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# A Few $\sigma$-Ideals of Measure Zero Sets Related to their Covers

Abstract: After extending Borel's classification of measure zero sets in previous papers, we now look at some collections of measure zero sets naturally defined from properties of their covers and prove they actually form $\sigma$-ideals; we also study the parameters involved in their definitions, using some methods of Hausdorff measure introduced by Zakrzewski.

## 1 Introduction

Following Borel, we define a sequence of intervals $<I_n : n \in \mathbb{N}>$ to be a cover of $\mathcal{X}$ if both $\mathcal{X} \subseteq \bigcap_n \bigcup_{k \geq n} I_k$ and $\sum_{n \in \mathbb{N}} |I_n|$ converges. Therefore a set $\mathcal{X}$ has such a cover if and only if it has measure zero. Borel was interested in the rate of convergence of the above series of lengths for the various covers a measure zero set may have; that is, given a cover $<I_n : n \in \mathbb{N}>$ of $\mathcal{X}$, what can be said about the nonincreasing sequence converging to zero $f_I(n) = \sum_{k \geq n} |I_k|$ ? In [1], he studied the class of strong measure zero sets, that is those with arbitrarily fast rate of convergence. The general investigation was pursued by Fréchet in [2] and various authors of his time, but more recently in [3], [5], [6] and [9].

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We now know at least four different types of measure zero sets regarding the behaviour of their covers as above:

1. $\mathcal{X}$ is of strong measure zero, the $\mathcal{S}$-class.
   I.e., for all sequences of positive reals $<\epsilon_n : n \in \mathbb{N}>$, there is a cover $\mathcal{I} = <I_n : n \in \mathbb{N}>$ of $\mathcal{X}$ such that $\sum_{k \geq n} |I_k| < \epsilon_n$ for each $n$.

2. $\mathcal{X}$ belongs to the $\mathcal{H}$-class:

   **i)** There is a sequence of positive reals $<\epsilon_n : n \in \mathbb{N}>$ such that for all covers $\mathcal{I} = <I_n : n \in \mathbb{N}>$ of $\mathcal{X}$, $\sum_{k \geq n} |I_k| \geq \epsilon_n$ for infinitely many $n$.

   **ii)** There is a sequence of positive reals $<\delta_n : n \in \mathbb{N}>$ converging to zero such that for all (nonincreasing) sequences $<\gamma_n : n \in \mathbb{N}>$, if $\gamma_n \geq \delta_n$ infinitely often then $\mathcal{X}$ has a cover $\mathcal{I}$ satisfying $\sum_{k \geq n} |I_k| \leq \gamma_n$ for each $n$.

3. $\mathcal{X}$ belongs to the $\mathcal{U}$-class:

   **i)** There is an ultrafilter $\mathcal{U}_1$ and a sequence of positive reals $<\epsilon_n : n \in \mathbb{N}>$ such that for all covers $\mathcal{I} = <I_n : n \in \mathbb{N}>$ of $\mathcal{X}, \sum_{k \geq n} |I_k| \geq \epsilon_n$ for a set of $n$ in the ultrafilter $\mathcal{U}_1$.

   **ii)** There is an ultrafilter $\mathcal{U}_2$ and a sequence of positive reals $<\delta_n : n \in \mathbb{N}>$ converging to zero such that for all (nonincreasing) sequences $<\gamma_n : n \in \mathbb{N}>$, if $\gamma_n \geq \delta_n$ on a set of $n$ in $\mathcal{U}_2$, then $\mathcal{X}$ has a cover $\mathcal{I}$ satisfying $\sum_{k \geq n} |I_k| \leq \gamma_n$ for each $n$.

4. $\mathcal{X}$ belongs to the $\mathcal{L}$-class:
   There is a sequence of positive reals $<\epsilon_n : n \in \mathbb{N}>$ such that all covers $\mathcal{I}$ of $\mathcal{X}$ satisfy $\sum_{k \geq n} |I_k| \geq \epsilon_n$ for all but finitely many $n$.

All those classes have been shown to be nonempty in [3]; in particular any countable set is of strong measure zero, any uncountable closed set of measure zero belongs to the $\mathcal{H}$-class and a comeager set of measure zero belongs to the $\mathcal{L}$-class. We shall be interested in this note mostly to those three classes, although an example of a set $\mathcal{X}$ falling in the $\mathcal{U}$-class was also built in [3].

