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ON TAU-SMOOTH MEASURE SPACES WITHOUT THICK LINDELÖF SUBSETS

1. Introduction. Let X be a completely regular Hausdorff space. The smallest σ–algebra of subsets of X making all the real valued continuous functions on X measurable (or, equivalently, the σ–algebra generated by the cozero sets of X) is denoted by $Ba(X)$, the Baire sets of X; and the σ–algebra generated by the open sets of X is denoted by $Bo(X)$, the Borel sets of X.

We say that a collection $\mathcal{N}$ of subsets of X is directed upwards if whenever $O_1, O_2 \in \mathcal{N}$, there exists a set $O_3 \in \mathcal{N}$ with $O_1 \cup O_2 \subset O_3$. A countably additive measure μ defined on $Ba(X)$ is said to be *τ–smooth* or *τ–additive*, if for any upwards directed collection $\mathcal{N}$ of cozero subsets of X such that $\cup \mathcal{N} = X$, we have $\sup_{O \in \mathcal{N}} \mu(O) = \mu(X)$. A Borel measure μ is *τ–smooth* if for any upwards directed collection $\mathcal{N}$ of open subsets of X, we have $\sup_{O \in \mathcal{N}} \mu(O) = \mu(\cup \mathcal{N})$.

In [1] Robert F. Wheeler asks the following question (Problem 8.14): Is it true that if μ is a finite τ–smooth Baire measure on X, then there is a Lindelöf subset of X with full μ–outer measure? For the sake of brevity we say that a τ–smooth measure space $(X, Ba(X), \mu)$ has the L property if there is a μ–thick Lindelöf subset of X. A set $B \subset X$ is thick (or equivalently, has full outer measure) if the inner measure of the complement is zero. Clearly, a measure space with the L property is τ–smooth, since every cover of X by cozero sets contains a countable subcollection whose union has full measure; so Wheeler's question amounts to asking whether the L property characterizes the measure spaces that are τ–smooth. In this note we show that a negative answer is consistent with **ZFC**. We investigate the specific case of the Sorgenfrey plane with Lebesgue measure, and prove that whether it has the L property or not is undecidable in **ZF** (assuming inaccesible cardinals exist). More precisely, it will be shown that

i) there is a model of **ZF** where the Sorgenfrey plane with Lebesgue measure lacks the L property;

ii) under **ZFC + CH** the Sorgenfrey plane with Lebesgue measure has the L property;

iii) the existence of a τ–smooth measure space without the L property is consistent with **ZFC**.

It is well known that every τ–smooth Baire measure μ has a unique τ–smooth Borel extension $\hat{\mu}$. Clearly, a Lindelöf subset of a space X with full μ–outer measure will also have full $\hat{\mu}$–outer measure, so with respect to Wheeler's question it makes no difference whether we are working with a Baire measure or its Borel extension.

2. A model of ZF in which the L property fails.

Let **ZFC** be Zermelo-Frankel set theory together with the Axiom of Choice, and let **I** be the statement "there is an inaccessible cardinal". Under the assumption that there is a transitive ε–model of **ZFC + I**, Robert M. Solovay has proven (see [2], Theorems 1 and 4.1) that there exists a transitive ε–model $\mathcal{M}$ of **ZF**, in which the following hold:

i) The principle of dependent choice

ii) Every subset of the real line $\mathbf{R}$ is Lebesgue measurable

iii) Every subset of the plane $\mathbf{R}^2$ is Lebesgue measurable.

Denote the planar Lebesgue measure by λ^2. We are going to show that in $\mathcal{M}$, the Sorgenfrey plane $\mathbf{R}_\ell^2$ has no Lindelöf subsets with positive (outer) measure. Wheeler asks his question in the context of finite Baire measures. Lebesgue measure is not finite, but this does not represent a problem, since we can always restrict our attention to some appropriately chosen subset of the plane with finite measure. The symbol $\mathbf{R}_\ell$ denotes the real line with the Sorgenfrey topology (the topology generated by the semiopen intervals $[a, b)$), and $\mathbf{R}_\ell^2$ stands for the plane with the product Sorgenfrey topology. Clearly $Bo(\mathbf{R}^2) \subset Ba(\mathbf{R}_\ell^2)$, since $Ba(\mathbf{R}^2) = Bo(\mathbf{R}^2)$ (actually Bade has proven that $Ba(\mathbf{R}_\ell^2) = Bo(\mathbf{R}^2)$, cf. [3]); thus every Lebesgue subset of the plane can be approximated from inside by a Baire set (in the Sorgenfrey topology) with the same measure, and hence there is no need to restrict λ^2 to $Ba(\mathbf{R}_\ell^2)$, for the answer to Wheeler's question does not change.

