Real Analysis Exchange Vol. 17 (1991-92)

S. Agronsky, Department of Mathematics, California Polytechnic State University, San Luis Obispo, CA 93401
J. G. Ceder, Department of Mathematics, University of California, Santa Barbar, CA 93106

EACH PEANO SUBSPACE OF E^k IS AN ω-limit SET

As established in [ABCP] and [BS] a nonvoid closed subset F of $I = [0, 1]$ is anω-limit set for some continuous $f : R \to R$ if and only if F is nowhere dense or a union of finitely many nondegenerate closed intervals. In [AC] we tried to extend this result to E^k asking what compact sets in E^k can be ω-limit sets for some continuous function from E^k into E^k. In particular it was found that each totally disconnected compact set in E^k is such an ω-limit. However, we were unable to characterize those continua which can be ω-limit sets.

As a partial answer towards settling this problem we prove that any locally connected continuum in E^k is an ω-limit set. This answers a number of questions raised in [AC]. We also give some examples which suggest an attractive conjectural characterization.

Terminology and Notation:
Suppose $A \subseteq E^k$ and $f : A \to A$ and f is continuous. We define $f^0(x) = x$ and $f^{n+1}(x) = f(f^n(x))$ for each $x \in A$ and natural number n. An ω-limit set, $\omega(x, f)$ is defined to be the set of limit points of the sequence $\{f^n(x)\}_{n=0}^{\infty}$. In this paper we will be considering only bounded sequences $\{f^n(x)\}_{n=0}^{\infty}$ or compact ω-limit sets. We will use $\gamma(x, f)$ to denote the range of the sequence $\{f^n(x)\}_{n=0}^{\infty}$. Note that there exists a k such that either $\gamma(x_k, f) \subseteq \omega(x_k, f)$ or $\gamma(x_k, f) \cap \omega(x_k, f) = \phi$ where $x_k = f^k(x)$. We will say that an ω-limit set A is orbit enclosing if there exist f and x for which $\gamma(x, f) \subseteq \omega(x, f)$.

By a continuum we mean any nonvoid compact connected set. A continuum M is locally connected if M in the relative topology has a basis consisting of connected open sets. A locally connected continuum is usually called a Peano space and it is well known that when a Peano space is metrizable it is the continuous image of I. A set M is arcwise connected if each two points in M belong to some homeomorph of I. A set M is totally disconnected if the largest connected subsets of M are singleton sets. A continuum is indecomposable if it is not the union of two proper subcontinua.

An ordered n-tuple $\langle A_1, \ldots, A_n\rangle$ is called a chain if $A_i \cap A_{i+1} \neq \phi$ for each i. We say that a set B is chainable if there exist $A_1, \ldots, A_n$ such that $B = \{A_1, \ldots, A_n\}$ and $\langle A_1, \ldots, A_n\rangle$ is a chain.

The first result exploits the construction found in the proof of the existence of the "space filling curve" appearing in [N; p. 90]. In fact it is a strengthening of the well known theorem that a Peano space is the continuous image of I.

Theorem 1. *Suppose S is a nondegenerate Peano subspace of E^k and I is a closed line segment in E^k disjoint from S. Then $I \cup S$ is an orbit enclosing ω-limit set.*

Proof. When $k = 1$ the result is obvious from [ABCP]. We will carry out the proof for $k = 2$ from which the general case will be clear. Since $diam\ S > 0$ we may assume $diam\ S = 1$.