M. Goldstern recently remarked that the results of [3] show that all Borel sets of measure zero (actually all sets of measure zero with the Baire property) must belong to either the $\mathcal{S}$, $\mathcal{H}$ or the $\mathcal{L}$-class; we have also shown in [5] and [6] the consistency (relative to ZFC) that any measure zero set falls into one of these four classes, where even a fixed ultrafilter can be chosen for both cases of the $\mathcal{U}$-class. We do not know if, in ZFC alone, sets of measure zero must belong to one of the four classes.

We shall be interested here in ideals defined by those classes and the parameters involved in the definition of the $\mathcal{H}$ and $\mathcal{L}$-classes.

## 2 A few $\sigma$-ideals

It is well known that the collection $\mathsf{S}$ of strong measure zero sets, the $\mathcal{S}$-class, is a $\sigma$-ideal. We naturally consider the following collections.

By extension of the $\mathcal{H}$-class, we define:

**Definition 1** $\mathsf{H} = \{\mathcal{X}$ : *there is a sequence of nonnegative reals* $<\epsilon_n : n \in \mathbb{N}>$ *converging to 0 such that for all nonincreasing sequences* $<\delta_n : n \in \mathbb{N}>$*, if* $\delta_n \geq \epsilon_n$ *infinitely often then* $\mathcal{X}$ *has a cover* $I = <I_n : n \in \mathbb{N}>$ *satisfying* $\sum_{k \geq n} |I_k| \leq \delta_n$ *for each* $n$.

**Proposition 2** $\mathsf{H}$ *is a* $\sigma$*-ideal.*

**Proof:** $\mathsf{H}$ is clearly closed under subsets, so it suffices to show that given $\mathcal{X}_m$ in $\mathsf{H}$ with associated parameters $<\epsilon_n^m : n \in \mathbb{N}>$ for $m \in \mathbb{N}$, their union $\mathcal{X} = \bigcup \mathcal{X}_m$ is also in $\mathsf{H}$.

First choose a sequence $<a_n : n \in \mathbb{N}>$ of natural numbers such that:

1. $(\forall n)(\forall k \geq a_n) \sum_{m=1}^{n} \epsilon_k^m < 1/n$.
2. $(\forall n) n(a_{n-1}) \leq a_n$.

Now define $\epsilon_k = 2 \sum_{m=1}^{n} \epsilon_{a_n}^m$ for $a_{n+1} \leq k < a_{n+2}$.

Since $\epsilon_k < 2/n$ for $a_{n+1} \leq k < a_{n+2}$, the sequence $<\epsilon_n : n \in \mathbb{N}>$ converges to 0; we show that it witnesses membership of$\mathcal{X}$ in $\mathsf{H}$.

So we are given a nonincreasing sequence $<\delta_n : n \in \mathbb{N}>$ such that $\delta_n \geq \epsilon_n$ for infinitely many $n$. By construction of the $\epsilon_k$, we can find a sequence of natural numbers $<b_n : n \in \mathbb{N}>$ such that $\delta_{a_{b_n}} \geq \epsilon_{a_{b_n}}$ for each $n$. We then define nonincreasing sequences $<\delta_k^m : k \in \mathbb{N}>$ for $m \in \mathbb{N}$ by:

$$\delta_k^m = min\{\epsilon^m_{a_{b_{n+1}-1}}, \frac{\delta_{(a_{b_{j+1}-1})(b_{j+1})}}{2^{1+i}} : b_j + i \leq m\}$$

where $n$ is chosen so that $a_{b_n-1} < k \leq a_{b_{n+1}-1}$.

Since $\delta^m_{a_{b_n-1}} = \epsilon^m_{a_{b_n-1}}$ for all but finitely many $n$, we certainly have $\delta_k^m \geq \epsilon_k^m$ for infinitely many $k$ and hence we can use our assumption and obtain a cover $<I_n^m : n \in \mathbb{N}>$ of $\mathcal{X}_m$ for each $m$ such that $\sum_{k \geq n} |I_k^m| \leq \delta_n^m$ for each $n$.