2.1 THEOREM. *In the model $\mathcal{M}$, the Sorgenfrey plane lacks the L property.*

Proof: The Lebesgue measure λ^2 on $\mathbf{R}_\ell^2$ is τ–smooth because linear Lebesgue measure λ is τ–smooth on $\mathbf{R}_\ell$, and a product of two τ–smooth measures is τ–smooth (in fact, the product of an arbitrary number of τ–smooth probability measures is τ–smooth, cf. [4]). Let E be a subset of the plane with positive λ^2–measure. If we regard λ^2 on $\mathbf{R}_\ell^2$ as the product measure obtained by rotating the usual coordinate axes by forty five degrees (either clockwise or counterclockwise) and then assigning to each rotated axis the measure λ, then it follows from the Fubini-Tonelli Theorem that there is a line with slope –1 whose intersection with E is uncountable. Therefore E is not Lindelöf. Note that the Fubini-Tonelli Theorem holds in $\mathcal{M}$, since its proof does not use the full Axiom of Choice (Dependent Choice suffices). ⧫

Next, we consider the situation in **ZFC**. The symbol $\mathcal{L}$ stands for the Lebesgue sets of the plane.

3. In ZFC + CH the measure space $(\mathbf{R}^2_\ell, \mathcal{L}, \lambda^2)$ has the L property.

Let **CH** be the Continuum Hypothesis. Under **ZFC + CH** it is easy to show that $\mathbf{R}^2_\ell$ has a Lindelöf subset S with full λ^2–outer measure. This follows from the existence of a thick Sierpinski subset of the plane. A set is *Sierpinski* if it is uncountable and its intersection with every set of measure zero is at most countable. We remind the reader of how to produce a Sierpinski set S with full outer measure. The cardinality of the Borel sets (with the usual topology on the plane) is the same as the cardinality of the continuum, and therefore equal to $\aleph_1$ by the Continuum Hypothesis. Let $\{B_\alpha : \alpha < \aleph_1\}$ be the collection of all Borel sets of planar measure zero, and let $\{C_\alpha : \alpha < \aleph_1\}$ be the collection of all closed sets with positive planar measure. Inductively select $z_\alpha \in C_\alpha \setminus \cup \{B_\beta : \beta < \alpha\}$. Such choice is always possible, since $\cup\{B_\beta : \beta < \alpha\}$ is a countable union of sets of measure zero, while C_α has positive measure. Then the set $S := \{z_\alpha : \alpha < \aleph_1\}$ has the desired properties.

Let Λ be an uncountable index set. We call the set $\{(x_\alpha, y_\alpha) : \alpha \in \Lambda\}$ *uncountable and strictly decreasing* if the following conditions are satisfied: i) if $\alpha \neq \beta$, then $x_\alpha \neq x_\beta$; ii) if $x_\alpha < x_\beta$, then $y_\alpha > y_\beta$; iii) if $x_\alpha > x_\beta$, then $y_\alpha < y_\beta$.

3.1. THEOREM. *Under* **ZFC + CH**, *the Sorgenfrey plane has the L property.*

Proof: Every uncountable decreasing set $A \subset \mathbf{R}^2_\ell$ is contained in the graph of a monotone decreasing function f, defined on some interval. To see this, simply set $f(x) = \inf\{y_\alpha : (x_\alpha, y_\alpha) \in A \text{ and } x_\alpha \leq x\}$, provided that the infimum exists and that the set $\{y_\alpha : (x_\alpha, y_\alpha) \in A \text{ and } x_\alpha \leq x\}$ is not empty; otherwise, leave $f(x)$ undefined. Hence, A is a subset of a Borel set (in the usual topology) of planar measure zero. Let S be a Sierpinski set with full outer measure. Then S contains no uncountable decreasing set, and by the next lemma, it is Lindelöf. ⧫

We mention that the proof of Lemma 3.2 does not use the Continuum Hypothesis.

3.2. LEMMA. *A subset of $\mathbf{R}^2_\ell$ is Lindelöf if and only if it does not contain an uncountable, strictly decreasing subset. Furthermore, every Borel subset of $\mathbf{R}^2_\ell$ is Lebesgue measurable.*

Proof: Suppose $A \subset \mathbf{R}^2_\ell$ contains a strictly decreasing uncountable subset D. The relative closure $\overline{D}$ of D in A is not Lindelöf, since the collection $\{[a,b)\times[c,d) : (a,c) \in \overline{D}\}$ has not countable subcover. But every closed subset of a Lindelöf space is Lindelöf, so A is not Lindelöf.

