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COMPARING THE RANGES OF CONTINUOUS FUNCTIONS

If f and g are differentiable real valued functions on $I = [0,1]$ such that $|f'| \geq |g'|$ on I, someone might expect that the length of the interval $g(I)$ cannot exceed the length of $f(I)$. This need not be true, as the counterexample $f(x) = |x - \frac{1}{2}|^3$ and $g(x) = (x - \frac{1}{2})^3$ shows. The trouble seems to be that some points u and v $(u \neq v)$ satisfy $f(u) = f(v)$ and $g(u) \neq g(v)$. Suppose we require that the equation $f(u) = f(v)$ implies $g(u) = g(v)$. Then must length $f(I) \geq$ length $g(I)$? Under reasonable circumstances the answer is yes, as we now show.

We will use continuous real valued functions on I. We will say that a function f is an N-function if f maps each set of Lebesgue measure zero to a set of measure zero. Let λ denote Lebesgue outer measure. We offer:

THEOREM 1. *Let f and g be continuous N-functions, differentiable almost everywhere on I. Let $|f'(x)| \geq |g'(x)|$ almost everywhere on I. For $u, v \in I$, let the equation $f(u) = f(v)$ imply the equation $g(u) = g(v)$. Then $\lambda f(I) \geq \lambda g(I)$.*

THEOREM 2. *Let f and g be continuous N-functions, differentiable almost everywhere on I. Let $|f'(x)| = |g'(x)|$ almost everywhere on I. For $u, v \in I$, let each equation $f(u) = f(v)$ and $g(u) = g(v)$ imply the other. Then either $f + g$ is constant or $f - g$ is constant on I.*

For functions f and g that are absolutely continuous or everywhere differentiable on I, we deduce from Theorem 2 that if $|f'(x)| = |g'(x)|$ almost everywhere on I, if each equation $f(u) = f(v)$, $g(u) = g(v)$ $(u, v \in I)$ implies the other, then either $f + g$ is constant or $f - g$ is constant on I.

N-functions are essential in Theorems 1 and 2. For example, let h be a continuous nondecreasing nonconstant function such that $h'(x) = 0$ almost everywhere on I, let $f(x) = x$ and $g(x) = x + h(x)$ on I. Then $\lambda g(I) > \lambda f(I)$. But all the hypotheses of Theorem 2 are satisfied, except that g is not an N-function.

We require five Lemmas.

LEMMA 1. *Let the hypothesis be as in Theorem 1. Then for any subinterval J of I, $\lambda f(J) \geq \lambda g(J)$.*

Proof. Put $U = \{x \in J : f \text{ is differentiable at } x\}$, and $V = \{x \in J : g \text{ is differentiable at } x\}$

For each real number y, let $B(f)(y)$ denote the number of points in the set $U \cap f^{-1}(y)$ and let $B(g)(y)$ denote the number of points in the set $V \cap g^{-1}(y)$. For each $x \in I$, let $h_1(x) = 1/B(f)(f(x))$ and $h_2(x) = 1/B(g)(g(x))$. (Here we understand that $1/\infty$ means 0 and $1/0$ means ∞.)

Let $X = \{x \in I : f \text{ is not differentiable at } x\}$ and $X_0 = \{x \in I : f'(x) = 0\}$. Then $\lambda f(X_0) = 0$ by [HS, Exercise (18.48)] and $\lambda f(X) = 0$ because $\lambda(X) = 0$ and f is an N-function. Now if $B(f)(y)$ is infinite, then $f^{-1}(y)$ contains an accumulation point x of $f^{-1}(y)$, and x must be in $X \cup X_0$. Thus $y \in f(X \cup X_0)$. Hence $\{y : B(f)(y) \text{ is infinite}\} \subset f(X \cup X_0)$ and $\{y : B(f)(y) \text{ is infinite}\}$ has measure zero. Likewise $\{y : B(g)(y) \text{ is infinite}\}$ has measure zero. From [C, Theorem 9] we infer that the functions $h_1(x)|f'(x)|$ and $h_2(x)|g'(x)|$ are measurable on J, and

$$\lambda f(J) = \int_J h_1(x)|f'(x)|dx,$$

$$\lambda g(J) = \int_J h_2(x)|g'(x)|dx.$$

But from the hypothesis it follows that $h_1(x)|f'(x)| \geq h_2(x)|g'(x)|$ almost everywhere on J, so $\lambda f(J) \geq \lambda g(J)$. $\square$

To prove Theorem 1, put $J = I$ in Lemma 1. To prove Theorem 2, we require more Lemmas.

