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STRONG DENSITY TOPOLOGIES WITH RESPECT TO MEASURE AND CATEGORY

In [2] the notions of $\mathcal{I}$-density point and strong $\mathcal{I}$-density point of a plane set have been introduced and studied in comparison with the one-dimensional case. Moreover, the topologies, called $\mathcal{I}$-density topology and strong $\mathcal{I}$-density topology, associated with $\mathcal{I}$-approximately continuous functions and strong $\mathcal{I}$-approximately continuous functions respectively, have been introduced and investigated. In particular, the problem of finding the coarsest topology under which $\mathcal{I}$-approximately continuous functions are continuous has been solved. That topology has been called the deep $\mathcal{I}$-density topology.

An essential role is played in this question by a Lusin-Menchoff theorem in the sense of category.

A similar problem concerning the coarsest topology under which all strongly approximately continuous functions (with respect to Lebesgue measure) are continuous, is unsolved.

The strong density topology, i.e. the topology associated with strongly approximately continuous functions, is not the coarsest topology under which all strongly approximately continuous functions are continuous (see [3]).

For this purpose, a topology $\tilde{d}$ which is strictly coarser than the strong density topology and which also makes continuous all strongly approximately continuous functions, has been introduced in fact in [3]. It is unknown whether or not the $\tilde{d}$ topology is the coarsest topology under which all strongly approximately continuous functions are continuous.

In this paper the deep strong $\mathcal{I}$-density topology is studied and the question if it is the coarsest topology under which all strongly $\mathcal{I}$-approximately continuous functions are continuous is solved with negative answer (see th. 6).

At the same time it is proved that a Lusin-Menchoff theorem doesn't hold either for the deep strong $\mathcal{I}$-density topology (see th. 5) or for the $\tilde{d}$ topology (see remark 2).

Throughout the paper S will denote the class of plane sets having the Baire property. $\mathcal{I}$ will denote the σ-ideal of sets of R^2 of the first category.

We shall say that a property holds $\mathcal{I}$-a.e. in X if it holds for all points of X except a set of the first category. The symmetric difference of the sets A and B is denoted by $A \triangle B$. For $A, B \in S$ $A \sim B$ will always mean $A \triangle B \in \mathcal{I}$.

χ_A will mean the characteristic function of the set A. The euclidean distance in R^2 will be denoted by ρ.

For $n, m \in N$ and $A \in S$ we put $(n,m) \cdot A = \{(nx, my) : (x,y) \in A\}$.

Def 1: *$(0,0)$ is a strong $\mathcal{I}$-density point of $A \in S$ if and only if for every increasing sequences $\{k'_n\}_{n\in N}$ and $\{k''_n\}_{n\in N}$ of natural numbers there exist subsequences $\{k'_{n_p}\}_{p\in N}$ and $\{k''_{n_p}\}_{p\in N}$ for which* $\{\chi_{((k'_{n_p},k''_{n_p})\cdot A\cap[-1,1]^2}\}_{p\in N}$ converges to 1 $\mathcal{I}$-a.e.

Def 2: *A point (x_0, y_0) is a strong $\mathcal{I}$-density point of $A \in S$ if and only if $(0,0)$ is a strong $\mathcal{I}$-density point of the set $A - (x_0, y_0) = \{(x - x_0, y - y_0) : (x,y) \in A\}$.*

Def 3: *A point (x_0, y_0) is a strong $\mathcal{I}$-dispersion point of $A \in S$ if and only if (x_0, y_0) is a strong $\mathcal{I}$-density point of $R^2 - A$.*

For $A \in S$ $\varphi_s(A)$ will denote the set of all strong $\mathcal{I}$-density points of the set A. For $A, B \in S$ the following properties hold (see [2] th. 3'):

1. $\varphi_s(A) \sim A$
2. If $A \sim B$, then $\varphi_s(A) = \varphi_s(B)$
3. $\varphi_s(\emptyset) = \emptyset$, $\varphi_s(R^2) = R^2$
4. $\varphi_s(A \cap B) = \varphi_s(A) \cap \varphi_s(B)$.

Def 4: *A point (x_0, y_0) is a deep strong $\mathcal{I}$-density point of A if and only if there exists a set B, open in the natural topology, such that $B \supset R^2 - A$ and (x_0, y_0) is a strong $\mathcal{I}$-dispersion point of B.*

The following result holds (see [1], lemma 1.5):

Lemma 1: *If (x_0, y_0) is a deep strong $\mathcal{I}$-density point of the set $A \in S$, then (x_0, y_0) is a deep strong $\mathcal{I}$-density of A in the direction of x and y-axes.*

Let $\mathcal{T}_{\mathcal{I}}^s$ denote the topology introduced in [2] as the family of sets $A \in S$ such that $A \subset \varphi_s(A)$. Denote by $\mathcal{T}_s$ the family of sets $A \in S$ such that each point of A is a deep strong $\mathcal{I}$-density point of A. We omit an easy proof of the following:

Theorem 1: *$\mathcal{T}_s$ is a topology (on the plane) coarser than $\mathcal{T}_{\mathcal{I}}^s$.*

Observe that the set $A = R^2 - \{(0, +\infty) \times \{0\}\} \in \mathcal{T}_{\mathcal{I}}^s$. Since $(0,0)$ cannot be a deep strong $\mathcal{I}$-density point of A (by Lemma 3), $A \notin \mathcal{T}_s$.