Let us now define a sequence of finite coverings of S by subcontinua, $\{\mathcal{G}_n\}_{n=0}^{\infty}$, as follows: Put $\mathcal{G}_0 = \{S\}$. Supposing $\mathcal{G}_n$ has been defined, the local connectedness of S allows us to construct a $\mathcal{G}_{n+1}$ (see [N] or [HY]) which is a finite covering of S by subcontinua such that

(1) each member $G \in \mathcal{G}_n$ is the union of a chainable subfamily $\mathcal{H}(G)$ of $\mathcal{G}_{n+1}$

(2) $0 <\ diam\ H < 2^{-n-1}$ for each $H \in \mathcal{G}_{n+1}$

Since $\bigcup_{n=0}^{\infty} \mathcal{G}_n$ is countable we may find a line L so that $diam\ \Pi(G) > 0$ for all $G \in \bigcup_{n=0}^{\infty} \mathcal{G}_n$ where Π is the projection mapping onto L. From Theorem 2 of [AC] if two sets are homeomorphic and one is an orbit enclosing ω-limit set, then so is the other. Therefore without loss of generality we may assume that S is a subset of the planar disk $\{z : |\ z - \frac{1}{2} - 2i\ | \leq \frac{1}{2}\}$, a and b belong to S where $a = 2i$ and $b = 1 + 2i$, L is the x-axis and $I = [0,1] \times \{0\}$. Put $\delta_n = \min\{diam\ \Pi(G) : G \in \mathcal{G}_n\}$. Then $\delta_n > 0$ for each n.

Next we will define a sequence $\{N_m\}_{m=0}^{\infty}$ of positive integers and a sequence of chains $\{\mathcal{C}_m\}_{m=0}^{\infty}$ of subcontinua of S as follows: Choose N_0 so that $N_0\delta_1 > 2$ and put $K_0^i = S$ for each $i \leq N_0$ and put $\mathcal{C}_0 = \langle K_0^1, \ldots, K_0^{N_0}\rangle$. Now suppose we have defined for each $i \leq m$ a positive integer N_i such that $N_i\delta_{i+1} > 2$ and a chain $\mathcal{C}_i = \langle K_i^1, \ldots, K_i^{N_i}\rangle$ where each $K_i^j \in \mathcal{G}_i$ and $a \in K_i^1$ and $b \in K_i^{N_i}$.

Pick $a_i \in K_m^i \cap K_m^{i+1}$ for each $i < N_m$ and put $a_0 = a$ and $a_{N_m} = b$. Since $\mathcal{H}(K_m^i)$ is chainable there exists a chain $\mathcal{C}_{im}$ whose coordinates are members of $\mathcal{H}(K_m^i)$ and whose first coordinate contains a_{i-1} and whose last coordinate contains a_i. Let $\alpha(i,m)$ be the number of coordinates in $\mathcal{C}_{im}$. Choose n_m such that $n_m N_m \delta_{m+2} > 2$ and $n_m \geq \max\{\alpha(i,m) : i \leq N_m\}$. Put $N_{m+1} = n_m N_m$. By including repetitions of coordinates in $\mathcal{C}_{im}$ we may assume each $\mathcal{H}(K_m^j)$ is chainable by $\langle K_{m+1}^{\beta}, \ldots, K_{m+1}^{\gamma}\rangle$ where $\beta = n_m(j-1)+1$ and $\gamma = n_m j$. Let $\mathcal{C}_{m+1}$ be the concatenation of the above chains so that the set of coordinates of $\mathcal{C}_{m+1}$ is $\{K_{m+1}^i : i \leq N_{m+1}\}$. Clearly the inductive hypothesis is satisfied.

Note that if $n_m(j-1) < i \leq n_m\ j$ then $K_{m+1}^i \subseteq K_m^j$. For $x \in [0,1)$ there exists for each m a unique j such that $j - 1 \leq xN_m < j$. Denote this j by x_m and put $A_m(x) = K_m^{x_m}$. Also put $A_m(1) = K_m^{N_m}$ for each m.

Let us show that $A_{m+1}(x) \subseteq A_m(x)$ for all m and x. For $x = 1$ it is obvious. If $x \neq 1$ we have to show $K_{m+1}^{x_{m+1}} \subseteq K_m^{x_m}$. For this it suffices to show $n_m(x_m - 1) < x_{m+1} \leq n_m x_m$. It is easy to see that

$$\frac{x_m - 1}{N_m} < \frac{x_{m+1} - 1}{N_{m+1}} \leq x < \frac{x_{m+1}}{N_{m+1}} \leq \frac{x_m}{N_m} .$$

By multiplying this inequality by N_{m+1} we obtain $n_m(x_m - 1) < x_{m+1} \leq n_m x_m$.

