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Paradoxical decompositions using Lipschitz functions[1]

Let G_k denote the group of isometries of $\mathbf{R}^k$. The sets $A, B \subset \mathbf{R}^k$ are called equidecomposable, if there are partitions $A = A_1 \cup \ldots \cup A_n$, $B = B_1 \cup \ldots \cup B_n$ and isometries $f_1, \ldots, f_n \in G_k$ such that $f_i(A_i) = B_i$ $(i = 1, \ldots, n)$. We shall denote this by $A \sim B$. By a well-known theorem of S. Banach and A. Tarski [3], if A, B are bounded subsets of $\mathbf{R}^k$ $(k \geq 3)$ with non-empty interior, then $A \sim B$. The proof is based on the fact that for $k \geq 3$ the group G_k contains free subgroups [13].

In $\mathbf{R}^2$ such a paradox does not exist. Banach proved in 1923 that for $k = 1, 2$ the Lebesgue measure on $\mathbf{R}^k$ can be extended to the power set of $\mathbf{R}^k$ as an invariant and finitely additive measure [1]. This implies that if $A, B \subset \mathbf{R}^k$ $(k = 1, 2)$ are measurable and $A \sim B$, then $\lambda(A) = \lambda(B)$. As it was realized later by J. von Neumann, the existence of the Banach measures in $\mathbf{R}$ and $\mathbf{R}^2$ is the consequence of the fact that the groups G_1 and G_2 are solvable, and hence *amenable:* they support a finitely additive invariant measure (see [13], Chapter 10).

Still, paradoxical sets do exist in $\mathbf{R}^2$. S. Mazurkiewicz and W. Sierpiński showed in [8] that there is a non-empty set $A \subset \mathbf{R}^2$ which can be decomposed into two disjoint subsets congruent to A. (Proof: let c be a transcendental complex number with $|c| = 1$, and put $A = \{a_n c^n + \ldots a_0 : a_i \in \mathbf{N}\}$. Then $A = (c \cdot A) \cup (A + 1)$ is a partition of A into sets congruent to A.)

Sierpiński proved in 1946 that such a paradox does not exist in $\mathbf{R}$. Moreover, no set $A \subset \mathbf{R}$ can be partitioned into two subsets which are equidecomposable to A (see [12], p. 56). The underlying fact is that the group G_1 is not only amenable, but is also *supramenable:* for every $H \subset G_1$, $H \neq \emptyset$ there is a finitely additive invariant measure μ on G_1 such that $\mu(H) = 1$ (see [13], Chapter 12).

In spite of the fact that paradoxical sets do not exist in $\mathbf{R}$ if only isometries can be used, there are paradoxical decompositions in $\mathbf{R}$ which use Lipschitz functions (in particular, contractions).

A map $f : A \to \mathbf{R}$ $(A \subset \mathbf{R})$ is called piecewise contractive if there is a finite partition $A = A_1 \cup \ldots \cup A_n$ such that the restriction $f|A_i$ is a contraction for every $i = 1, \ldots, n$. Von Neumann proved in [9] that every interval can be mapped, using a piecewise contractive map, onto a longer interval. This easily

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implies that whenever A, B are bounded subsets of $\mathbf{R}$ with nonempty interior, then A can be mapped, using a piecewise contractive map, onto B (see [13], Theorem 7.12, p. 105). Indeed, let I and J be intervals such that $I \subset A$ and $B \subset J$. Let ϕ be an injective contraction mapping A into $\operatorname{int} B$. By von Neumann's theorem, there is a piecewise contractive bijection ψ from I onto J. Let ψ_0 denote the restriction of ψ to $\psi^{-1}(B)$, then ψ_0^{-1} is an injective map from B into A. By a theorem of Banach [2], there are partitions $A = A_1 \cup A_2$, $B = B_1 \cup B_2$ such that $\phi(A_1) = B_1$ and $\psi_0(A_2) = B_2$. Thus the map f defined by $f(x) = \phi(x)$ $(x \in A_1)$, $f(x) = \psi_0(x)$ $(x \in A_2)$ is a piecewise contractive bijection from A onto B.

In the next theorem we determine the range of the Lebesgue measure of B, supposing that the number of pieces in the partition of A is given. The Lebesgue outer measure will be denoted by λ. If I is an interval then we write $|I| = \lambda(I)$.

Theorem 1.

(i) *Let $A, B \subset \mathbf{R}$ be measurable and suppose that there is a map $f : A \to \mathbf{R}$ and a partition $A = A_1 \cup \ldots \cup A_n$ such that $B = f(A)$ and $f|A_i$ is a contraction for every $i = 1, \ldots, n$. Then $\lambda(B) < n \cdot \lambda(A)/2$.*

(ii) *Let $A \subset \mathbf{R}$ be measurable and let J be an interval with $|J| < n \cdot \lambda(A)/2$, where n is a positive integer. Then there is a map $f : A \to \mathbf{R}$ and a partition $A = A_1 \cup \ldots \cup A_n$ such that $f(A) = J$ and $f|A_i$ is a contraction for every $i = 1, \ldots, n$. If A is an interval, then f can be chosen to be a bijection between A and J.*

Let J be an interval with $1 < |J| < 3/2$. Then (ii) of Theorem 1 implies that there is a bijection from $[0, 1]$ onto J which consists of three contractions; that is, von Neumann's paradox can be realized using three pieces. On the other hand, such a paradoxical decomposition does not exist if only two pieces can be used, as (i) of Theorem 1 shows.