Lemma 2: *If $A \in \mathcal{T}_s$, then $A = G_1 \cup P_1$, where G_1 is an open set and P_1 is a nowhere dense set.*

Proof. It is obvious if $A = \emptyset$. If not, let $(x_0, y_0) \in A$. Then there exists a closed set $F \subset A$ such that $(x_0, y_0) \in \varphi_s(F)$. Thus $\mathring{F} \neq \emptyset$. Let $G_1 = \mathring{A}$ and $P_1 = A - \mathring{A}$. Since $P_1 \subset Fr(G_1)$, P_1 is nowhere dense.

Lemma 3: *If $A = G_1 \cup P_1$, where G_1 is an open set, P_1 is a nowhere dense set and $A \subset \varphi_s(A)$, then $R^2 - A = G_2 \cup P_2$, where G_2 is an open set and P_2 is a nowhere dense set.*

Proof: It is similar to the proof of the analogous lemma 3 of [2].

Def 5: *A point (x_0, y_0) is a deep strong $\mathcal{I}$-density point of $A \in S$ in the direction of x-axis (y-axis) if and only if x_0 (y_0) is a deep $\mathcal{I}$-density point of the set*

$$\{t \in R : (t, y_0) \in A\} \quad (\{t \in R : (x_0, t) \in A\})^1$$

Lemma 4: *Let $(0,0)$ be a deep strong $\mathcal{I}$-density point of $A \in S$. Then, for each $P \subset R^2 - A$ we have: $(*)$ for any increasing sequences $\{n_m\}_{m \in N}$ and $\{k_m\}_{m \in N}$ of natural numbers, there exist subsequences $\{n_{m_p}\}_{p \in N}$ and $\{k_{m_p}\}_{p \in N}$ such that for each rectangle $(a', b') \times (a'', b'') \subset [-1, 1]^2$ there exist a rectangle $(c', d') \times (c'', d'') \subset (a', b') \times (a'', b'')$ and a natural number $r \in N$ such that:*

1. $\bigcup_{p=r}^{\infty} ((n_{m_p}, k_{m_p}) \cdot P) \cap ((c', d') \times (c'', d'')) = \emptyset$;
2. $\bigcup_{p=r}^{\infty} ((n_{m_p}, 1) \cdot P) \cap ((c', d') \times \{0\}) = \emptyset$;
3. $\bigcup_{p=r}^{\infty} ((1, k_{m_p}) \cdot P) \cap (\{0\} \times (c'', d'')) = \emptyset$.

Proof: Suppose that the above statement is not true. Then there exist $P \subset R^2 - A$ and two increasing sequences $\{n_m\}_{m \in N}$ and $\{k_m\}_{m \in N}$ such that for all subsequences $\{n_{m_p}\}_{p \in N}$ and $\{k_{m_p}\}_{p \in N}$ there exist a rectangle $(a', b') \times (a'', b'') \subset$

[1] The definition of deep $\mathcal{I}$-density point of a set on the line is, with obvious changes, similar to the def. 4 (see also [6], def. 6).

$[-1,1]^2$ such that for each $r \in N$ and for each rectangle $(c',d') \times (c'',d'') \subset (a',b') \times (a'',b'')$ we have not 1. or not 2. or not 3. .

There are two possibilities: for each $r \in N$ the set $\bigcup_{p=r}^{\infty}((n_{m_p}, k_{m_p}) \cdot P)$ is dense in $(a',b') \times (a'',b'')$ or for some $r_0 \in N$ $\bigcup_{p=r}^{\infty}((n_{m_p}, k_{m_p}) \cdot P)$ is not dense in $(a',b') \times (a'',b'')$ for $r \geq r_0$.

In the first case, if B is an open set which contains P, then for each $r \in N$ the set $\bigcup_{p=r}^{\infty}(n_{m_p}, k_{m_p}) \cdot B$ is residual in $(a',b') \times (a'',b'')$, a contradiction.

