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INTEGRATION BY PARTS FOR THE FORAN INTEGRAL

Definition 1. ([1], p.360). Given a natural number N and a set E, a function F is said to be $A(N)$ on E, if for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $I_1, \dots, I_k, \dots$ are nonoverlapping intervals with $E \cap U_k \neq \emptyset$ and $\sum |I_k| < \delta$ then there exist intervals J_{kn}, $n = 1, \dots, N$ for which

$$B(F; E \cap \cup I_k) \subset \bigcup_k \bigcup_{n=1}^{N} I_k \times J_{kn} \text{ and } \sum_k \sum_{n=1}^{N} |J_{kn}| < \varepsilon.$$

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Proposition 1. *Definition 1 and Definition 2 are equivalent.*

Proof. "Definition 1 $\Rightarrow$ Definition 2": Since $B(F; E \cap \cup I_k) \subset \bigcup_k \bigcup_{n=1}^{N} I_k \times J_{kn}$ it follows that $F(E \cap I_k) \subset \bigcup_{n=1}^{N} J_{kn}$, $k \geq 1$. Let $E_{kn} = (E \cap I_k) \cap F^{-1}(J_{kn})$. Then $\bigcup_{n=1}^{N} E_{kn} = E \cap I_k$ and $0(F; E_{kn}) \leq |J_{kn}|$.

"Definition 2 $\Rightarrow$ Definition 1": Let $J_{kn} = [\inf_{x \in E_{kn}} F(x),\ \sup_{x \in E_{kn}} F(x)]$. Then $0(F; E_{kn}) = |J_{kn}|$ and $F(E \cap I_k) \subset \bigcup_{n=1}^{N} J_{kn}$.

Definition 3. ([1], p.360). The class $\mathcal{F}$ will consist of all continuous functions F defined on a closed interval I for which there exist a sequence of sets E_n and natural numbers N_n such that $I = \cup E_n$ and F is $A(N_n)$ on E_n.

Definition 4. A function $f : [a, b] \to \bar{R}$ is Foran integrable if there exists a function $F \in \mathcal{F}$, F approximately differentiable a.e., such that $F'_{ap}(x) = f(x)$ a.e. on $[a, b]$ and

$$(\mathcal{F}) \int_a^b f(x)dx = F(b) - F(a).$$

Lemma 1. *Let $F, G, H : [a,b] \to R$, $F \in VB$, G continuous and $H(x) = F(x) \cdot G(x) - (S)\int_a^x G(t)dF(t)$. Let $E \subset [a,b]$ such that G is $A(N)$ on E. Then H is continuous on $[a,b]$ and H is $A(N^2)$ on E. Moreover, if F is monotone then H is $A(N)$ on E. (Here $(S)\int_a^x G(t)dF(t)$ is the Riemann-Stieltjes integral.)*

Proof. Suppose that F is increasing on $[a,b]$, $F(a) = 0$, $F(b) = M_0$. Let $[c,d] \subset [a,b]$. By [2] (Theorem 2.1, (i), p.244) we have $H(d) - H(c) = (G(d) - G(c)) \cdot F(d) + (F(d) - F(c)) \cdot G(c) - (S)\int_c^d G(t)dF(t) = (G(d) - G(c)) \cdot F(d) + (F(d) - F(c)) \cdot (G(c) - A)$, where A is a number between the bounds of G on $[c,d]$. Hence

$$|H(d) - H(c)| \le M_0 \cdot |G(d) - G(c)| + 0(G; [c,d]) \cdot (F(d) - F(c)). \tag{1}$$