For the other direction, assume $A \subset \mathbf{R}^2_\ell$ is not Lindelöf. Then there exists a cover $\mathcal{C} = \{[a_\alpha, b_\alpha)\times[c_\alpha, d_\alpha) : \alpha \in \Lambda\}$ of A without a countable subcover. Since the plane with the usual topology is Lindelöf, it follows that the set $\cup_\alpha(a_\alpha, b_\alpha)\times(c_\alpha, d_\alpha)$ can be covered with countably many elements from $\mathcal{C}$. Therefore the set $Z := (\cup_\alpha[a_\alpha, b_\alpha) \times [c_\alpha, d_\alpha))\backslash(\cup_\alpha(a_\alpha, b_\alpha) \times (c_\alpha, d_\alpha))$ is not Lindelöf. Note that Z is a countable union of sets none of which contains a strictly increasing pair of points (i.e., a pair (x_0, y_0), (x_1, y_1) with $x_0 < x_1$ and $y_0 < y_1$). To see why this is true, for each $n \in \mathbf{N}$ let $\{B^i(1/2n) : i \in \mathbf{N}\}$ be a countable cover of the plane by balls of radius $1/2n$, and let Z_n be the set $\{(x,y) \in Z : [x, x+1/n) \times [y, y+1/n) \subset [a_\alpha, b_\alpha)\times[c_\alpha, d_\alpha)$ for some $\alpha \in \Lambda\}$. Then for all $i, n \in \mathbf{N}$, $Z_n \cap B^i(1/2n)$ contains no increasing pair. Select i and n so that $Z_n \cap B^i(1/2n)$ is not Lindelöf. The number of horizontal lines that intersect $Z_n \cap B^i(1/2n)$ in more than one point is at most countable, as the following argument shows. For each such line H, select two points in $H \cap Z_n \cap B^i(1/2n)$. These two points are the endpoints of an open interval I_H. The projections of these intervals onto the x-axis must be disjoint, for else we would have an strictly increasing pair in $Z_n \cap B^i(1/2n)$. Thus, there are at most countably many intervals I_H. The same argument shows that the number of vertical lines that intersect $Z_n \cap B^i(1/2n)$ in more than one point is at most countable. By removing these horizontal and vertical lines from $Z_n \cap B^i(1/2n)$, we are left with a strictly decreasing set, which must be uncountable, since $Z_n \cap B^i(1/2n)$ is not Lindelöf.

To show that every set in $Bo(\mathbf{R}^2_\ell)$ is Lebesgue measurable, it is enough to prove it for the open sets in the Sorgenfrey topology. Let $\mathcal{C} = \{[a_\alpha, b_\alpha)\times[c_\alpha, d_\alpha) : \alpha \in \Lambda\}$ be an arbitrary collection of basic open sets. The result will follow if we show that

$$Z := (\cup_\alpha[a_\alpha, b_\alpha) \times [c_\alpha, d_\alpha))\backslash(\cup_\alpha(a_\alpha, b_\alpha) \times (c_\alpha, d_\alpha))$$

has measure zero. But we saw in the previous paragraph that Z can be expressed as a countable union of sets with no strictly increasing pairs. Since each such set is the union of a strictly decreasing set with a subset of a countable union of lines, it follows that Z has measure zero. ♦

Assuming the consistency of **ZFC + I**, it follows from Theorems 2.1 and 3.1

that one cannot decide in **ZF** whether or not $(\mathbb{R}^2_\ell, \mathcal{L}, \lambda^2)$ has the L property. But most mathematics is carried out within **ZFC**, and it is conceivable that by using the Axiom of Choice, one might be able to produce a thick Lindelöf subset in each τ-smooth measure space. We show next that this is not the case.

4. The existence of a subset of the Sorgenfrey plane which lacks the L property is consistent with ZFC.

Let $(\mathcal{P}(\mathbb{N}), \leq)$ denote the collection of all subsets of the natural numbers ordered by inclusion, and let **(A)** be the statement "every uncountable subset of $\mathcal{P}(\mathbb{N})$ contains an uncountable chain or an uncountable antichain". James E. Baumgartner has proven that if **ZF** is consistent, then so is **ZFC** + **MA** + **(A)** + $2^{\aleph_0} = \aleph_2$ (cf. Theorem 1 of [5]), where **MA** stands for Martin's Axiom. The reason why **(A)** is useful for our purposes is that it entails the following proposition: Every injective function from an uncountable set of reals into the reals is (strictly) monotone on an uncountable set (cf. [6], Theorem 6.14, page 947).