Therefore for each x, $\{A_m(x)\}_{m=0}^{\infty}$ is a descending sequence of compact sets whose
diameters tend to 0. Hence define $f(x)$ so that $\{f(x)\} = \bigcap_{m=0}^{\infty} A_m(x)$.

Now we show that $K_m^j \subseteq f([\frac{j-1}{N_m}, \frac{j}{N_m}])$ for each m and $j \leq N_m$. Letting $y \in K_m^j$ we may pick i_1 such that $y \in K_{m+1}^{i_1} \subseteq K_m^j$. Then there exists i_2 such that $y \in K_{m+2}^{i_2} \subseteq K_{m+1}^{i_1}$. Continuing in this way we obtain a sequence $\{i_k\}_{k=1}^{\infty}$ such that $y \in K_{m+k}^{i_k} \subseteq K_{m+k-1}^{i_{k-1}}$ and $n_{m+k}(i_{k-1} - 1) < i_k \leq n_{m+k}(i_k - 1)$ for all k. However $\{[\frac{i_k - 1}{N_{m+k}}, \frac{i_k}{N_{m+k}}]\}_{k=1}^{\infty}$ is a descending sequence whose intersection consists of a single point $x \in [\frac{j-1}{N_m}, \frac{j}{N_m}]$. Hence $y \in \bigcap_{k=1}^{\infty} K_{m+k}^{i_k} \subseteq \bigcap_{m=0}^{\infty} K_m^{x_m} = \{f(x)\}$.

If $x \in [\frac{j-1}{N_m}, \frac{j}{N_m})$, then $f(x) \in K_m^{x_m} = K_m^j$. If $x = \frac{j}{N_m}$, then $x \in [\frac{j}{N_m}, \frac{j+1}{N_m})$ so that $f(x) \in K_m^{j+1}$. Hence $K_m^j \subseteq f([\frac{j-1}{N_m}, \frac{j}{N_m}]) \subseteq K_m^j \cup K_m^{j+1}$ for all j and m.

If $| x - y | < \frac{1}{N_m}$ then there exists i such that both x and y belong to $[\frac{i-1}{N_m}, \frac{i+1}{N_m}]$. Hence, $f(x)$ and $f(y)$ belong to $K_m^i \cup K_m^{i+1} \cup K_m^{i+2}$. Hence, $| f(x) - f(y) | \leq 3 \cdot 2^{-m}$. It follows that f is continuous on I.

Since $f([\frac{j-1}{N_m}, \frac{j}{N_m})) \subseteq K_m^j$ and f is continuous $f(\frac{j}{N_m})$ must be a limit point of K_m^j. Hence, for all m and j

$$K_m^j = f([\frac{j-1}{N_m}, \frac{j}{N_m}]) .$$

From this it follows that $f(I) = S$.

Now extend f to $I \cup S$ by putting $f(x) = \Pi(x)$ for $x \in S$. Hence f is continuous from $I \cup S$ onto $I \cup S$ with $f(I) = S$ and $f(S) = I$. Moreover, using the Tietze extension theorem we may extend f continuously to all of E^2.

