \to Y$ satisfies the hypotheses of Corollary 2.6, and if A, $B \in \Lambda(f)$ such that there is an x with $x \in \operatorname{int} A \cap \operatorname{int} B$, then $A = B$. However, a function can be Darboux, Baire 1and not have the dense mapping property. Such a function may have distinct ω-limit sets that share interior points.

There are countable collections of nowhere dense sets that cannot belong to the set of ω-limit sets for any Darboux, Baire 1function. For example, suppose we have $f : [0,1] \to [0,1]$ with $\{ \{0,p\}, \{q,1\} : p, q \in \mathbb{Q} \} \subseteq \Lambda(f)$. Since $\operatorname{osc}(f,x) = 1$ for each $x \in \mathbb{Q}$, f is nowhere continuous, hence not Baire 1.

Given a countable collection of real nonempty closed sets $\{ K_m : 1 \leq m < \omega_0 \}$, can we create, for each m, a sequence $\{x_{m,n}\}_{n=0}^{\infty}$ with limit set $= K_m$ such that there is a function $f \in \mathcal{DB}_1$ with $f(x_{m,n-1}) = x_{m,n}$ for each m, $n \geq 1$? If so, then $\omega(x_{m,0}, f) = K_m$ for each m. The key to answering such questions is the following lemma.

Lemma 4.1 (Bruckner) *Suppose I and J are bounded closed intervals with nonempty $D \subseteq I$. Consider $h : D \to J$. For each positive integer n, let*

$$D_n = \left\{ x \in D : osc(h,x) \geq \frac{1}{n} \right\}. \tag{5}$$

If, for each n, $cl D_n$ is countable, then there is a Darboux, Baire 1function $\varphi : I \to J$ with φ an extension of h. Furthermore, if the Lebesgue measure of D is 0, then h can be extended to a bounded approximately continuous function f on I (which is also a derivative).

PROOF. For $h : D \to J$ such that for each n, $\operatorname{cl} D_n$ is countable, define $g : \operatorname{cl} D \to J$ with g an extension of h given by

$$g(x) = \begin{cases} h(x) & \text{if } x \in D \\ \limsup\limits_{D \ni t \to x} h(t) & \text{if } x \in \operatorname{cl} D \setminus D. \end{cases}$$

Suppose $g \notin \mathcal{B}_1$. Then there is a perfect set P with $\emptyset \neq P \subseteq \operatorname{cl} D$ and such that $g|P$ is nowhere continuous, that is to say $\operatorname{osc}(g|P, x) > 0$ for every $x \in P$. By the Baire Category Theorem, there is a positive integer n and an open interval E intersecting P with

$$\operatorname{osc}(g|P, x) \geq \frac{1}{n} \qquad \text{for every } x \in E \cap P. \tag{6}$$

Since D_{3n} is nowhere dense in P there is an open interval $U \subseteq E$ intersecting P with

$$D_{3n} \cap U \cap P = \emptyset. \tag{7}$$

Let $\alpha = \inf\{\, g(x) : x \in U \cap P \,\}$. Pick $x_0 \in U \cap P$ with

$$g(x_0) < \alpha + \frac{1}{3n}. \tag{8}$$

Then $\limsup_{D \ni t \to x_0} g(t) < \alpha + \frac{2}{3n}$.

Suppose $x_0 \in D$. By (7) and (8), there is a neighborhood L of x_0 with $h(x) < \alpha + \frac{2}{3n}$ whenever $x \in D \cap L$. But then $g(x) \leq \alpha + \frac{2}{3n}$ for all $x \in L \cap P$, contradicting (6) at x_0.

Now suppose $x_0 \notin D$. By (6), given any $\varepsilon > 0$, there is an $x \in P \cap (x_0 - \varepsilon, x_0 + \varepsilon)$ with $g(x) > \alpha + \frac{2}{3n}$. From the definition of g we see that there is a $t \in D \cap (x_0 - \varepsilon, x_0 + \varepsilon)$ with $h(t) > \alpha + \frac{2}{3n}$. But then $g(x_0) = \limsup_{D \ni t \to x_0} h(t) \geq \alpha + \frac{2}{3n}$, contradicting (8). Thus $g \in \mathcal{B}_1$.

We can extend g to a Baire 1 function $\hat{\varphi} : I \to J$ by applying a linear interpolation on each of the intervals of $[\min \operatorname{cl} D, \max \operatorname{cl} D] \setminus \operatorname{cl} D$ and by applying a constant extrapolation on the remaining at most 2 intervals of $I \setminus \operatorname{cl} D$.