LEMMA 2. *Let the hypothesis be as in Theorem 2. Then for every subinterval J of I, $\lambda f(J) = \lambda g(J)$.*

Proof. This follows from Lemma 1. $\square$

In the Lemmas that follow, f and g need not be N–functions and need not be differentiable.

LEMMA 3. *Let f and g be continuous functions on I such that for any subinterval J of I, $\lambda f(J) = \lambda g(J)$. Let (a, b) be a subinterval of I such that $f(x) > f(a)$ for $a < x < b$. Then $|f(a) - f(b)| = |g(a) - g(b)|$.*

Proof. We assume, without loss of generality, that f is not constant on I; for otherwise $f(I)$ and $g(I)$ have length zero, and g is constant also.

Let $x_0 = a$ and let x_1 be the smallest value in $[x_0, b]$ for which $f(x_1) = \max f[x_0, b]$. Note that $f[x_0, x_1] = f[x_0, b]$. Let x_2 be the smallest value in $[x_1, b]$ such that $f(x_2) = \min f[x_1, b]$. Note that $f[x_1, x_2] = f[x_1, b]$. In general we let x_k be the smallest value in $[x_{k-1}, b]$ such that $f(x_k) = \max f[x_{k-1}, b]$ if k is odd, and $f(x_k) = \min f[x_{k-1}, b]$ if k is even. By induction we define x_k for all positive integers k, (x_k) is nondecreasing, and $f[x_{k-1}, x_k] = f[x_{k-1}, b]$ for all $k \geq 1$. Let $x' \in [a, b]$ be the limit of the sequence (x_k) in $[a, b]$. It follows from the construction that f is constant on $[x', b]$, and $(f(x_k))$ converges to $f(x') = f(b)$. Likewise, g is constant on $[x', b]$ because $f[x', b]$ and $g[x', b]$ both have length zero. So $g(x') = g(b)$ also.

We assume, without loss of generality, that $g(x_1) \geq g(x_0)$. For if $g(x_1) < g(x_0)$, we replace g with $-g$ in the argument. We assume, without loss of generality, that $g(x_0) = f(x_0)$. Otherwise use $g + f(x_0) - g(x_0)$ in place of g.

Now for any $u \in (x_0, x_1)$ we have $f(x_0) < f(u) < f(x_1)$ and consequently $\lambda f[x_0, x_1] > \lambda f[x_0, u]$ and $\lambda f[x_0, x_1] > \lambda f[u, x_1]$. Hence $\lambda g[x_0, x_1] > \lambda g[x_0, u]$ and $\lambda g[x_0, x_1] > \lambda g[u, x_1]$. It follows from this, together with $g(x_1) \geq g(x_0)$, that $g(x_0) = \min g[x_0, x_1]$ and $g(x_1) = \max g[x_0, x_1]$. Thus

$$g(x_1) - g(x_0) = \lambda g[x_0, x_1] = \lambda f[x_0, x_1] = f(x_1) - f(x_0).$$

But $g(x_0) = f(x_0)$, so

$$f(x_1) = g(x_1). \tag{1}$$

Moreover, $g(x) \leq f(x_1) = g(x_1)$ for $x \in [x_1, b]$; for otherwise $g(x) > f(x_1)$ and $g[a, b]$ would contain $g(x)$ and $g(x_0) = f(x_0)$, and

$$\lambda g[a, b] > f(x_1) - f(x_0) = \lambda f[a, b].$$

Also $g(x) \geq f(x_2)$ for $x \in [x_1, b]$; for otherwise $g(x) < f(x_2)$ and $g[x_1, b]$ would contain $g(x)$ and $g(x_1) = f(x_1)$, and

$$\lambda g[x_1, b] > f(x_1) - f(x_2) = \lambda f[x_1, b].$$

It follows that

$$g[x_1, b] \subset [f(x_2), f(x_1)] = f[x_1, b]$$

and from $\lambda g[x_1, b] = \lambda f[x_1, b]$ we infer that

$$g[x_1, b] = f[x_1, b]. \tag{2}$$

It follows from (2) that $g(x_2) \geq f(x_2)$. It also follows that $g(x_2) \leq f(x_2)$; for otherwise $g(x_2) > f(x_2)$ and from this and

$$\lambda g[x_1, x_2] = \lambda f[x_1, x_2] = f(x_1) - f(x_2)$$

we infer that $g(v) = f(x_2)$ for some $v \in (x_1, x_2)$. Thus

$$\lambda f[x_1, v] = \lambda g[x_1, v] \geq f(x_1) - f(x_2)$$

and hence $f(w) = f(x_2)$ for some $x \in [x_1, v]$, contrary to the choice of x_2. So