The first statement of the theorem is an immediate consequence of the following result. Let $\operatorname{Lip} g = \sup\{|(g(x) - g(y))/(x - y)| : x, y \in C,\ x \neq y\}$ denote the Lipschitz constant of the function $g : C \to \mathbf{R}$. Let $A = A_1 \cup \ldots \cup A_n$ be a partition of the set $A \subset \mathbf{R}$ and let $f : A \to \mathbf{R}$ be a map such that $f|A_i$ is a Lipschitz function with $\operatorname{Lip}(f|A_i) \leq M_i$ for every $i = 1, \ldots, n$. Then the inner Lebesgue measure of $f(A)$ is at most $M \cdot \lambda(A)$, where

$$M = \max\left(M_1, \ldots, M_n, \frac{1}{2}\sum_{i=1}^{n} M_i\right) \tag{1}$$

(see [6], Theorem 4).

If the restrictions $f|A_i$ are contractions, then we may apply this estimate with
$M_1 = \ldots = M_n = 1 - \varepsilon$, and obtain (i) of Theorem 1.

As the next theorem shows, the estimate given above is sharp. Then (ii) of Theorem 1 will follow again by putting $M_1 = \ldots = M_n = 1 - \varepsilon$.

Theorem 2. *Let $M_1, \ldots, M_n$ be positive numbers and let M be defined by* (1). *Then for every measurable set $A \subset \mathbf{R}$ and for every $0 < d < M \cdot \lambda(A)$ there is a partition $A = A_1 \cup \ldots \cup A_n$ and there is a function $f : A \to \mathbf{R}$ such that $f|A_i$ is a Lipschitz function with* Lip $(f|A_i) \leq M_i$ *for every $i = 1, \ldots, n$ and $f(A)$ is an interval of length d.*

If A is an interval, then f can be chosen to be a bijection.

The proof of Theorem 2 can be found in [7]. Here we only mention two features of the proof: the use of graph theory and of locally commutative actions.

Graph theory was first applied to problems of equidecomposability by D. König, when he proved the "cancellation law" for the semigroup of equidecomposability types. This semigroup, introduced by Tarski, can be defined as follows. We may assign to every $A \subset \mathbf{R}^k$ a type $[A]$ such that $[A] = [B]$ if and only if $A \sim B$. (We may define $[A] = \{B : A \sim B\}$.) If $[A] = a$ and $[b] = b$ and $A \cap B = \emptyset$, then we define $a + b = [A \cup B]$. This is a well-defined operation on the set of types of bounded subsets of $\mathbf{R}^k$ (and can be extended to the types of all subsets, see [13], Chapter 8) which makes the set of types a commutative semigroup S with identity element $0 = [\emptyset]$. In 1924 Tarski asked whether or not $2a = 2b$ implies $a = b$ for every $a, b \in S$. An affirmative answer was given by Kuratowski in [5]. The general cancellation law ($na = nb \Longrightarrow a = b$) was proved by D. König and S. Valkó in [4]. They realized that this is an easy consequence of the following theorem: every n-regular bipartite graph contains a perfect matching. (A bipartite graph is a subset Γ of $X \times Y$, where X and Y are disjoint sets. The elements of $X \cup Y$ are called points, the pairs $(x, y) \in \Gamma$ are called lines; the number of lines containing a point $u \in X \cup Y$ is called the degree of u. Γ is n-regular, if the degree of every point is n. $M \subset \Gamma$ is a perfect matching if there is a bijection f from X onto Y such that $M = \{(x, f(x)) : x \in X\}$.)

In the proof of Theorem 2 we use the following condition for the existence of a perfect matching: if Γ is connected, the degree of each point of Γ is finite and at least two, and if Γ contains at most one cycle, then Γ contains a perfect matching.

Let X be a non-empty set, and let G be a group of bijections of X onto itself. We say that G is *locally commutative* provided that whenever two elements of G have a common fixed point then they commute. The role of local commutativity in the theory of equidecomposability was discovered by R. M. Robinson in [10]. In this paper he finds the minimal number of pieces which are needed to duplicate a ball. Banach and Tarski in their paper [3] did not specify the number of pieces to obtain a paradoxical decomposition of the ball. In 1929 von Neumann remarked that 9 pieces suffice. Sierpiński used 8 pieces in [11]. Finally, Robinson showed in [10] that the minimal number is 5. His proof is based on the fact that the group of the rotations of a sphere is locally commutative: if two rotations have a common fixed point then they have the same axis and hence they commute. As for the applications of local commutativity in questions of equidecomposability, see [13], Chapter 4.