Let W be any open set hitting S. Then $K_m^i \subseteq W$ for some i and $m > 1$. Since $diam\ K_m^i \geq \delta_m > \frac{2}{N_{m-1}}$ the line segment $f(K_m^i)$ contains some $[\frac{j-1}{N_{m-1}}, \frac{j}{N_{m-1}}]$. Hence $[\frac{j-1}{N_{m-1}}, \frac{j}{N_{m-1}}] \subseteq f(K_m^i) \subseteq f(W)$. Then $K_{m-1}^j = f([\frac{j-1}{N_{m-1}}, \frac{j}{N_{m-1}}]) \subseteq f^2(W)$ and again since $diam\ K_{m-1}^j \geq \delta_{m-1} > \frac{2}{N_{m-2}}$, $f(K_{m-1}^j)$ contains some $[\frac{t-1}{N_{m-2}}, \frac{t}{N_{m-2}}]$ and $K_{m-2}^t = f([\frac{t-1}{N_{m-2}}, \frac{t}{N_{m-2}}]) \subseteq f^2(K_{m-1}^j) \subseteq f^4(W)$. Continuing in this way we eventually get α and β such that $K_0^{\alpha} = f([\frac{\alpha-1}{N_0}, \frac{\alpha}{N_0}]) \subseteq f^{\beta}(W)$. However since $S = K_0^{\alpha}$ we obtain $f^{\beta}(W) = S$ and $f^{\beta+1}(W) = I$.

Hence for each open sets U and V hitting $S \cup I$ there exists n such that $f^n(U) \cap V \neq \phi$. This is a well-known sufficient condition for $S \cup I$ to be an ω-limit set for f. (See [S])

Theorem 2. *If $\mathcal{A}$ consists of finitely many mutually disjoint nondegenerate Peano subspaces of E^k, then $U\mathcal{A}$ is an orbit enclosing ω-limit set. In particular any Peano subspace of E^k is an orbit enclosing ω-limit set.*

Proof. If $\mathcal{A}$ has one member A, choose a line segment I disjoint from A. From Theorem 1 there exists $x_0 \in A$ and a continuous f such that $I \cup A = \omega(x_0, f)$ and $\omega(x_0, f)$ is orbit enclosing. Clearly $A = \omega(x_0, f^2)$ and $I = \omega(f(x_0), f^2)$ and both are orbit enclosing.

Now let us do it when $\mathcal{A} = \{A, B\}$ with $A \neq B$. From this the general proof will be clear. Choose a line segment I disjoint from $A \cup B$. Carry out the construction of the proof of Theorem 1 where A^j_m and B^j_m play the role of K^j_m. Clearly we may select the sequence $\{N_m\}_{m=0}^{\infty}$ to be the same for both constructions. Suppose $\omega(x_o, f) = I \cup A$ and $\omega(y_0, g) = I \cup B$.

Define $h = g \circ f$ on A and $h = f \circ g$ on B. Then we may extend h continuously to E^k so that $h(A) = B$ and $h(B) = A$.

Let W be any open set hitting A. Pick j and m so that $A^j_m \subseteq W$. Then as in the proof of Theorem 1 there exists t such that $[\frac{t-1}{N_{m-1}}, \frac{t}{N_{m-1}}] \subseteq f(A^j_m) \subseteq f(W)$. Since $g([\frac{t-1}{N_{m-1}}, \frac{t}{N_{m-1}}]) = B^t_{m-1}$ we have $B^t_{m-1} = g([\frac{t-1}{N_{m-1}}, \frac{t}{N_{m-1}}]) \subseteq g\, f(W) = h(W)$. Then there exists s such that $[\frac{s-1}{N_{m-2}}, \frac{s}{N_{m-2}}] \subseteq g(B^t_{m-1})$. Hence $A^s_{m-2} = f([\frac{s-1}{N_{m-2}}, \frac{s}{N_{m-2}}]) \subseteq f\, g(B^t_{m-1}) = h(B^t_{m-1}) \subseteq h^2(W)$.

Continuing in this way we eventually get α and β for which $A^\alpha_0 \subseteq h^\beta(W)$. Since $f(A^\alpha_0) = I$ and $h(A^\alpha_0) = g\, f(A^\alpha_0) = g(I) = B$ we have $B = h^{\beta+1}(W)$ and $A = h^{\beta+2}(W)$. The same relationship holds when W is open and hits B.

Therefore for any open U and V hitting $A \cup B$ there exists n such that $h^n(U) \cap V \neq \phi$ and this is sufficient to make $A \cup B$ into an orbit enclosing ω-limit set.

An interesting consequence of Theorem 1 is the following.