$$f(x_2) = g(x_2). \tag{3}$$

Just as we proved (2) we also prove $g[x_2, b] = f[x_2, b]$, and just as we proved (3) we also prove $f(x_3) = g(x_3)$. By induction, we prove that $g[x_k, b] = f[x_k, b]$ and $f(x_k) = g(x_k)$ for all k. From the second paragraph of this proof and from the continuity of f and g we infer

$$f(b) = f(x') = \lim f(x_k) = \lim g(x_k) = g(x') = g(b),$$

and from the third paragraph of this proof we infer $f(a) = g(a)$. In view of the modification of g we made, we conclude that in general

$$|f(a) - f(b)| = |g(a) - g(b)|. \tag{4}$$

□

We improve on Lemma 3.

LEMMA 4. *Let the hypothesis be as in Lemma 3. Then either $f + g$ or $f - g$ is constant on the subinterval $[a, b]$.*

Proof. As in the proof of Lemma 3, we can assume without loss of generality that $g(a) = f(a)$ and that $g(x_1) \geq g(a)$. Just as in that argument we obtain $f(x_1) = g(x_1)$.

Moreover, $g(x) \geq g(a)$ for $x \in (a, b)$; for otherwise $g(x) < g(a)$ and $g[a, b]$ contains $g(x)$ and $g(x_1) = f(x_1)$, and $\lambda g[a, b] > f(x_1) - f(a) = \lambda f[a, b]$. We apply Lemma 3 to the interval $[a, x]$ and obtain

$$|f(a) - f(x)| = |g(a) - g(x)|.$$

But both $f(a) - f(x)$ and $g(a) - g(x)$ are nonpositive, so

$$f(a) - f(x) = g(a) - g(x).$$

Finally, $f(a) = g(a)$ and it follows that $f(x) = g(x)$. In view of the modification of g we made, the desired conclusion follows. □

If $f+g$ or $f-g$ is constant on a subinterval J of I, we say that f concurs with g on J. By Lemma 4, if f and g are continuous on I, if $\lambda f(J) = \lambda g(J)$ for every subinterval J of I and if $f(x) > f(a)$ for $a < x < b$, then f concurs with g on $[a, b]$. This works when $f(x) > f(b)$ replaces $f(x) > f(a)$. Just use $f(a+b-x)$ and $g(a+b-x)$. This also works when the inequalities $f(x) > f(a)$, etc., are reversed. Use $-f$ and $-g$.

LEMMA 5. *Let f and g be continuous functions on I such that for any subinterval J of I, $\lambda f(J) = \lambda g(J)$. Then f concurs with g on I.*

Proof. Without loss of generality we assume that f is not constant on I. Say $f(0) < \max f(I)$. (Otherwise use $-f$ in place of f.) Let u_1 be the smallest number in I for which $f(u_1) = \max f(I)$. Let u_2 be the smallest number in $(u_1, 1]$ for which $f(u_2) = \min f[u_1, 1]$, if there is one. Let u_3 be the smallest number in $(u_2, 1]$ for which $f(u_3) = \max f(I)$, if there is one. Let u_4 be the smallest number in $(u_3, 1]$ for which $f(u_4) = \min f[u_1, 1]$, if there is one. We continue this way so that in general $f(u_j) = \max f(I)$ if j is odd, and $f(u_j) = \min f[u_1, 1]$ if j is even. But this process must conclude with some u_q because f is continuous on I. Put $u_0 = 0$. For each $j = 1, \ldots, q$, let v_j be the largest number in $[u_{j-1}, u_j)$ for which $f(v_j) = f(u_{j-1})$. Put $u_{q+1} = 1$.

Note that $f(x) < f(u_1)$ for $u_0 < x < u_1$, and $f(x) > \min\, (f(v_1), f(u_2))$ for $v_1 < x < u_2$. When $j \geq 3$, $f(x) < f(v_{j-1})$ for $v_{j-1} < x < u_j$ and j odd, and $f(x) > f(v_{j-1})$ for $v_{j-1} < x < u_j$ and j even. By Lemma 4, f concurs with g on the intervals $[u_0, u_1], [v_1, u_2], [v_2, u_3], [v_3, u_4], \ldots, [v_q, u_{q+1}]$. Now if f concurs with g on overlapping intervals J_1 and J_2 and if f is nonconstant on $J_1 \cap J_2$, then f concurs with g on $J_1 \cup J_2$. It follows that f concurs with g on the intervals $[u_0, u_2], [u_0, u_3], [u_0, u_4], \ldots, [u_0, u_q], [u_0, u_{q+1}] = I$. $\square$

Now Theorem 2 follows from Lemmas 2 and 5.

References

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