Real Analysis Exchange Vol. 17 (1991–92)

A. M. Bruckner, Department of Mathematics, University of California, Santa Barbara, CA 93106

Three Forms of Chaos and Their Associated Attractors[1]

1. Introduction.

What strange behavior does the term *chaos* suggest? The existence of strange attractors? Some form of sensitive dependence on initial conditions? Actually, the two notions are intimately connected as we shall see. One finds a variety of definitions of chaos for continuous self-maps of an interval in the literature. These all carry the notion, in some form or another, that points close together can have orbits or ω-limit sets (attractors) that spread apart or are far apart. Here we discuss three such notions, two standard ones, and a new one that is intermediate to the other two in a sense we shall make precise. We also point out how the notions of chaos relate to the kinds of attractors possible.

Throughout this paper, we shall consider continuous functions f that map $I = [0,1]$ into itself. A set Ω is called an ω-limit set for f there exists $x \in [0,1]$ such that Ω is the cluster set of the sequence $\{f^n(x)\}$. (Here, as usual, $f^1 = f$ and $f^{n+1} = f \circ f^n$, $n = 1,2,3\ldots$). We write $\omega(x,f) = \Omega$ to indicate Ω is the ω-limit set of x under f.

2. An Example.

We illustrate the ideas we develop with a rudimentary example. Let

$$g(x) = \begin{cases} 3x & \text{on } \left[0, \frac{1}{3}\right] \\ 1 & \text{on } \left[\frac{1}{3}, \frac{2}{3}\right] \\ 3(1-x) & \text{on } \left[\frac{2}{3}, 1\right]. \end{cases}$$

To analyze the iterative behavior, we represent points by their ternary expansions,

$$x = .x_1x_2x_3\ldots, \quad (x_i = 0,1,2).$$

It is easy to verify that for $x \in \left[0, \frac{1}{3}\right]$, $g(x) = .x_2x_3\ldots$ while for $x \in \left[\frac{1}{3}, \frac{2}{3}\right]$, $g(x) = 1$ (so $g^2(x) = 0$), and for $x \in \left[\frac{2}{3}, 1\right]$, $g(x) = .x_2^*x_3^*\ldots$, where $x_i^* = 2 - x_i$.

[1] This paper is a transcription of a talk presented at the Fifteenth Summer Symposium in Real Analysis, Smolenice, Czechoslovakia

Thus, if a ternary expansion of x contains a 1, then for some n, $g^n(x) = g^{n+1}(x) = 0$. All but countably many points of the Cantor set Q have unique expansions containing only 0's and 2's. Thus the orbits of most points of the set Q miss the interval $\left[\frac{1}{3}, \frac{2}{3}\right]$, and one finds various sorts of ω-limit sets within Q; e.g. periodic orbits of all periods, countable ω-limit sets, and the entire set Q.

For example, $\frac{9}{10} = .\overline{2200}$ has period 2. All other periods can be obtained by considering points whose ternary expansions are of the form $x = .\overline{200\ldots0}$; if there are $n > 1$ 0's in the block, x has period $n+1$. If a point $x \in Q$ has a ternary expansion in which <u>every</u> block of 0's and 2's appears, then the trajectory of x is dense in $Q : \omega(x,g) = Q$.

To find an x with $\omega(x,g)$ countable, let x be of the form $x = .0\ldots02\ldots20\ldots 02\ldots2\ldots$, the lengths of the blocks of 0's as well as the lengths of the blocks of 2's approaching infinity. One finds that x is attracted to the countable set

$$\{0\} \cup \bigcup_{k=0}^{\infty} \left(\frac{1}{3}\right)^k \quad \text{(decimal notation).}$$

Suppose now that y is another point with a similar ternary expansion, and the blocks of 0's and 2's are properly mismatched.

$$\begin{aligned} \text{Then} \quad & \limsup_{n\to\infty} \mid g^n(x) - g^n(y) \mid = 1 \\ \text{while} \quad & \liminf_{n\to\infty} \mid g^n(x) - g^n(y) \mid = 0. \end{aligned}$$

We say x and y belong to a scrambled set.

Definition 2.1. *A set S is called a* **scrambled** *set for f if for $x, y \in S$, $(x \neq y)$*

$$\begin{aligned} & \limsup_{n\to\infty} \mid f^n(x) - f^n(y) \mid > 0 \\ \textit{and} \quad & \liminf_{n\to\infty} \mid f^n(x) - f^n(y) \mid = 0. \end{aligned}$$

Now, it was noted by Jankova and Smital [JS] that if f possesses a two-point scrambled set, then f possesses an uncountable scrambled set. (See [KS] for a proof.)

Definition 2.2. *The function f is* **chaotic** *if f possesses an uncountable scrambled set.*

Thus the function g under consideration is chaotic.