In the second case there exists $(a_1',b_1') \times (a_1'',b_1'') \subset (a',b') \times (a'',b'')$ such that $\bigcup_{p=r}^{\infty}((n_{m_p}, k_{m_p}) \cdot P \cap ((a_1',b_1') \times (a_1'',b_1'')) = \emptyset$ for each $r \geq r_0$. Therefore for each $(c',d') \times (c'',d'') \subset (a_1',b_1') \times (a_1'',b_1'')$ and for each $r \geq r_0$ 1. holds. If the set $\bigcup_{p=r}^{\infty}(n_{m_p}, 1) \cdot P$ is dense in $(a_1',b_1') \times \{0\}$ for each $r \geq r_0$, then for any open set $B \supset P$ and for each $r \in N$ $\bigcup_{p=r}^{\infty}(n_{m_p}, 1) \cdot B$ is residual in $(a_1',b_1') \times \{0\}$, a contradiction by Lemma 1. Then the set $\bigcup_{p=r}^{\infty}((n_{m_p}, 1) \cdot P)$ is not dense in $(a_1',b_1') \times \{0\}$, for some $r_1 \geq r_0$. Consequently, there exists $(a_2',b_2') \subset (a_1',b_1')$ such that $\bigcup_{p=r}^{\infty}((n_{m_p}, 1) \cdot P) \cap (a_2',b_2') \times \{0\} = \emptyset$ for $r \geq r_1$.

Thus for any rectangle $(c',d') \times (c'',d'') \subset (a_2',b_2') \times (a_1'',b_1'')$ and for $r \geq r_1$ 1. and 2. hold.

Therefore, for each $r \geq r_1$ and for each $(c',d') \times (c'',d'') \subset (a_2',b_2') \times (a_1'',b_1'')$ we get $\bigcup_{p=r}^{\infty}((1, k_{m_p}) \cdot P) \cap (\{0\} \times (c'',d'')) \neq \emptyset$. Thus, for each $r \geq r_1$ $\bigcup_{p=r}^{\infty}(1, k_{m_p}) \cdot P$ is dense in $\{0\} \times (a_1'',b_1'')$ and $\bigcup_{p=r}^{\infty}(1, k_{m_p}) \cdot B$ is dense in $\{0\} \times (a_1'',b_1'')$, for any open set $B \supset P$, a contradiction by Lemma 1.

Theorem 2: *Let $A \in S$ such that $R^2 - A = G \cup P$, where G is open (in the natural topology) and P is a nowhere dense set, and let $(0,0) \in A$.*

Then $(0,0)$ is a deep strong $\mathcal{I}$-density point of A if and only if the condition $(*)$ of the lemma 4 is satisfied (for the set P).

Proof: Since $P \subset R^2 - A$ and by lemma 4, it is obvious that, if $(0,0)$ is a deep strong $\mathcal{I}$-density point of A, then P satisfies the condition $(*)$.

Conversely, if the condition $(*)$ of the lemma 4 is fulfilled, we shall construct an open set $B \supset R^2 - A$ such that $(0,0)$ is a strong $\mathcal{I}$-dispersion point of B.

With the denotation of lemma 4 we are looking for a set B_1 such that $B_1 \supset P$ and $(0,0)$ is a strong $\mathcal{I}$-dispersion point of B_1. The requested set B will be $B_1 \cup G$.

In order to do that, let $(x,y) \in P$. Put

$$B_{x,y} = (x - \delta_{x,y}, x + \delta_{x,y}) \times (y - \delta_{x,y}, y + \delta_{x,y}),$$

where $0 < \delta_{x,y} < \min\{\rho^2((x,0),(0,0)), \rho^2((0,y),(0,0))\}$ if $x \neq 0$ and $y \neq 0$, $0 < \delta_{x,y} < \rho^2((0,0),(0,y))$ if $x = 0$ and $0 < \delta_{x,y} < \rho^2((x,0),(0,0))$ if $y = 0$.

Let $B_1 = \bigcup_{(x,y) \in P} B_{x,y}$. Obviously B_1 is open and $B_1 \supset P$.

Let $\{n_m\}_{m\in N}$ and $\{k_m\}_{m\in N}$ be arbitrary increasing sequences of natural numbers. Let $\{n_{m_p}\}_{p\in N}$ and $\{k_{m_p}\}_{p\in N}$ be the subsequences of $\{n_m\}_{m\in N}$ and $\{k_m\}_{m\in N}$ from the condition $(*)$ of lemma 4. We shall show that $\lim_p \sup((n_{m_p}, k_{m_p})\cdot B_1)\cap [-1,1]^2$ is a nowhere dense set and, consequently, that $(0,0)$ is an $\mathcal{I}$-dispersion point of B_1. Let $(a',b')\times(a'',b'')\subset[-1,1]^2$. From condition $(*)$ there exist $(c',d')\times(c'',d'')\subset(a',b')\times(a'',b'')$ and $r\in N$ such that 1., 2. of lemma 4 are satisfied. We shall prove that $\lim_p \sup((n_{m_p}, k_{m_p})\cdot B_1)\cap((c',d')\times(c'',d''))=\emptyset$.