Corollary. There exists $x_0 \in I$ and a continuous function $g : I \to I$ for which $I = \omega(x_0, g)$ and each level set of g is uncountable with the exception of possibly two.

Proof. Let S be the disk of Theorem 1. Then for each $x \in I$ $f^{-1}(x)$ is a segment. But $I = \omega(x_o, f^2)$ and clearly $g^{-1}(\lambda)$ is uncountable whenever $\lambda \in (0,1)$ where $g = f^2$.

Let us return to the question of what continua can be ω-limit sets. First of all not all continua are ω-limit sets. For example, as shown in [AC] adjoining an indecomposable continuum to a disk yields a continuum which is not an ω-limit set.

On the other hand [AC] proved that any continuum with empty interior is an ω-limit set. But it may not be orbit enclosing as shown by example 1 below. Note that an ω-limit set with nonempty interior must be orbit enclosing.

Example 1. There exists in E^2 a non arcwise connected continuum which is an ω-limit set but not an orbit enclosing ω-limit set.

Proof. Let S be the perimeter of the unit square with R its right edge. Let a be interior to S and let T be a spiral starting at a which approaches S (i.e. $S \subseteq T'$). Let W be a "sin $\frac{1}{x}$ curve" approaching R from outside S and beginning at b. Then X is a continuum which is not arcwise connected. By Theorem 7 of [AC] X is an ω-limit set.

Assume now that X is an orbit enclosing ω-limit set $\omega(x_0, f)$. Hence $f(X) = X$ and $f^n(x_0) \in X$ for all n. Since the continuous image of an arc must be locally connected, $f(A)$ is an arc whenever A is an arc. From this it follows that $f(W) \subseteq W \cup S$ or $f(W) \subseteq T \cup S$; $f(T) \subseteq T \cup S$ or $f(T) \subseteq W \cup S$; $f(S) \subseteq S$ or $f(S)$ is an arc in W or T.

Let us now show $f(S) = S$. Suppose $s \in S$ and $t \in T$ with $f(s) = t$. Let $t_n \in T$ with $t_n \to s$. Then $f(t_n) \to t$. Therefore $f(T) \subseteq T \cup S$ and it follows that $f^k(T) \subseteq T \cup S$ for all k. Hence $\{f^n(x_0)\}_{n=0}^{\infty}$ is eventually outside the open set W and can't be dense in W, a contradiction. Likewise we obtain a contradiction when $f(S) \cap W \neq \phi$.

Since $f(S) = S$ it follows that $f(W) \subseteq T \cup S$ and $f(T) \subseteq W \cup S$. Otherwise, say, $f(W) \subseteq W \cup S$ and $f^k(W) \subseteq W \cup S$ for all k, leading to a contradiction.

Suppose $f(W) \cap S \neq \phi$. Choose $w \in W$ and $s \in S$ for which $f(w) = s$. Then for any arc A containing w, $f(A)$ is an arc containing s and $f(A) \cap T = \phi$. It follows that $f(W) \cap T = \phi$ and $f(W) = S$. Therefore $T \cap f(X) = \phi$, a contradiction. Hence, we must have $f(W) = T$ and likewise $f(T) = W$.

Suppose $x \in S$. Choose $t_n \in T$ such that $t_n \to x$. Then $f(t_n) \in W$ and $f(t_n) \to f(x) \in \overline{W}$. Hence, $f(x) \in R$. Therefore $f(S) \subseteq R$ and $(S - R) \cap f(X) = \phi$, a contradiction.

This finishes the proof.

So the question should be divided into two questions: What are necessary and sufficient conditions for a continuum to be an ω-limit set? What are necessary and sufficient conditions for a continuum to be an orbit enclosing ω-limit set?

The converse of Theorem 2 is not true, that is, local connectedness is not a necessary and sufficient condition for being an orbit enclosing ω-limit set. This follows from the following example.

Example 2. There is an orbit enclosing ω-limit set in E^2 which is arcwise connected but not locally connected.