Now list$\{I_n^m : m, n \in \mathbb{N}\}$ as $<I_n : n \in \mathbb{N}>$ in such a way that

$$\{I_k^m : m \leq b_n, k \leq a_{b_n-1}\} = \{I_k : k \leq (a_{b_n-1})(b_n)\}.$$

We certainly have $\mathcal{X} \subseteq \bigcap_n \bigcup_{k \geq n} I_k$ but also if $k$ satisfies $(a_{b_n-1})(b_n) < k \leq (a_{b_{n+1}-1})(b_{n+1})$, then

$$\begin{aligned}
\textstyle\sum_{m \geq k} |I_m| &\leq \textstyle\sum_{m \leq b_n; j \geq a_{b_n-1}+1} |I_j^m| &&+ \textstyle\sum_{m \geq b_n+1; j \in \mathbb{N}} |I_j^m| \\
&\leq \textstyle\sum_{m \leq b_n} \delta^m_{a_{b_n-1}+1} &&+ \textstyle\sum_{m \geq b_n+1} \delta_1^m \\
&\leq \textstyle\sum_{m \leq b_n} \epsilon^m_{a_{b_{n+1}-1}} &&+ \textstyle\sum_{j \in \mathbb{N}} \frac{\delta_{(a_{b_{n+1}-1})(b_{n+1})}}{2^{j+2}} \\
&\leq \frac{\epsilon_{a_{b_{n+1}}}}{2} &&+ \frac{\delta_{(a_{b_{n+1}-1})(b_{n+1})}}{2} \\
&\leq \delta_{a_{b_{n+1}}}/2 &&+ \frac{\delta_{(a_{b_{n+1}-1})(b_{n+1})}}{2} \\
&\leq \delta_{(a_{b_{n+1}-1})(b_{n+1})} &&\leq \delta_k
\end{aligned}$$

since $(a_{b_{n+1}-1})(b_{n+1}) \leq a_{b_{n+1}}$ by construction.

This shows that $<I_N : n \in \mathbb{N}>$ is the required cover of $\mathcal{X} = \bigcup \mathcal{X}_m$. □

The next collection consists of the measure zero sets not in $\mathcal{L}$, which we call the unbouded sets:

**Definition 3** $\mathsf{U} = \{\mathcal{X}$ : *for all sequences* $<\epsilon_n : n \in \mathbb{N}>$ *of positive reals, there is a cover* $<I_n : n \in \mathbb{N}>$ *of* $\mathcal{X}$ *such that* $\sum_{k \geq n} |I_k| \leq \epsilon_n$ *for infinitely many* $n$ $\}$.

As we mentionned before, $\mathsf{U}$ consistently consists of the $\mathcal{S}$, $\mathcal{H}$ and $\mathcal{U}$ classes but I do not know if this can be proved in ZFC alone.

**Proposition 4** $\mathsf{U}$ *is a* $\sigma$*-ideal.*

**Proof:** We again only need to show that given $\mathcal{X}_m$ in $\mathsf{U}$ for $m \in \mathsf{N}$, their union $\mathcal{X}$ is also in $\mathsf{U}$.

We thus fix a sequence $<\epsilon_n : n \in \mathsf{N}>$ of positive reals, and we may as well assume that $\epsilon_n \geq \epsilon_{n+1}$ for each $n$. First define a decreasing sequence $<\delta_n^0 : n \in \mathsf{N}>$ of positive rea ls such that $(n+2)\delta_n^0 \leq \epsilon_{\frac{n(n+1)}{2}}$ for each $n$.

Since $\mathcal{X}_0 \in \mathsf{U}$, we can choose a cover $<I_n^0 : n \in \mathsf{N}>$ of it and an infinite set of natural numbers $X_0 = \{x_n^0 : n \in \mathsf{N}\}$ containing without loss of generality 0 and such that $\sum_{k \geq x_n^0} |I_k^0| \leq \delta_{x_n^0}^0$ for each $n$.

Now to continue the construction inductively, assume that we have defined a positive decreasing sequence $<\delta_n^m : n \in \mathsf{N}>$, that we have a cover $<I_n^m : n \in \mathsf{N}>$ of $\mathcal{X}_m$ and an infinite set of natural numbers $X_m = \{x_n^m : n \in \mathsf{N}\}$ containing 0 such that $\sum_{k \geq x_n^m} |I_k^m| \leq \delta_{x_n^m}^m$ for each $n$.