Suppose the contrary. Then there exists $(x',y')\in \lim_p\sup((n_{m_p},k_{m_p})\cdot B_1\cap ((c',d')\times(c'',d''))$ and, consequently, there exists a subsequence $\{(n_{m_{p_s}},k_{m_{p_s}})\}_{s\in N}$ such that for each $s\in N$ $(x',y')\in(n_{m_{p_s}},k_{m_{p_s}})\cdot B_1$. Then we can construct a sequence $(x_s,y_s)_{s\in N}$ of points of P such that for each $s\in N$ we have that $(x',y')\in(n_{m_{p_s}},k_{m_{p_s}})\cdot(x_s-\delta_{x_s,y_s},x_s+\delta_{x_s,y_s})\times(y_s-\delta_{x_s,y_s},y_s+\delta_{x_s,y_s})$.

Obviously $\{(x_s,y_s)\}_{s\in N}$ tends to $(0,0)$.

Suppose that there exists a subsequence $\{(x_{s_r},y_{s_r})\}_{r\in N}$ of $\{(x_s,y_s)\}_{s\in N}$ such that, for each $r\in N$ $x_{s_r}\neq 0$ and $y_{s_r}\neq 0$. Since x' is also of the form $x'=n_{m_{p_s}}\cdot x'_s$ and $y'=k_{m_{p_s}}\cdot y'_s$, where $x'_s\in(x_s-\delta_{x_s,y_s},y_s+\delta_{x_s,y_s})$ and $y'_s\in(y_s-\delta_{x_s,y_s},y_s+\delta_{x_s,y_s})$ for every $s\in N$, we have for any $r\in N$

$$\frac{|x'-n_{m_{p_{s_r}}}x_{s_r}|}{|n_{m_{p_{s_r}}}x_{s_r}|}\le\frac{|x'_{s_r}-x_{s_r}|}{|x_{s_r}|}\le\frac{\delta_{x_{s_r},y_{s_r}}}{|x_{s_r}|}\le\frac{|x_{s_r}|^2}{|x_{s_r}|}=|x_{s_r}|.$$

So $\{n_{m_{p_{s_r}}}x_{s_r}\}_{r\in N}$ tends to x'.

Analogously we can prove that $\{k_{m_{p_{s_r}}}y_{s_r}\}_{r\in N}$ tends to y'. Thus, $(n_{m_{p_{s_r}}},k_{m_{p_{s_r}}})\cdot(x_{s_r},y_{s_r})\in(c',d')\times(c'',d'')$ for sufficiently big $r\in N$, but that contradicts 1. of the condition $(*)$.

Suppose now that there exists a subsequence $\{(x_{s_r},y_{s_r})\}_{r\in N}$ of $\{(x_s,y_s)\}_{s\in N}$ such that $y_{s_r}=0$ for each $r\in N$, then, as above we may conclude that $\{n_{m_{p_{s_r}}}x_{s_r}\}_{r\in N}$ tends to x' and, consequently, that $n_{m_{p_{s_r}}}x_{s_r}\in(c',d')$ and $y_{s_r}=0$ for sufficiently big $r\in N$.

Therefore $(n_{m_{p_{s_r}}}x_{s_r},y_{s_r})\in(c',d')\times\{0\}$ but this contradicts 2. of condition $(*)$.

Suppose, finally, that there exists a subsequence $\{(x_{s_r},y_{s_r})\}_{r\in N}$ such that $x_{s_r}=0$ for each $r\in N$. With analogous argument we obtain a contradiction too.

<u>Def. 6</u>: *We say that $(0,0)$ is a strong $\mathcal{I}$-density point of $A\in S$ with respect to the first quarter of the plane if, for all increasing sequences $\{n_m\}_{m\in N}$ and $\{k_m\}_{m\in N}$ of positive integers, there exist subsequences $\{n_{m_p}\}_{p\in N}$ and $\{k_{m_p}\}_{p\in N}$ for which $\{\chi_{((n_{m_p},k_{m_p})\cdot A)\cap[0,1]^2}\}_{p\in N}$ converges to 1 $\mathcal{I}$-a.e.*

<u>Def. 7</u>: *We say that $(0,0)$ is a strong $\mathcal{I}$-dispersion point of $A\in S$ with respect to the first quarter of the plane if and only if $(0,0)$ is a strong $\mathcal{I}$-density point of R^2-A with respect to the first quarter of the plane.*

In the standard way we extend the above definitions to the points different than $(0,0)$. Analogous definitions can be formulated for the remaining quarters.