By Proposition 1 of [BCK], we see that there is a Darboux, Baire 1function $\varphi : I \to J$ such that $\varphi(x) = \hat{\varphi}(x)$ whenever $x \in D$, a first category subset of I. If the Lebesgue measure of D is 0, then, by Theorem 3.2 of [PL], we can extend h to a bounded approximately continuous function f on I which is also a derivative (since $f(x) = \lim\limits_{h \to 0} \int_x^{x+h} f(t)\,dt$). □

Proposition 4.2 *Suppose $\{\, K_m : 1 \leq m \leq M \,\}$ is a finite collection of nonempty closed subsets of $[0,1]$. Then there is a bounded approximately continuous function $f : [0,1] \to [0,1]$ with $\{\, K_m : 1 \leq m \leq M \,\} \subseteq \Lambda(f)$.*

PROOF. The idea is to identify the endpoints of the unit interval and then have M soldiers march around this circle taking smaller and smaller steps such that there is no limit to the number of circuits taken and such that no point is stepped on more than once. The m^{th} soldier thus defines a sequence, a subsequence of which has K_m as a limit set.

For each m, there is a sequence $\{t_{m,k}\}_{k=0}^{\infty}$ such that, for $k \geq 1$,

$$\begin{aligned} \frac{1}{k+1} &\leq t_{m,k+1} - t_{m,k} \leq \frac{1}{k} &\qquad& \text{for } t_{m,k} + \tfrac{1}{k} \leq 1, \\ \frac{1}{k+1} &\leq (t_{m,k+1} + 1) - t_{m,k} \leq \frac{1}{k} && \text{for } t_{m,k} + \tfrac{1}{k} > 1 \end{aligned}$$

and such that $t_{a,b} = t_{c,d}$ only if $(a,b) = (c,d)$.

For each sequence $\{t_{m,k}\}_{k=0}^{\infty}$, the subsequence $\{x_{m,n}\}_{n=0}^{\infty} = \{t_{m,n(m,k)}\}_{k=0}^{\infty}$ has limit set K_m. So, for each m, let $n(m,0) = 0$, and for $k > 0$, recursively define $n(m,k)$ to be the least integer satisfying $n(m,k) > n(m,k-1)$ and also satisfying distance $d(t_{m,n(m,k)}, K_m) \le \frac{1}{n(m,k)}$.

Let $h(x_{m,n}) = x_{m,n+1}$ for $1 \le m \le M$ and $n \ge 0$. If h satisfies the hypotheses of Lemma 4.1, then there is an approximately continuous function $f : [0,1] \to [0,1]$ such that for each m, $\omega(x_{m,0}, f) = K_m$. Here $D = \{ x_{m,n} : 1 \le m \le M \text{ and } n \ge 0 \}$. Recalling (5) we see that limit points of D_k occur only at 1 and at the left endpoints of complementary intervals of $\mathbb{R} \setminus K_m$, $1 \le m \le M$, of length $\ge \frac{1}{k}$. Since there are only a finite number of such limit points, $\operatorname{cl} D_k$ is countable for each k. □

A corollary to Proposition 4.2 is that any nonempty closed subset of $[0,1]$ is an ω-limit set for some $f \in \mathcal{DB}_1$. The original proof, owing to Ceder, is in [BCP].

For proofs of the following two Propositions, which are similar to but more tedious than the proof of Proposition 4.2, see [K].

Proposition 4.3 *Suppose* $\{ K_m : 1 \le m < \omega_0 \}$ *is a countable collection of nonempty closed subintervals of* $[0,1]$. *Then there is a bounded approximately continuous function* $f : [0,1] \to [0,1]$ *with* $\{ K_m : 1 \le m < \omega_0 \} \subseteq \Lambda(f)$.

Proposition 4.4 *Given* $\varepsilon > 0$, *suppose, for any positive integer* m, $K_m \subseteq [0,1]$ *is closed and is a nonempty union of intervals of diameter* ε. *Then there is a bounded approximately continuous function* $f : [0,1] \to [0,1]$ *with* $\{ K_m : 1 \le m < \omega_0 \} \subseteq \Lambda(f)$.