4.1. THEOREM. *It is consistent with* **ZFC** + **MA** + **(A)** $+2^{\aleph_0} = \aleph_2$ *that there is a* τ*-smooth measure space without the* L *property.*

Proof. If it is consistent with **ZFC** + **MA** + **(A)** $+2^{\aleph_0} = \aleph_2$ that $\mathbb{R}^2_\ell$ has no λ^2–thick Lindelöf subset, then we are done. So we assume that under **ZFC** + **MA** + **(A)** $+2^{\aleph_0} = \aleph_2$, $\mathbb{R}^2_\ell$ does have a Lindelöf subset E with full outer measure. By Lemma 3.2, E contains no uncountable strictly decreasing set. Let R denote the image of E under a 90 degree rotation (say, clockwise, for definiteness). Then R has full outer measure, and since every Borel subset of R is of the form $B \cap R$ for some B in $Bo(\mathbb{R}^2_\ell)$, it follows that the measure λ^2_R defined by $\lambda^2_R(B \cap R) = \lambda^2(B)$ is a τ-smooth Borel measure on the completely regular and Hausdorff space R (with the subspace topology from the Sorgenfrey plane).

Let $T \subset R$ be λ^2_R–thick. Select an injective function f defined on an uncountable subset of $\mathbb{R}$, such that its graph is contained in T. Such a function (an uncountable set of ordered pairs) can be defined by induction: Let $\beta < \omega_1$, and suppose that for $\gamma < \beta$, (x_γ, y_γ) has already been chosen. The countable union of lines $\cup_{\gamma<\beta}(\{x = x_\gamma\} \cup \{y = y_\gamma\})$ has measure zero, so we select $(x_\beta, y_\beta) \in T \setminus \cup_{\gamma<\beta} (\{x = x_\gamma\} \cup \{y = y_\gamma\})$, and then set $f = \cup\{(x_\beta, y_\beta) : \beta < \omega_1\}$. The function f is strictly monotone on an uncountable set of reals (by **(A)**). Since $f \subset R$, it is monotone decreasing on that set, for R contains no uncountable, strictly increasing subset. Thus, by Lemma 3.2, T is not Lindelöf. ♦

4.2. COROLLARY. *The existence of a* τ*-smooth measure space without the* L *property is consistent with* **ZFC** + **MA** $+2^{\aleph_0} = \aleph_2$.

Proof. By Theorem 4.1 and Baumgartner's result (Theorem 1 of [5]). ♦

Often one can prove results that hold in **ZFC + CH** by using only **ZFC + MA** (recall that the Continuum Hypothesis implies Martin's Axiom). Here we have an example where this is not the case. We have seen that the existence of a thick set $R \subset \mathbf{R}^2_\ell$ such that $(R, \lambda^2_R, Bo(R))$ lacks the L property is consistent with **ZFC + MA**. In **ZFC + CH**, however, given any thick set $R \subset \mathbf{R}^2_\ell$, the measure space $(R, \lambda^2_R, Bo(R))$ has the L property, by the same argument given for the plane in Section 3: Select a Sierpinski set $S \subset R$ with full λ^2_R–outer measure. Then S is Lindelöf.

5. Final remarks.

We have shown that Wheeler's question does not have a positive answer in **ZFC**, but it is still possible that by adding new axioms consistent with **ZFC** one might be able to prove that all τ–smooth measure spaces have the L property. To exclude this possibility it is necessary to exhibit a counterexample in **ZFC** without the use of any special axioms. Axiom **(A)** was utilized in Section 4 to produce a "large" (uncountable) set with a desired property (strictly decreasing), inside an arbitrary subset of the plane with full outer measure. Perhaps if one considers an uncountabale product of probability measure spaces such "large" sets occur naturally. But characterizing the Lindelöf subsets of uncountable products appears to be a difficult task. I do not know whether the L property holds or fails even for products of nice spaces, such as $(0,1)^{\aleph_1}$ with the product Lebesgue measure and the usual topology on $(0,1)$, or $\mathbf{N}^{\aleph_1}$ with the product measure determined by a probability measure on $\mathbf{N}$ whose support is infinite.

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