The set of strong $\mathcal{I}$-density ($\mathcal{I}$-dispersion) points of the set A with respect to the first quarter is denoted by $\varphi_s^{++}(A)$ ($\psi_s^{++}(A)$). For the remainig quarters we use the symbols $\varphi_s^{-+}(A)$, $\varphi_s^{--}(A)$, $\varphi_s^{+-}(A)$, ($\psi_s^{-+}(A)$, $\psi_s^{--}(A)$, $\psi_s^{+-}(A)$).

Lemma 5: *Let $G \subset R^2$ be an open set. Then $(0,0) \in \psi_s^{++}(G)$ if and only if for each $n \in N$ there exist $k \in N$ and a real number $\delta > 0$ such that for any $h, h' \in (0,\delta)$ and $i, i' \in \{1,\ldots,n\}$ there are $j, j' \in \{1,\ldots,k\}$ such that*

$$G \cap \left[\frac{(i-1)k+j-1}{nk}h, \frac{(i-1)k+j}{nk}h\right] \times \left[\frac{(i'-1)k+j'-1}{nk}h', \frac{(i'-1)k+j'}{nk}h'\right] = \emptyset$$

Proof: Similar to the linear case (see [5], th 1.)

Def. 8: *We say that $f : R^2 \to R$ is strongly $\mathcal{I}$-approximately continuous if f is continuous with respect to the topology $\mathcal{T}_{\mathcal{I}}^s$.*

Theorem 4: *If $f : R^2 \to R$ is a strongly $\mathcal{I}$-approximately continuous function and (α,β) is an open interval, each point of the set $f^{-1}((\alpha,\beta))$ is its deep strong $\mathcal{I}$-density point.*

Proof: It suffices to show that if $f(0,0) > 0$, then there exists a closed set $T \subset f^{-1}((0,+\infty))$ such that $(0,0)$ is a strong $\mathcal{I}$-density point of T. Choose $p \in N$ for which $f(0,0) > \frac{1}{p}$. Then, $(0,0)$ is a strong $\mathcal{I}$-density point of $f^{-1}((\frac{1}{p},+\infty))$. Since $f^{-1}((\frac{1}{p},+\infty))$ has the Baire property, it can be expressed in the form $F \triangle E$, where F is closed and E is meager on the plane. Obviously $(0,0)$ is a strong $\mathcal{I}$-density point of F. Thus, by lemma 5 we may assume that for each $n \in N$ the numbers k and δ are such that for any $h, h' \in (0,\delta)$ and $i, i' \in \{1,\ldots,n\}$ the numbers j_0 and $j_0' \in \{1,\ldots,k\}$ are chosen in order to have

$$\left[\frac{(i-1)k+j_0-1}{nk}h, \frac{(i-1)k+j}{nk}h\right] \times \left[\frac{(i'-1)k+j_0'-1}{nk}h', \frac{(i'-1)k+j_0'}{nk}h'\right] \subset F.$$

Now, fix $n \in N$ and let k and δ be with the above meaning. For any $m, m' \in N$, $i, i' \in \{1,\ldots,n\}$ and $j, j' \in \{1,\ldots,k\}$, by $P_{mm'}^{nii'jj'}$ we denote the set of all points $(h,h') \in R^2$ for which the following conditions hold

$$\left(\frac{(i-1)k+j-1}{(i-1)k+j}\right)^m \delta \le h < \left(\frac{(i-1)k+j-1}{(i-1)k+j}\right)^{m-1} \delta,$$

$$\left(\frac{(i'-1)k+j'-1}{(i'-1)k+j'}\right)^{m'} \delta \le h' < \left(\frac{(i'-1)k+j'-1}{(i'-1)k+j'}\right)^{m'-1} \delta.$$

Thus, for any $i, i' \in \{1, \dots, n\}$ and $j, j' \in \{1, \dots, k\}$ we have $\bigcup_{m=1}^{\infty} \bigcup_{m'=1}^{\infty} P_{mm'}^{nii'jj'} = (0, \delta)^2$.

Let $n \in N$, $i, i \in \{1, \dots, n\}$, $j, j' \in \{1, \dots, k\}$ be fixed. Put $u = (i-1)k + j$, $u' = (i'-1)k + j'$ and define $H_{ii'jj'}^{n}$ as the closure of the union

$$\bigcup_{m=1}^{\infty} \bigcup_{m'=1}^{\infty} \bigcup_{(h,h') \in P_{mm'}^{nii'jj'}} \left[\frac{u-1}{nk}h, \frac{u}{nk}h\right] \times \left[\frac{u'-1}{nk}h', \frac{u'}{nk}h'\right].$$

We shall show that $H_{ii'jj'}^{n} \subset f^{-1}([\frac{1}{p}, +\infty))$. To this end, let $(x_0, y_0) \in H_{ii'jj'}^{n}$. Then there are sequences $\{x_s\}_{s \in N}$ and $\{y_s\}_{s \in N}$ tending to x_0 and y_0, respectively, such that, for each $s \in N$ there are $m_s, m_s' \in N$ and $(h_s, h_s') \in P_{m_s m_s'}^{nii'jj'}$ such that $x_s \in \left[\frac{u-1}{nk}h_s, \frac{u}{nk}h_s\right]$ and $y_s \in \left[\frac{u'-1}{nk}h_s', \frac{u'}{nk}h_s'\right]$.