We now show that g is also chaotic in a different sense. Let n be a positive integer, and let

$$X_n = \{x : x_i = 0 \text{ for all } i > n\} \cap Q$$

Suppose $x, y \in X_n$, $(x \neq y)$. Let k be the smallest integer for which $x_k \neq y_k$, say $x_k = 0$ and $y_k = 2$. Thus the ternary expansions take the form

$$x = .x_1 x_2 x_3 \ldots x_{k-1} 0 x_{k+1} \ldots x_n \overline{0},$$
and
$$y = .x_1 x_2 x_3 \ldots x_{k-1} 2 y_{k+1} \ldots y_n \overline{0}.$$

One finds

$$g^{k-1}(x) = \begin{cases} .0 x_{k+1} \ldots x_n \overline{0}, \\ \text{or} \\ .2 x^*_{k+1} \ldots x^*_n \overline{2} \end{cases}$$

while

$$g^{k-1}(y) = \begin{cases} .2 y_{k+1} \ldots y_n \overline{0}, \\ \text{or} \\ .0 y^*_{k+1} \ldots y^*_n \overline{2} \end{cases}$$

Thus

$$| g^{k-1}(x) - g^{k-1}(y) | \geq .1$$

We have seen that for each n there is a set X_n of cardinality 2^n such that for $x, y \in X_n$, $(x \neq y)$, one of the first n iterates of g separates x and y by at least .1. This shows that the topological entropy of g is positive, according to the following definition.

For given $\epsilon > 0$ and positive integer n, let $S = S(f, \epsilon, n)$ be a set of maximal cardinality such that for $x, y \in S$ and $x \neq y$, there is an integer k with $0 \leq k \leq n$ such that $| f^k(x) - f^k(y) | > \epsilon$. With the preceding notation we define the topological entropy $\mathbf{h}(f)$ of a function f as follows:

Definition 2.3. $\mathbf{h}(f) = \lim\limits_{\epsilon \to 0} \limsup\limits_{n \to \infty} \dfrac{1}{n} \log \operatorname{card} S(f, \epsilon, n).$

Thus the function g has $\mathbf{h}(g) \geq \log 2$.

Positive topological entropy represents another form of **chaos** that is more severe than the form first mentoned involving scrambled sets. We discuss this in the next section. At the moment, we mention only that if $\mathbf{h}(f) > 0$, then f possesses scrambled sets, but the converse is not true.

Both forms of chaos we have discussed involve the separation of trajectories of points. Perhaps the most direct method of measuring this separation of trajectories can be obtained via the map $x \to \omega(x, f)$. We furnish the space of compact subsets $\mathcal{K}$ of I with the Hausdorff metric, and study the continuity properties of $\omega_f : I \to \mathcal{K}$ defined by $\omega_f(x) = \omega(x, f)$.

To illustrate with our function g, we observe first that $\omega(x, g) = \{0\}$ for all $x \notin Q$. Thus ω_g is continuous on a dense, open set. This suggests nonchaotic behavior. On the other hand, the restriction of ω_g to the Cantor set Q is discontinuous everywhere on Q. To see this, observe first that for $x, y \in Q$, say $x = .x_1x_2x_3\ldots$ and $y = .y_1y_2y_3\ldots$, a point $z = .x_1x_2x_3\ldots x_n0y_1y_2y_3\ldots$ can be chosen arbitrarily close to x with $g^{n+1}(x) = y$. Thus $\omega(z, g) = \omega(y, g)$. It follows that ω_g takes all of the values it takes in Q in every portion of Q. Hence $\omega_g \mid_Q$ is everywhere discontinuous. (Incidentally, for the tent function t, ω_t is everywhere discontinuous.)

Thus g is chaotic in the strong sense that $\mathbf{h}(g) > 0$, yet ω_g is continuous on a dense open set of full measure. If we wish to use continuity criteria for chaotic behavior, we should be able to detect the chaotic behavior that may occur on "small" sets such as Cantor sets. A useful notion here involves the Baire class of ω_f. Our function g has $\omega_g \mid_Q$ everywhere discontinuous, thus ω_g is not in $\mathcal{B}_1$, the first Baire class. Just how the Baire class of ω_f fits into the picture of chaos can be described briefly by the following result [BC].

Theorem 2.4. *Let f be continuous, $f : I \to I$. Then*

1) *$\omega_f \in \mathcal{B}_2$, the second Baire class.*
2) *If f is nonchaotic, then $\omega_f \in \mathcal{B}_1$ (but the converse fails).*
3) *If $\omega_f \in \mathcal{B}_1$, then $\mathbf{h}(f) = 0$ (but the converse fails).*

Thus the condition that $\omega_f \in \mathcal{B}_1$ is a form of nonchaos strictly intermediate to the other forms discussed.