Define then a decreasing sequence $<\delta_n^{m+1} : n \in \mathsf{N}>$ of positive reals such that $\delta_0^{m+1} \leq \delta_0^m/2$, and $\delta_n^{m+1} \leq \delta_{x_{n+1}^m}^m$ for each $n$. But because $\mathcal{X}_{m+1} \in \mathsf{U}$, we can find a cover $<I_n^{m+1} : n \in \mathsf{N}>$ of it and an infinite set of natural numbers $X_{m+1} = \{x_m^{m+1} : n \in \mathsf{N}\}$ containing 0 such that $\sum_{k \geq x_n^{m+1}} |I_k^{m+1}| \leq \delta_{x_n^{m+1}}^{m+1}$ for each $n$.

This completes the construction. List the obtained covers $<I_n^m : m, n \in \mathsf{N}>$ as $<J_n : n \in \mathsf{N}>$ in the usual Cantor fashion; that is given $n$, choose $m$ to be the largest integer such that $\frac{m(m+1)}{2} \leq n$, put $p = n - \frac{m(m+1)}{2}$ and define $J_n = I_{m-p}^p$.

Clearly $<J_n : n \in \mathsf{N}>$ is a cover of $\mathcal{X}$. We verify that $\sum_{k \geq n} |J_k| \leq \epsilon_n$ for infinitely many $n$.

So fix $m$ and put $y_0 = 0$; given $y_k$, for $k<m$, put

$$y_{k+1} = x_{y_k+1}^{m-k-1} \in X_{m-k-1}$$

and observe that $y_k<y_{k+1}$ (since we can always assume that $n<x_{n+1}^k$ for each $k, n$) and also $\delta_{y_k}^{m-k} \leq \delta_{y_m}^0$ for each $k$.

Then for each $m$, we get:

$$\begin{aligned}
\sum_{k \geq \frac{y_m(y_m+1)}{2}} |J_k| &= \sum_{k=0}^{y_m-1} \sum_{n \geq y_m-k} |I_n^k| && + \sum_{k=y_m}^{\infty} \sum_{n=0}^{\infty} |I_n^k| \\
&\leq \sum_{k=0}^{m-1} \sum_{n \geq y_{m-k}} |I_n^k| && + \sum_{k=m}^{\infty} \sum_{n=0}^{\infty} |I_n^k| \\
&\leq \sum_{k=0}^{m-1} \delta_{y_{m-k}}^k && + \sum_{k=m}^{\infty} \delta_0^k \\
&\leq m\delta_{y_m}^0 && + \delta_{y_m}^0 + \delta_{y_m}^0/2 + \delta_{y_m}^0/4 + \ldots \\
&= (m+2)\delta_{y_m}^0 \\
&\leq (y_m+2)\delta_{y_m}^0 \\
&\leq \epsilon_{\frac{y_m(y_m+1)}{2}}
\end{aligned}$$

as desired. □

The next collection would naturally be $\mathbf{L}$, the full ideal of measure zero sets which is of course a $\sigma$-ideal. This last ideal is Borel supported, that is contains a base of Borel sets; but the three $\sigma$-idaels $\mathbf{S}$, $\mathbf{H}$ and $\mathbf{U}$ are not quite as nice. Indeed any Borel set of measure zero containing a Luzin set (that is an uncountable set with countable intersection with every meager set) certainly cannot be meager itself, and therefore must be comeager in some interval and hence belongs to the $\mathcal{L}$-class. Since the Continuum Hypothesis allows one to build such a Luzin set, these three ideals are consistently not Borel supported. I do not know if this is provable in ZFC for $\mathbf{H}$ and $\mathbf{U}$, but Laver [7] has shown that the strong measure zero sets consistently consist of precisely the countable sets, hence $\mathbf{S}$ consistently is Borel supported. Another well know $\sigma$-ideal of measure zero sets is the ideal $\mathbf{M}$ of meager sets of measure zero, contained in $\mathbf{H}$. Consistently, these are different as again a Luzin set has strong measure zero but clearly does not belong to $\mathbf{M}$. I do not know if $\mathbf{M}$ could be equal to $\mathbf{H}$.