Before getting $f(x_0, y_0) \geq \frac{1}{p}$ we need to construct a subsequence $\{h_{s_r}\}_{r \in N}$ of $\{h_s\}_{s \in N}$ and a sequence $\{a_r\}_{r \in N}$ of real numbers different than 0, convergent to 0 such that for each $\ell \in N$ there exists $t \in \{1, \dots, 2nk\}$ such that for all $z \in \{1, \dots, \ell\}$ and for almost all $r \in N$ we have

$$\left[x_0 + \frac{(t-1)l + z - 1}{2nkl}a_r, x_0 + \frac{(t-1)l + z}{2nkl}a_r\right] \subset \left[\frac{u-1}{nk}h_{s_r}, \frac{u}{nk}h_{s_r}\right]$$

(if $a_r < 0$, the endpoints of the interval on the left-hand side in this inclusion must be written conversely; this remark should be also repeated further in the proof).

Consider some cases. First assume that there is a subsequence $\{m_{s_r}\}_{r \in N}$ of $\{m_s\}_{s \in N}$ tending to infinity. Put $a_r = h_{s_r}$ for $r \in N$. Then for all $r \in N$, we have

$$0 \leq x_{s_r} \leq \frac{u}{nk}h_{s_r} \leq \frac{u}{nk}\left(\frac{u-1}{u}\right)^{m_{s_r}-1}\delta$$

so $\lim_r h_{s_r} = \lim_r x_{s_r} = 0$. Hence we have $x_0 = 0$.

Put $t = 2u - 1$. For $1 \in N$ and for any $z \in \{1, \dots, \ell\}$ and $r \in N$, we get

$$\begin{aligned} &\left[\frac{(t-1)l + z - 1}{2nkl}h_{s_r}, \frac{(t-1)l + z}{2nkl}h_{s_r}\right] = \\ &\left[\left(\frac{u-1}{nk} + \frac{z-1}{2nkl}\right)h_{s_r}, \left(\frac{u-1}{nk} + \frac{z}{2nkl}\right)h_{s_r}\right] \subset \\ &\left[\frac{u-1}{nk}h_{s_r}, \frac{u}{nk}h_{s_r}\right] \end{aligned}$$

which has been desired.

The remaining case is when $\{m_s\}_{s\in N}$ contains a constant subsequence. For simplicity assume that all terms m_s are equal to m_{s_r}. Consider the first subcase when there exists $s_* \in N$ such that $x_0 \in \left[\frac{u-1}{nk}h_s, \frac{u}{nk}h_s\right]$ for all $s \geq s_*$. For $r \in N$ put $h_{s_r} = h_{s_*+r}$ and choose $a_r \neq 0$ such that $x_0 + a_r \in \left[\frac{u-1}{nk}h_{s_r}, \frac{u}{nk}h_{s_r}\right]$ and $\lim_r a_r = 0$. Let $l \in N$ and put $t = 1$. Then for any $z \in \{1, \ldots, l\}$ and $r \in N$ we get

$$\left[x_0 + \frac{z-1}{2nkl}a_r, x_0 + \frac{z}{2nkl}a_r\right] \subset [x_0, x_0 + a_r] \subset \left[\frac{u-1}{nk}h_{s_r}, \frac{u}{nk}h_{s_r}\right].$$

The remaining subcase means that there is a subsequence $\{h_{s_r}\}_{r\in N}$ of $\{h_s\}_{s\in N}$ such that $x_0 \notin \left[\frac{u-1}{nk}h_{s_r}, \frac{u}{nk}h_{s_r}\right]$ for all $r \in N$. We may assume (choosing a subsequence if necessary) that either $x_0 < \frac{u-1}{nk}h_{s_r}$ for all $r \in N$ or $x_0 > \frac{u}{nk}h_{s_r}$ for all $r \in N$. Consider for example the first situation. Since $x_0 < \frac{u-1}{nk}h_{s_r} \leq x_{s_r}$ for all $r \in N$, we have that $\lim_r \frac{u-1}{nk}h_{s_r} = x_0$. Then there is a $r_* \in N$ such that $\frac{u-1}{nk}h_{s_r} - x_0 < \frac{1}{nk}\left(\frac{u-1}{u}\right)^{m_{s_0}}\delta$ for each $r \geq r_*$. Define $a_r = 2\left(\frac{u-1}{nk}h_{s_r} - x_0\right)$ for $r \in N$. Let $\ell \in N$ and put $t = nk+1$. Since in this case $\left(\frac{u-1}{nk}\right)^{m_{s_0}}\delta \leq h_{s_r}$, we have