Von Neumann proved his paradox by observing that a system of fractional linear transformations $(a_i x + b_i)/(c_i x + d_i)$ generates a free group, if the coefficients a_i, b_i, c_i, d_i are algebraically independent over the field of rationals [9]. The proof of Theorem 2 uses the fact that this group is also locally commutative.

Using these results, the proof of Theorem 2 is the following. Suppose first that A is an interval, and let J be an interval such that $|J| = d < M \cdot |A|$. Then there are functions $f_i : A \to J$ such that Lip $f_i \leq M_i$ for every i and the sets $f_i(A)$ cover each point of J twice (let f_i consist of two linear functions of slopes M_i and $-M_i$). Using fractional linear transformations with algebraically independent coefficients, we can construct functions $g_i : A \to J$ $(i = 1, \ldots, n)$ approximating the functions f_i in such a way that Lip $g_i \leq M_i$ for every i, and the sets $g_i(A)$ cover each point of J twice. Then, applying the local commutativity of the group of the fractional linear transformations we can check that each connected component of the graph

$$\Gamma = \{(x, y) : x \in A,\ y \in J,\ y = f_i(x) \text{ for some } i = 1, \ldots, n\}$$

satisfies the condition given above for the existence of a perfect matching. These matchings together constitute a perfect matching M for Γ. Then we define $A_i = \{x \in A : (x, f_i(x)) \in M\}$ $(i = 1, \ldots, n)$ and $f(x) = f_i(x)$ $(x \in A_i,\ i = 1, \ldots, n)$; it is easy to see that f satisfies the requirements of Theorem 2.

If $A \subset \mathbf{R}$ is measurable, then we argue as follows. Let J be an interval of length $d < M \cdot \lambda(A)$. Let K be a compact subset of A such that $d < M \cdot \lambda(K)$, and let $I = [0, \lambda(K)]$. As we proved above, there is a partition $I = C_1 \cup \ldots \cup C_n$ and there is a function $g : I \to \mathbf{R}$ such that $g(I) = J$ and Lip $(g|C_i) \leq M_i$ for every $i = 1, \ldots, n$.

Let $h(x) = \lambda(K \cap (-\infty, x])$ $(x \in A)$, then Lip $h \leq 1$. Since K is compact, it is easy to see that $h(K) = I$. Thus $h(K) \subset h(A) \subset I$ implies that h maps A onto I. Therefore the sets $A_i = h^{-1}(C_i)$ $(i = 1, \ldots, n)$ and the function $f = g \circ h$ satisfy the requirements of Theorem 2.

We conclude by mentioning some problems concerning the higher dimensional analogues of the previous results. Let $A \subset \mathbf{R}^k$ be measurable, and let $f : A \to \mathbf{R}^k$ be a map such that Lip $(f|A_i) \leq M_i$ for every $i = 1, \ldots, n$. Then the inner Lebesgue measure of $f(A)$ is at most $M \cdot \lambda(A)$, where

$$M = \max\left(M_1^k, \dots, M_n^k, \frac{1}{2} \sum_{i=1}^{n} M_i^k\right) \tag{2}$$

(see [6], Theorem 4). This implies that (i) of Theorem 1 remains valid in every dimension. We do not know, however, whether or not (ii) of Theorem 1 or Theorem 2 remain valid in $\mathbf{R}^k$. It is easy to see that the value of M given by (2) is sharp; for every measurable $A \subset \mathbf{R}^k$ and $d < M \cdot \lambda(A)$ there is a function $f : A \to \mathbf{R}^k$ and a partition $A = A_1 \cup \dots \cup A_n$ such that $\text{Lip } f|A_i \le M_i$ for every $i = 1, \dots, n$ and $f(A)$ is a measurable set of measure d. The problem is that we cannot ensure that $f(A)$ is an interval. (Even if A is an interval, the proof only gives a set $f(A)$ which is a finite union of intervals.) Therefore the following problem remains open.

Problem 1. Let $I \subset \mathbf{R}^k$ be an interval and let $M_1, \dots, M_n$ be given positive numbers. What is the supremum of the measures of those intervals J for which there is a function $f : I \to \mathbf{R}^k$ and a partition $I = A_1 \cup \dots \cup A_n$ such that $\text{Lip } f|A_i \le M_i$ for every $i = 1, \dots, n$ and $f(I) = J$?

In particular, what is the supremum of the measures of those intervals J for which there is a piecewise contractive map of I onto J using n pieces?

Of course, we may ask the same question for every measurable $A \subset \mathbf{R}^k$ instead of an interval I. In this case, however, we do not know even the existence of a piecewise Lipschitz map of the set A onto an interval. (As we saw above, in $\mathbf{R}$ every measurable set of positive measure can be mapped, using a Lipschitz function, onto an interval.) Therefore we face the following question.

Problem 2. Let $A \subset \mathbf{R}^k$ be a measurable set of positive measure. Does there exist a Lipschitz map $f : A \to \mathbf{R}^k$ such that $f(A)$ is an interval?

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