Since G is continuous, by (1) it follows that H is continuous on $[a,b]$. Let $\varepsilon > 0$. Since G is $A(N)$ on E it follows that there exists a $\delta > 0$ such that if $I_1, \dots, I_k, \dots$ are nonoverlapping intervals with $E \cap I_k \ne \emptyset$ and $\sum |I_k| < \delta$ then there exist sets E_{kn}, $n = 1, \dots, N$ such that $\bigcup_{n=1}^N E_{kn} = E \cap I_k$ and $\sum_k \sum_{n=1}^N 0(G; E_{kn}) < \varepsilon/(2M_0)$. Let $\eta > 0$ such that $0(G; I) \le \varepsilon/(2NM_0) = \varepsilon'$, for each interval $I \subset [a,b]$ with $|I| < \eta$. (This is possible since G is continuous on $[a,b]$.) Let $\delta_1 = \min\{\delta, \eta\}$, then $0(G; I_k) \le \varepsilon'$, for each k. By (1) it follows that $0(H; E_{kn}) \le M_0 \cdot 0(G; E_{kn}) + 0(G; I_k) \cdot |F(I_k)|$. Hence $\sum_{k=1}^\infty \sum_{n=1}^N 0(H; E_{kn}) \le M_0 \cdot \sum_{k=1}^\infty \sum_{n=1}^N 0(G; E_{kn}) + N \cdot \varepsilon' \cdot \sum_{k=1}^\infty |F(I_k)| \le M_0 \cdot (\varepsilon/2M_0) + N \cdot \varepsilon' \cdot M_0 < \varepsilon$. Hence H is $A(N)$ on E. Suppose that F is VB on $[a,b]$. Then $F = F_1 - F_2$, where F_1 and F_2 are increasing and bounded on $[a,b]$. Let

$$H_1(x) = F_1(x) \cdot G(x) - (S)\int_a^x G(t)dF_1(t) \quad \text{and}$$

$$H_2(x) = F_2(x) \cdot G(x) - (S)\int_a^x G(t)dF_2(t).$$

Then $H(x) = H_1(x) - H_2(x)$ and $H_1, H_2 \in A(N)$ on E. By [1] ((ii), p.360) it follows that H is $A(N^2)$ on E.

Theorem 1. **(An extension of Theorem 2.5 of [2], p.246).** *Let $F, g : [a,b] \to R$ be such that $F \in VB$ and g is an $(\mathcal{F})$-integrable function on $[a,b]$. Then $F(x) \cdot g(x)$ is $(\mathcal{F})$-integrable and denoting the $(\mathcal{F})$-indefinite integral of g by G, we have*

$$(\mathcal{F})\int_a^b F(x)g(x)dx = G(b) \cdot F(b) - G(a) \cdot F(a) - (S)\int_a^b G(x)dF(x).$$

Proof. G is continuous, $G \in \mathcal{F}$ and $G'_{ap}(x) = g(x)$ a.e. on $[a,b]$. By Lemma 1, $H(x) = F(x) \cdot G(x) - (S)\int_a^x G(t)dF(t)$ is in $\mathcal{F}$ on $[a,b]$. By [2] (Theorem 2.1, (ii), p.244) it follows that $H'_{ap}(x) = F'(x)\cdot G(x)+F(x)\cdot G'_{ap}(x)-G(x)\cdot F'(x) = F(x)\cdot g(x)$ a.e. on $[a,b]$. It follows that $F(x)\cdot g(x)$ is $(\mathcal{F})$-integrable on $[a,b]$ and

$$(\mathcal{F})\int_a^b F(x)\cdot g(x)dx = H(b)-H(a) = F(b)\cdot G(b)-F(a)\cdot G(a)-(S)\int_a^b G(t)dF(t).$$

<u>**Theorem 2**</u>. *Let $F:[a,b]\to R$ be an increasing function and let $g:[a,b]\to \bar{R}$ be an $(\mathcal{F})$-integrable function. Then there exists $c\in[a,b]$ such that*

$$(\mathcal{F})\int_a^b g(x)\cdot F(x)dx = F(a)\cdot(\mathcal{F})\int_a^c g(x)dx + F(b)\cdot(\mathcal{F})\int_c^b g(x)dx.$$

Proof. The proof is similar to that of Theorem 2.6 of [2] (p.246), using Theorem 1 instead of Theorem 2.5 of [2] (p.246).

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References

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