$$\begin{aligned}
&\left[x_0 + \frac{(t-1)l + z - 1}{2nkl}a_r, x_0 + \frac{(t-1)l + z}{2nkl}a_r\right] = \\
&\left[x_0 + \left(\frac{1}{2} + \frac{z-1}{2nkl}\right)a_r, x_0 + \left(\frac{1}{2} + \frac{z}{2nkl}\right)a_r\right] \subset \\
&\left[x_0 + \frac{1}{2}a_r, x_0 + \left(\frac{1}{2} + \frac{1}{2nk}\right)a_r\right] \subset \\
&\left[\frac{u-1}{nk}h_{s_r}, \frac{u-1}{nk}h_{s_r} + \frac{1}{nk}\left(\frac{u-1}{nk}\right)^{m_{s_0}}\delta\right] \subset \\
&\left[\frac{u-1}{nk}h_{s_r}, \frac{u-1}{nk}h_{s_r} + \frac{1}{nk}h_{s_r}\right].
\end{aligned}$$

Analogously, we show that there is a subsequence $\{h'_{s_{r_q}}\}_{q\in N}$ of $\{h'_{s_r}\}_{r\in N}$ and a sequence $\{b_q\}_{q\in N}$ of real numbers different than 0 convergent to 0 such that for each $\ell \in N$ there exists $t' \in \{1, \ldots, 2nk\}$ such that for all $z' \in \{1, \ldots, l\}$ and for almost all $q \in N$ we have

$$\left[y_0 + \frac{(t'-1)l + z' - 1}{2nkl}b_q, y_0 + \frac{(t'-1)l + z'}{2nkl}b_q\right] \subset \left[\frac{u'-1}{nk}h'_{s_{r_q}}, \frac{u'}{nk}h'_{s_{r_q}}\right].$$

From the above facts it easily follows that for any $\ell \in N$ and $\eta > 0$ there are $a_{r_q}, b_q \in (-\eta, 0) \cup (0, \eta)$ and $t, t' \in \{1, \ldots, 2nk\}$ such that for all $z, z' \in \{1, \ldots, \ell\}$

we have

$$\left[x_0 + \frac{(t-1)l+z-1}{2nkl}a_{r_q}, x_0 + \frac{(t-1)l+z}{2nkl}a_{r_q}\right] \times$$

$$\left[y_0 + \frac{(t'-1)l+z'-1}{2nkl}b_q, y_0 + \frac{(t'-1)+z'}{2nkl}b_q\right] \subset$$

$$\left[\frac{u-1}{nk}h_{s_{r_q}}, \frac{u}{nk}h_{s_{r_q}}\right] \times \left[\frac{u'-1}{nk}h'_{s_{r_q}}, \frac{u'}{nk}h'_{s_{r_q}}\right] \subset F.$$

Hence, by Lemma 5, we may conclude that (x_0, y_0) is not a strong $\mathcal{I}$-dispersion point of F and then it is not a strong $\mathcal{I}$ dispersion point of $f^{-1}((\frac{1}{p}, +\infty))$. Therefore (x_0, y_0) is not a strong $\mathcal{I}$-density point of $f^{-1}((-\infty, \frac{1}{p}))$. Since f is $\mathcal{I}$-approximately continuous, it follows that $f(x_0, y_0) \geq \frac{1}{p}$, what proves the inclusion $H^n_{ii'jj'} \subset f^{-1}([\frac{1}{p}, +\infty))$.

Now, for $m \in N$ let $D_m = \left[0, \frac{1}{m}\right]^2 - \left[0, \frac{1}{m+1}\right]^2$ and $T^{++} = \bigcup_{m=1}^{\infty}(D_m \cap \bigcup_{n=1}^{m} \bigcup_{i=1}^{n} \bigcup_{i'=1}^{n} \bigcup_{j=1}^{k} \bigcup_{j'=1}^{k} H^n_{ii'jj'}) \cup \{(0,0)\}$.

T^{++} is closed and $T^{++} \subset f^{-1}([\frac{1}{p}, +\infty))$. We shall show that $(0,0) \in \varphi_s^{++}(T^{++})$ with use of lemma 5.