<u>First version:</u> Let h be the piecewise linear function from I into I with vertices $(0,0)$, $(\frac{1}{5},1)$, $(\frac{2}{5},0)$, $(\frac{3}{5},1)$, $(\frac{4}{5},0)$ and $(1,1)$. Using the equations of the line

segments of h it is easy to show in terms of "decimals" to base 5 that

$$h(.x_1x_2x_3\ \dots) = \begin{cases} .x_2x_3\dots & \text{if } x_1 \text{ is even} \\ .\overline{x}_2\overline{x}_3\dots & \text{if } x_1 \text{ is odd} \end{cases}$$

where $\overline{x}_k = 4 - x_k$.

Let Z be the set of integers and define a function α on Z as follows: $\alpha_0 = 0$, $\alpha_n = \frac{\pi}{2} - \frac{1}{n}$ when $n > 0$ and $\alpha_n = -\frac{\pi}{2} - \frac{1}{n}$ when $n < 0$. Let J be the closed line segment joining i to $-i$. For $n \in Z$ put $I_n = \{r\, e^{i\alpha_n} : 0 \leq r \leq 1\}$. Put $X = J \cup U\{I_n : n \in Z\}$.

Then X is arcwise connected but not locally connected. Define $f : X \to X$ by $f(y) = y$ if $y \in J$

$$f(e^{i\alpha_n}x) = \begin{cases} e^{i\alpha_{n+1}}h(x) & \text{if } x \in [0, \frac{2}{5}] \\ e^{i\alpha_{n-1}}h(x) & \text{if } x \in (\frac{2}{5}, 1] \end{cases}$$

It is easily verified that f is continuous.

If $x = e^{i\alpha_n}(.a_1a_2\ \dots\ )$, we will call n the <u>indicator</u> <u>of</u> x and $.a_1a_2\ \dots$ the <u>decimal</u> <u>part</u> <u>of</u> x. If $x \in X - J$, then $f(x) \in X - J$ and f increases the indicator by 1 if $a_1 = 0$ or 1 and decreases the indicator by 1 if $a_1 = 2, 3$ or 4. If a_1 is even, then the decimal part of $f(x)$ is $.a_2a_3\ \dots$ while if a_1 is odd the decimal part is $.\overline{a}_2\overline{a}_3\ \dots$. We will say an iterate $f^m(x)$ is <u>neutral</u> if the number of "ones" plus the number of "threes" in its first m digits is even.

Now we will construct a $z \in I_0$ such that $\omega(z, f) = X$. For this it will suffice to show for each $k \in Z$ and finite decimal $.y_1y_2\ \dots\ y_m$ there exists n and a sequence $\{\beta_j\}_{j=1}^{\infty}$ such that $f^n(z) = e^{i\alpha_k}.y_1y_2\ \dots\ y_m\ \beta_1\beta_2\ \dots$.

Let $\mathcal{B}$ consist of all finite strings whose elements are in $\{0, 1, 2, 3, 4\}$. Let $\{W_k\}_{k=0}^{\infty}$ be a $1-1$ enumeration of the denumerable set $Z \times \mathcal{B}$. We will construct the desired z by induction.

First suppose $W_0 = (n,\ .b_1\ \dots\ b_m)$. Let $z_1, z_2, \dots, z_{\xi_0}$ be given by $0, \dots, 0, b_1, b_2, \dots\ b_m$ where $\xi_0 = \mid n \mid + m$. Then $f^{|n|}(.z_1\ \dots\ z_{\xi_0}) = .b_1b_2\ \dots\ b_m e^{i\alpha_{|n|}}$.

Now suppose we have defined $\xi_0, \xi_1, \dots, \xi_n$ with $\xi_j < \xi_{j+1}$ and $\{z_j\}_{j=1}^{\xi_k}$ such that for each $j \leq k$ and $W_j = (n_j,\ .b_1\ \dots\ b_{m_j})$ we have $f^{|n_j|}(.z_1\ \dots\ z_{\xi_j}) = .b_1\ \dots\ b_{m_j} e^{i\alpha_{|n_j|}}$.