3. Chaos and ω-limit sets.

When a function exhibits any of the forms of chaos we have discussed, its iterative patterns are rather complex. One can also view "complexity" of the iterative patterns of f in terms of the types of ω-limit sets f possesses. Intuitively one may expect that uncomplicated iterative patterns should lead to the existence of relatively simple ω-limit sets, while complicated patterns should lead to more esoteric ω-limit sets. In fact, the Baire class of ω_f is closely linked to the type of ω-limit sets f can (or must) possess.

First we mention two facts:

Theorem 3.1. *[ABCP], [BS$_1$]*

A nonempty compact set S is an ω-limit set for some continuous function f if and only if S is either nowhere dense, or a finite union of intervals.

The second fact is that the existence of certain kinds of ω-limit sets for f implies the existence of certain other kinds. Sharkovski's famous theorem, for example, provides an order on the positive integers such that if p precedes q in the order, and $\omega(x, f)$ has cardinality q, then there exists $y \in I$ such that $\omega(y, f)$ has cardinality p. (This order begins with 1 and ends with 3: if f has a periodic point of period 3, it has periodic points of all periods.)

The chart below does two things: firstly it indicates an order on certain kinds of ω-limit sets relevant to our discussion. Secondly, it relates our three notions of chaos to the levels of complexity of ω-limit sets that are possible (or that must exist). The right column lists possible types of ω-limit sets in order of decreasing complexity, while the left column lists our three forms of chaos in order of decreasing severity. In this chart, Ω always denotes an ω-limit set.

Chart 3.2.

$$\begin{array}{lcl}
 & & \text{There exists } \Omega \text{ with interior} \\
 & & \downarrow \\
\mathbf{h}(f) > 0 & \longleftrightarrow & {}^{*}\ \text{There exists countably infinite } \Omega \\
\downarrow & & \downarrow \\
\omega_f \notin \mathcal{B}_1 & \longleftrightarrow & \text{There exists uncountable } \Omega \text{ with isolated points} \\
\downarrow & & \downarrow \\
f \text{ chaotic} & \longrightarrow & {}^{**}\ \text{There exists perfect } \Omega \\
 & & \downarrow \\
 & & \text{For every } n, \text{ there exists } \Omega \text{ with cardinality } 2^n
\end{array}$$

* This is equivalent to the existence of a finite ω-limit set of cardinality not a power of 2.

** There exist nonchaotic functions with perfect ω-limit sets.

It may be worth mentioning that the very desirable situation that ω_f be continuous rarely occurs. In terms of ω-limit sets one can say [BC] ω_f is continuous if and only if all ω-limit sets have cardinality 1 or 2, and the union of all ω-limit sets is connected. A more interesting situation occurs in connection with the condition $\omega_f \in \mathcal{B}_1^*$. Recall the class $\mathcal{B}_1^*$ consists of those functions whose restriction to each perfect set contains a portion (i.e. relative interval) of continuity. This can be compared with the analogous characterization of $\mathcal{B}_1$ in which the word ***portion*** is replaced by ***point***. In particular, a function in $\mathcal{B}_1^*$ is continuous on a dense open set rather than just on a dense set of type G_δ. One finds that if f has only finitely many ω-limit sets, then $\omega_f \in \mathcal{B}_1^*$, while if $\omega_f \in \mathcal{B}_1^*$, then every ω-limit set for f is finite.

It may be instructive to look at the well-studied logistic family $f_k(x) = kx(1-x)$, $0 \le k \le 4$, defined on I. The chart below relates this family to our forms of chaos.

Chart 3.3.

k	**ω-limit sets**	**Baire class of ω_f**	**classification of chaos**
$k \le 1$	$\{0\}$	continuous	nonchaotic
$1 < k < k_0$ ($k_0 \cong 3.57$)	finitely many, all representing periodic orbits whose period is a power of 2	$\mathcal{B}_1^*$, but not continuous	nonchaotic

$k = k_0$	A Cantor set and infinitely many (periodic) finite sets of arbitrarily high cardinality, all powers of 2	$\mathcal{B}_1$, not $\mathcal{B}_1^*$	nonchaotic
$k > k_0$	many complicated sets and finite sets of various cardinalities	$\mathcal{B}_2$, not $\mathcal{B}_1$	$\mathbf{h}(f) > 0$
$k = 4$	many types, including the entire interval $[0,1]$	$\mathcal{B}_2$, not $\mathcal{B}_1$ nowhere continuous	$\mathbf{h}(f) > 0$

For this particular family, as k increases through k_0, we pass suddenly from nonchaos to positive entropy.

