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THERE IS NO STRONGLY LOCALLY ANTISYMMETRIC SET

A function $f : R \to R$ (where R is a real line) is said to be locally symmetric if for each $x \in R$ there is a $\delta_x > 0$ such that $f(x+h) = f(x-h)$ holds for each h, $0 < h < \delta_x$. This notion was introduced in [F]. A full description of locally symmetric functions is known ([D], [R], [T]): these functions are constant on all but, at most a countable set.

We will say that the set $S \subset R$ is strongly locally symmetric, if its characteristic function is locally symmetric, i.e., for every $x \in R$ there is a $\delta_x > 0$ such that for each h, $0 < h < \delta_x$, $x + h \in S$ if and only if $x - h \in S$. Answering a query by Evans and Weil [EW], Rusza proved ([R]) that for every strongly locally symmetric subset S of R, either the closure of S or the closure of $R - S$ is countable.

In his paper [M] S. Marcus suggested investigating both symmetry and its polar opposite, antisymmetry, of sets and functions. This motivates the following definition:

Definition. A set $A \subset R$ is said to be strongly locally antisymmetric (SLA) if for every $x \in R$ there exists a $\delta_x > 0$ such that for each h, $0 < h < \delta_x$, $x + h \in A$ if and only if $x - h \notin A$.

If A is an SLA set, then it is obviously non-void, because both A and $R - A$ have to be dense in R. One can see that any SLA set A is necessarily Lebesgue non-measurable. Indeed, let A be a measurable SLA set and $x \in R$. Since $\lambda(A \cap (x-h, x+h)) = \lambda((R - A) \cap (x-h, x+h)) = h$ (where λ denotes Lebesgue measure) holds for each h, $0 < h < \delta_x$, we have $d(x, A) = \frac{1}{2}$ where $d(x, A)$ is the density of A at x. This contradicts the Lebesgue Density Theorem. In [M] S. Marcus asks whether an SLA set really exists. The following theorem gives a negative answer to his question.

Theorem. *There is no SLA set.*

Proof. Let A be an SLA set and let f be its characteristic function. Then for every $x \in R$ there is a $\delta_x > 0$ such that for each k, $0 < k < \delta_x$, we have $|f(x+k) - f(x-k)| = 1$. Put $E_n = \{x : \delta_x > \frac{1}{n}\}$ for $n = 1, 2, \dots$. Then

$\bigcup_{n=1}^{\infty} E_n = R$ and, according to the Baire Category Theorem, for some m there is an interval (a,b) such that E_m is dense in (a,b). Without loss of generality, we may assume that $b-a < \frac{1}{m}$. Let $x \in E_m$ and $I = [x-2h, x+2h] \subset (a,b)$. The notation is simplified if we assume that $I = [-2h, 2h]$, and that E_m is dense in $(-2h, 2h)$. If so, choose a negative x' in E_m so that $0 < x' - (-\frac{h}{2}) < \frac{1}{2}\delta_h$, and choose a positive x'' in E_m so that $0 < x'' - \frac{h}{2} < \frac{1}{2}\delta_{-h}$. This means that $0 < 2x' + h < \delta_h$ and $0 < 2x'' - h < \delta_{-h}$. Clearly, we can also arrange that $x'' + 2x' < 0$ and $2x'' + x' > 0$.

Let us define the following intervals:

$$\begin{aligned}
I_1 &= [-2h, 2x'+2h] = [x' - (x'+2h), x' + (x'+2h)],\\
I_2 &= [-2x', 2x'+2h] = [h - (2x'+h), h + (2x'+h)],\\
I_3 &= [2x''+2x', -2x'] = [x'' + (x''+2x'), x'' - (x''+2x')],\\
I_4 &= [-2x'', 2x'+2x''] = [x' - (2x''+x'), x' + (2x''+x')],\\
I_5 &= [-2x'', 2x''-2h] = [-h - (2x''-h), -h + (2x''-h)], \quad \text{and}\\
I_6 &= [2x''-2h, 2h] = [x'' - (2h - x''), x'' + (2h - x'')].
\end{aligned}$$

For $J = [r,s]$, let $F(J) = f(s) - f(r)$. It is straightforward to check that each I_i $(i = 1, \ldots, 6)$ is a subinterval of I, that $F(I_i) = +1$ or -1 for each I_i, and for f we have $f(2h) - f(-2h) = F(I_1) - F(I_2) - F(I_3) - F(I_4) + F(I_5) + F(I_6) = \sum_{i=1}^{6}(\pm 1) =$ even number. Hence $|f(2h) - f(-2h)| \neq 1$, a contradiction.

Remark. The family of intervals $\{I_i\}_{i=1}^{6}$ was used in [T] for an investigation of locally symmetric functions.

Problem. One can define the following polar opposite of locally symmetric function: A function f is said to be locally antisymmetric if for each $x \in R$ there is a $\delta_x > 0$ such that $|f(x+h) - f(x-h)| \geq \delta_x$ holds for each h, $0 < h < \delta_x$. It is an open problem whether such a function really exists.

References

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Received March 1, 1991