Suppose $W_{k+1} = (n,\ .b_1\ \dots\ b_m)$ and let v be the indicator of $f^{n_k}(.z_1\ \dots\ z_{\xi_k})$. Then we choose $z_{\xi_k+1}, \dots, z_{\xi_{k+1}}$ according to the cases given by the following table:

$f^{\|n_k\|}(.z_1 \ldots z_{\xi_n})$	$j = v - n$	$z_{\xi_k+1} \ldots z_{\xi_{k+1}}$
neutral	$j > 0$	$2\ 2 \ldots 2\ b_1 \ldots b_m$
neutral	$j < 0$	$0\ 0 \ldots 0\ b_1 \ldots b_m$
neutral	$j = 0$	$b_1 \ldots b_m$
non-neutral	$j > 0$	$2\ 2 \ldots 2\ 3\ b_1 \ldots b_m$
non-neutral	$j < 0$	$0\ 0 \ldots 0\ 1\ b_1 \ldots b_m$
non-neutral	$j = 0$	$1\,2\,b_1 \ldots b_m$

In the first 5 lines the numbers of 0's or 2's is determined by $\xi_{k+1} = \xi_k + \mid j \mid + m$

It is easily shown that $f^t(.z_1 \ldots z_{\xi_{k+1}}) = e^{i\alpha_n}\ .b_1 \ldots b_m$ where $t = \xi_{k+1} - m$.

Then $z = .z_1 z_2 \ldots$ obviously has the desired property, completing the proof.

Second Version: Let C be the unit circle in E^2 and K be a Cantor set in $(0, 2\pi)$. Put $X = \{r\,e^{iy} : 0 \leq r \leq 1,\ y \in K\}$. Define $h(x) = 2x$ if $0 \leq x \leq .5$ and $h(x) = 2(1-x)$ if $.5 < x \leq 1$. As is well-known there exists x_0 such that $\omega(x_0, h) = [0,1]$. By Theorem 13 of [ABCP] there exists a continuous $g : K \to K$ and a $y_0 \in K$ for which $\gamma(y_0, g) \subseteq \omega(y_0, g) = K$.

Both h and g have the following property (where $(f, A) = (h, C)$ or (g, K)): for all open sets U and V (relative to A) hitting A there exists m such that $f^{nm}(U) \cap V \neq \phi$ for all n. Now define $F : X \to X$ by $F(x, y) = h(x)e^{ig(y)}$. Then F is continuous and by the above property it follows that for each open sets U and V hitting X there exists n such that $F^n(U) \cap V \neq \phi$. Hence, $X = \omega(z_0, F)$ for some $z_0 \in X$.

Since the spaces in the two versions of Example 2 are radically different in that the first space has isolated "spokes" and the second space has no isolated "spokes" each version sheds some light on the following conjecture.

Conjecture. A continuum in E^k is an orbit enclosing ω-limit set if and only if it is arcwise connected.

Since a locally connected continuum is arcwise connected (see [N] or [HY]), the validity of the conjecture would imply our Theorem 2.

References

[ABCP] S. J. Agronsky, A. M. Bruckner, J. G. Ceder and T. L. Pearson, *The structure of ω-limit sets for continuous functions*, Real Analysis Exchange **15**, 483–510.

[AC] S. J. Agronsky and J. G. Ceder, *What sets can be ω-limit sets in E^n?*, to appear in Real Analysis Exchange.

[BS] A. M. Bruckner, J. Smital, *The structure of ω-limit sets for continuous maps of the interval*, to appear in Mat. Bohemica.

[HY] J. G. Hocking and G. S. Young, *Topology*, Addison-Wesley Publ. Co., 1961.

[N] M.H.A. Newman, *Topology of Plane Sets of Points*, Cambridge Univ. Press, 1961.

[S] A. N. Sharkovski, *Attracting and attracted sets*, Soviet Math. Dokl. **6** (1965), 268–270.

Received June 5, 1991