Let $\{ S_n : 1 \le n < \omega_0 \}$ be any collection of sets containing at most one element with $S = \bigcup_{n=1}^{\infty} S_n \subseteq [0,1]$. We can replace f of either Proposition 4.2 or Proposition 4.4 by $\tilde{f}$ such that $\tilde{f}$ has the properties required of f as well as having $\{ S_n : 1 \le n < \omega_0 \} \subseteq \Lambda(\tilde{f})$. In either case, modify D such that $D \cap S = \emptyset$ and then let $\tilde{D} = D \cup S$. We can extend the function h of the proof of Proposition 4.2 (and, it so happens, of the proof of Proposition 4.4) to $\tilde{h} : \tilde{D} \to [0,1]$ which still satisfies Lemma 4.1 if we let $\tilde{h}(x) = x$ for all $x \in S$. Thus we can extend $\tilde{h}$ to $\tilde{f}$ as desired.

Suppose, for some M with $1 \le M \le \omega_0$, $\{K_m\}_{m=1}^{M}$ is a collection of nonempty closed subsets of $[0,1]$ such that there is an $f : [0,1] \to [0,1]$ with $f \in \mathcal{DB}_1$ and $\{K_m\}_{m=1}^{M} \subseteq \Lambda(f)$. Suppose further that $\emptyset \ne K \subseteq [0,1]$ is closed. Unknown is whether there is always an $\tilde{f} : [0,1] \to [0,1]$ with $\tilde{f} \in \mathcal{DB}_1$ and with $\{K_m\}_{m=1}^{M} \cup \{K\} \subseteq \Lambda(\tilde{f})$.

Not much is known concerning nonnowhere dense ω-limit sets arising from Newton's method. By Theorem 2.13, if $\omega(x_0, N_f)$ has both singleton components and components with nonempty interior, then $f \notin C^1$, if indeed such a function exists. But, using an argument simpler than that found in [W], we can at least get an interval.

Example 4 Let

$$f(x) = \begin{cases} \left(\frac{2x+1}{x-1}\right)^{\frac{1}{3}} & \text{if } x \in \mathbb{R} \setminus \{1\} \\ 0 & \text{if } x = 1. \end{cases}$$

Then $N_f(x) = 2x^2 - 1$ and for $x \in [-1,1]$, $N_f(x) = \cos(2 \arccos x)$. To prove the well known fact that $[-1,1] \in \Lambda(N_f)$ without using the Baire Category Theorem, note that the orbit of $x \in [-1,1]$ under N_f is $\{\cos(2^n \arccos x)\}_{n=0}^{\infty}$. Consider complex z with $\mathrm{Re}(z) = x$ and $z \in S^1 = \{ z \in \mathbb{C} : |z| = 1 \}$. Then $\omega(x, N_f) = [-1,1]$ if $\{ z^{2^n} : 0 \le n < \omega_0 \}$ is dense in S^1.

If $z \in S^1$ then there is an $\alpha \in [0,1)$ with $z = e^{2\pi i \alpha}$. Express α as a binary number: $\alpha = 0.a_1a_2a_3\ldots$. Then, for $\beta = 0.a_{1+n}a_{2+n}a_{3+n}\ldots$, $z^{2^n} = e^{2\pi i \beta}$.

There is a sequence $\{a_n\}_{n=1}^{\infty}$ with range $\{0,1\}$ such that for any finite sequence $\{b_n\}_{n=1}^{N}$ with range $\{0,1\}$, there is a k with $\{b_n\}_{n=1}^{N} = \{a_{n+k}\}_{n=1}^{N}$. Let $\alpha = 0.a_1a_2a_3\ldots$ and let $\varepsilon > 0$. For any $z \in S^1$ there is a $c \in [0,1)$ with $z = e^{2\pi i c}$. Moreover, there is a terminating binary number $\gamma = 0.c_1c_2c_3\ldots c_m$ such that $|c-\gamma| < \frac{\varepsilon}{4\pi} > 2^{-m}$. By our choice of α, there is a j with $\{c_n\}_{n=1}^{m} = \{a_{n+j}\}_{n=1}^{m}$. Hence for $z_0 = e^{2\pi i \alpha}$, and for

$$\beta = 0.c_1c_2c_3\ldots c_m a_{m+j+1}a_{m+j+2}a_{m+j+3}\ldots = 0.a_{1+j}a_{2+j}a_{3+j}\ldots,$$

$z_0^{2^j} = e^{2\pi i \beta}$ and $|\gamma - \beta| < \frac{\varepsilon}{4\pi}$. Because $|c - \beta| < \frac{\varepsilon}{2\pi}$,we have $|z - z_0^{2^j}| < \varepsilon$, showing that $\{ z_0^{2^n} : 0 \le n < \omega_0 \}$ is dense in S^1. Thus, for $x_0 = \mathrm{Re}(z_0)$, $\omega(x_0, N_f) = [-1,1]$. □

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