I also do not know if the union of $\mathcal{S}$, $\mathcal{H}$ and $\mathcal{U}$ is a $\sigma$-ideal, even whether it is closed under subsets in ZFC. Given a fixed ultrafilter $\mathcal{U}$, I do not know if the collection

$\{\mathcal{X}$ : there is a sequence $<\epsilon_n : n \in \mathbf{N}>$ of positive reals converging to 0 such that for all nonincreasing sequences $<\delta_n : n \in \mathbf{N}>$, if $\{n : \delta_n \geq \epsilon_n\} \in \mathcal{U}$ then there exists a cover $<I_n : n \in \mathbf{N}>$ of $\mathcal{X}$ such that $\sum_{k\geq n} |I_n| \leq \delta_n$ for each $n$ $\}$ is a $\sigma$-ideal.

# 3 The Parameters involved

We now shift our interest to the parameters involved in the $\mathcal{H}$-class (and the ideal $\mathbf{H}$) and the $\mathcal{L}$-class.

Let $\mathcal{X}$ be a member of $\mathcal{H}$, and write $<\epsilon_n^{\mathcal{X}} : n \in \mathbf{N}>$ for the sequence of reals converging to 0 witnessing membership in the ideal $\mathbf{H}$. This sequence can be made to converge arbitrarily fast by choosing $\mathcal{X}$ to be an apropriately defined Cantor set type of uncountable closed set. But more interesting is how slow can it be forced to be; a partial answer is given by the following.

**Proposition 5** *Fix a sequence of nonnegative reals* $<\epsilon_n : n \in \mathbf{N}>$ *such that* $\sum \epsilon_n < \infty$. *Then there is* $\mathcal{X} \in \mathbf{H}$ *such that if* $<\epsilon_n^{\mathcal{X}} : n \in \mathbf{N}>$ *witnesses membership of* $\mathcal{X}$ *in* $\mathbf{H}$, *then* $\epsilon_n^{\mathcal{X}} \geq \epsilon_n$ *for all but finitely many* $n$.

**Proof:** We follow the method of Zakrzewski ([9]). Define a continuous and nondecreasing function $h : \mathbf{R}^+ \to \mathbf{R}^+$ such that:

$$\lim_{x \to 0} \frac{h(x)}{x} = \infty \quad (1)$$
$$\sum_n h(\epsilon_n) < \infty. \quad (2)$$

Recall that this defines a Hausdorff measure $\mu_h$ by:

$$\mu_h(\mathcal{X}) = \sup_{\delta>0} \inf\{\sum_{k=0}^{\infty} h(|I_k|) : \mathcal{X} \subseteq \cup I_k \text{ and } |I_k| < \delta\}.$$

¿From a result of Besicovitch (see [8]), there is a closed $\mathcal{X}$ of positive but finite measure, that is $0 < \mu_h(\mathcal{X}) < \infty$.

By (1) and the finite Hausdorff measure, $\mathcal{X}$ has (Lebesgue) measure zero; and being closed, $\mathcal{X}$ belongs to $\mathsf{H}$. Now suppose for the sake of a contradiction that $\epsilon_n^{\mathcal{X}} < \epsilon_n$ for infinitely many $n$; then by membership in $\mathsf{H}$, $\mathcal{X}$ has a cover $<I_n : n \in \mathbf{N}>$ such that $\sum_{k \geq n} |I_k| \leq \epsilon_n$ for each $n$. But then $|I_n| \leq \epsilon_n$ for each $n$ and therefore

$$\sum_{k \geq n} h(|I_k|) \leq \sum_{k \geq n} h(\epsilon_n) \to_{n \to \infty} 0$$

and hence $\mu_h(\mathcal{X}) = 0$, a contradiction. □

**Problem 1** *Can we hope for a similar result assuming only that* $<\epsilon_n : n \in \mathbf{N}>$ *converges to 0?*

The major difficulty here is that given $\sum_{k \geq n} \delta_k \leq \sum_{k \geq n} \epsilon_k$ for each $n$, we cannot say much about the individual terms $\delta_n$ and $\epsilon_n$.

**Problem 2** *Is there a measure zero set* $\mathcal{X}$ *such that* $\sum_n \sum_{k \geq n} |I_k| = +\infty$ *for each of its covers* $<I_n : n \in \mathbf{N}>$*?*

Such a set cannot be included in a $F_\sigma$-set of measure zero. But the most promising candidates are comeager sets of measure zero; so we move on to the study of the parameters involving the $\mathcal{L}$-class. Again, we write $<\epsilon_n^{\mathcal{X}} : n \in \mathbf{N}>$ for the sequence of positive reals witnessing membership of a set $\mathcal{X}$ in $\mathcal{L}$.