It may be helpful to discuss the situation as k approaches k_0 from below. If the periodic orbit of highest period for f_k has period $2^{p(k)}$, then $p(k) \to \infty$ as $k \to k_0^-$. For $k = k_0$ the situation is as follows: Let $f = f_k$. There is a sequence of compact intervals $\{J_n\}$ such that

i) $J_1 \supset J_2 \supset J_3 \supset \dots$
ii) J_n is periodic with period 2^n
iii) $Q = \bigcap\limits_{n=1}^{\infty} \bigcup\limits_{j=1}^{2^n} f^j(J_n)$

From ii) we see that each J_n contains a periodic orbit of period 2^n. If $x \in Q$, then for every n, x is in the periodic orbit of J_n, from which one sees that $\omega(x, f) = Q$, and the trajectory of x is approximatable by cycles of period 2^n. As $n \to \infty$, the approximation improves.

In general, nonchaotic functions do possess approximatable trajectories – every trajectory can be approximated by cycles when f is nonchaotic, but if f is chaotic, the sense of such approximations is much weaker. Suppose f is chaotic, but $\mathbf{h}(f) = 0$. There will be an infinite ω-limit set Ω. There will also be a system of periodic intervals $\{J_n\}$ as above but for

$$\delta_n = \max\{\text{diam}\,(f^j(J_n)) : j = 0, 1, \dots, 2^n - 1\}, \lim_{n\to\infty} \delta_n > 0.$$

Thus, condition iii) is replaced by $\Omega \subset \bigcap\limits_{n=1}^{\infty} \bigcup\limits_{j=1}^{2^n} f^j(J_n) = K$. The set K has interior.

We can write $K \setminus \text{int}\, K = Q \cup C$, where Q is a Cantor set and C is countable or empty. (If C is empty, $Q = \Omega$). There will be a scrambled set S with

$\omega(x, f) = Q$ for all $x \in S$ and $\limsup_{j \to \infty} \mid f^j(x) - f^j(y) \mid = \lim_{n \to \infty} \delta_n > 0$ for all $x \neq y \in S$. If $C \neq \phi$, then some intervals contiguous to Q will contain one or two components of int K. One endpoint of such a component will be in C. This point may but need not be in Ω. In any case, Ω can contain at most two isolated points in each interval contiguous to Q. The set Q is also an ω-limit set for f.

These and other comparisons of these forms of chaotic and nonchaotic behavior can be found in [FSS]. These remarks can form the basis for a striking contrast among the three types of chaotic behavior we have discussed.

Suppose Q is a perfect ω-limit set for a function f with $\mathbf{h}(f) = 0$. Then Q is minimal – Q contains no proper subsets that are ω-limit sets for f. If f is nonchaotic, Q is also maximal – Q is not properly contained in any ω-limit set. Furthermore, f is $1-1$ on Q. If f is chaotic, but $\omega_f \in \mathcal{B}_1$, then Q is still maximal, but f is not $1-1$ on Q.

When $\omega_f \notin \mathcal{B}_1$, the situation is more complicated. The set Q is <u>not</u> maximal. There will always be a countable set C such that $\Omega = Q \cup C$ is also an ω-limit set for f. Sharkovski [S] was the first to prove what we indicated earlier – that the set C can contain at most two points in each interval J contiguous to Q, with J contained in the convex hull $[a, b]$ of Q, and at most one point in $[0, a]$ or in $[b, 1]$. How many such sets Ω can there be? Uncountably many! In fact, the following holds:

Theorem 3.4. *Let Q be a Cantor set in $[0,1]$. Let C be a countable set satisfying the cardinality conditions stated above, such that $\overline{C} \supset Q$. Let $C = \bigcup_{k=1}^{\infty} C_k$ such that for each k, $\overline{C}_k \supset Q$ and $C_i \cap C_j = \phi (i \neq j)$. Then there exists a continuous function f with $\mathbf{h}(f) = 0$, and such that for every collection M of positive integers, the set $Q \cup \bigcup_{k \in M} C_k$ is an ω-limit set for f, and every C_k is a full orbit under f.*

These results as well as many refinements, can be found in [BS$_2$]. In particular, there are uncountably many ω-limit sets Ω for f satisfying $Q \subset \Omega \subset Q \cup C$, a fact that had already been known to Sharkovski.

We end with the following characterizations of those f for which $\omega_f \in \mathcal{B}_1$. It may be contrasted with the corresponding characterizations for nonchaos in the other two senses (see [FSS]).

Theorem 3.5. *Let $f : I \to I$ be continuous. The following are equivalent:*

1) $\omega_f \in \mathcal{B}_1$
2) *every ω-limit set is finite or a Cantor set*
3) *every ω-limit set is maximal*
4) *every ω-limit set is minimal*
5) *every ω-limit set is internally generated by every point*

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Received September 12, 1991