We extract from [3] the following proposition showing the existence of a set in $\mathcal{L}$, but carefully calculating the parameter involved. We work with functions $f \in 2^{\mathbf{N}}$ identified with points on the unit interval [0,1]; so in particular intervals of [0,1] correspond to intervals in the lexicographic order on $2^{\mathbf{N}}$.

**Proposition 6** *([3]) Given a sequence of positive integers $<v_n : n \in \mathbb{N}>$ such that $\sum_n 2^{-v_n} < \infty$, then there is a measure zero set $\mathcal{X}$ such that given any of its cover $<I_n : n \in \mathbb{N}>$ , we have $\sum_{k \geq n} |I_k| \geq 2^{-r_n - 1}$ for all but finitely many $n$, where $r_n = \sum_{k=0}^{n} v_k$.*

**Proof:** Let $\mathcal{X} = \{f \in 2^{\mathbb{N}} : f \restriction [r_k, r_{k+1}) \equiv 0$ for infinitely many $k$ $\}$. $\mathcal{X}$ is clearly (comeager) of measure zero as $\mathcal{X} = \bigcap_n \bigcup_{k \geq n} \{f : f \restriction [r_k, r_{k+1}) \equiv 0\}$. Now fix a cover $<I_n : n \in \mathbb{N}>$ of $\mathcal{X}$ and assume for the sake of a contradiction that an infinite sequence of natural numbers can be found such that $\sum_{k \geq n_i} |I_k| < 2^{-r_{n_i} - 1}$ for $i \in \mathbb{N}$. We may assume without loss of generality that $n_0 = 0$. Put $s_0 = \emptyset$. By induction, given $s_k \in 2^{r_{n_k}}$ such that $N_{s_k} \cap \bigcup_{m=0}^{n_k - 1} I_m = \emptyset$ (where $N_s = \{f \in 2^{\mathbb{N}} : s \subseteq f\}$), we try to extend $s_k$ to $s \in 2^{r_{k+1} - 1}$ such that

$$N_s \cap \bigcup_{m=n_k}^{n_{k+1}-1} I_m = \emptyset.$$

But this is possible since for each such $s$, $N_s$ has measure $2^{-r_{n_{k+1}-1}}$, so at most $2^{r_{n_{k+1}-1} - r_{n_k} - 1}$ are covered by $\bigcup_{m=n_k}^{n_{k+1}-1} I_m$. Also, at most $2(n_{k+1} - n_k)$ of them intersect the $I_m$ without beeing included. Hence remains at least $2^{r_{n_{k+1}-1} - r_{n_k}} - 2^{r_{n_{k+1}-1} - r_{n_k} - 1} - 2(n_{k+1} - n_k) > 0$ if $n_{k+1}$ is large enough compared to $n_k$.

Lastly, extend $s \in 2^{r_{n_{k+1}-1}}$ to $s_{k+1} \in 2^{r_{n_{k+1}}}$ so that $s_{k+1} \restriction [r_{n_{k+1}-1}, r_{n_{k+1}}) \equiv 0$.

We conclude that $f = \cup s_k \in \mathcal{X} \setminus \bigcup_{n=0}^{\infty} I_n$, a contradiction to our assumption on $\mathcal{X}$. □

**Corollary 7** *Suppose that $<\epsilon_n : n \in \mathbb{N}>$ is a sequence of positive reals (converging to 0) such that $\sum_n \frac{\epsilon_{n+1}}{\epsilon_n} < \infty$, then there is an $\mathcal{X}$ of measure zero such that for any of its covers $<I_n : n \in \mathbb{N}>$, $\sum_{k \geq n} |I_k| \geq \epsilon_n$ for all but finitely many $n$.*

**Proof:** Use the last proposition with $r_n = \lceil \log 1/\epsilon_n \rceil - 2$ and $v_n = r_n - r_{n-1}$. □

There seems to be however an extraordinary overkill, not only this does not answer problem 2, but a priori any sequence $<\epsilon_n : n \in \mathbb{N}>$ of positive reals converging to 0 is a possible candidate for a parameter in the $\mathcal{L}$-class.

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