Let $n \in N$ be arbitrary. Since $(0,0) \in \psi_s^{++}(R^2 - F)$, by lemma 5 we may choose k and δ. Put $\delta_1 = \min\{\delta, \frac{1}{n}\}$. Let $h, h' \in (0, \delta_1)$ and $i, i' \in \{1, \ldots, n\}$ be arbitrary. Choose $j, j' \in \{1, \ldots, k\}$ by lemma 5. There are $m, m' \in N$ such that $(h, h') \in P^{nii'jj'}_{mm'}$. By the definition of $H^n_{ii'jj'}$ we have

$$\left[\frac{u-1}{nk}h, \frac{u}{nk}h\right] \times \left[\frac{u'-1}{nk}h', \frac{u'}{nk}h'\right] \subset H^n_{ii'jj'} \cap \left[0, \frac{1}{n}\right]^2 \subset T^{++},$$

what means that $(0,0) \in \varphi_s^{++}(T^{++})$.

Analogously we can construct in the remaining quarters of the plane closed sets T^{+-}, T^{-+}, T^{--}, included in $f^{-1}([\frac{1}{p}, +\infty))$ and such that

$$(0,0) \in \varphi_s^{+-}(T^{+-}) \cap \varphi_s^{-+}(T^{-+}) \cap \varphi_s^{--}(T^{--}).$$

Put $T = T^{++} \cup T^{+-} \cup T^{-+} \cup T^{--}$. T is closed set included in $f^{-1}((0, +\infty))$ and $(0,0)$ is a deep strong $\mathcal{I}$-density point of $f^{-1}((0, +\infty))$, which ends the proof.

Remark 1: The theorem above implies that the strongly $\mathcal{I}$-approximately continuous functions are exactly the functions which are continuous with respect to the topology $\mathcal{T}_s$.

Next theorem shows that for the topology $\mathcal{T}_s$ the Lusin-Menchoff theorem doesn't hold.

Theorem 5: *There exist a set $E \in \mathcal{T}_s$ and a closed set $F \subset E$ such that, for each perfect set P such that $F \subset P \subset E$, there exists a point $(x_0, y_0) \in F$ which is not a strong $\mathcal{I}$-density point of P.*

Proof. First construct a set E $\mathcal{T}_s$-open. In order to do that, for $p \in N$ and for each odd $k \in \{1, 3, \dots, 2^p - 1\}$, denote by a_p^k the number $\frac{k}{2^p}$. Define a sequence $\{c_n\}_{n \in N}$ as follows: if $n < m$ and $c_n = a_p^k$, $c_m = a_q^h$, for some $p, q \in N$, $k \in \{1, 3, \dots, 2^p - 1\}$ and $h \in \{1, 3, \dots, 2^q - 1\}$, we have either $p < q$ or $p = q$ and $k < h$.

Let $\{[a_n, b_n]\}_{n \in N}$ be a sequence of closed intervals such that $a_1 = \frac{1}{2}$, $b_1 = 1$ and, for each natural $n \in N$ $b_{n+1} = \frac{1}{2} a_n$ and $a_{n+1} = \frac{1}{2} b_{n+1}$.

Let $D = \bigcup_{n=1}^{\infty} [a_n, b_n] \times \{c_n\}$ and $E = R^2 - D$. Put $F = \{0\} \times [0, 1]$. Then $F \subset E$ and F is closed. We shall show that E is an open set in the $\mathcal{T}_s$ topology.

Let $(x_0, y_0) \in E$. If $(x_0, y_0) \notin F$, then there exists $\delta > 0$ such that $x_0 \notin (-\delta, \delta)$ and there exists $n_0 \in N$ such that for each $n > n_0$, $[a_n, b_n] \subset (-\delta, \delta)$. Thus $D \cap (((-\infty, -\delta] \cup [\delta, +\infty)) \times R) = \bigcup_{n=1}^{n_0} [a_n, b_n] \times \{c_n\} \cap (((-\infty, -\delta] \cup [\delta, +\infty)) \times R)$ and $(x_0, y_0) \in \mathring{E}$.

Let $(x_0, y_0) \in \{0\} \times [0, 1]$. We shall show that for each $n \in N$ there exists $\delta > 0$ such that, for each $(h, h') \in (0, \delta) \times (0, \delta)$ and for each $(i, i') \in \{1, \dots, n\} \times \{1, \dots, n\}$ there exists $(j, j') \in \{1, 2, 3\} \times \{1, 2, 3\}$ such that

$$\left[\frac{3(i-1)+j-1}{3n} h, \frac{3(i-1)+j}{3n} h\right] \times \{y_0\} \subset E,$$

$$\{0\} \times \left[y_0 + \frac{3(i'-1)+j'-1}{3n} h', y_0 + \frac{3(i'-1)+j'}{3n} h'\right] \subset E,$$

$$\left[\frac{3(i-1)+j-1}{3n} h, \frac{3(i-1)+j}{3n} h\right] \times$$
$$\left[y_0 + \frac{3(i'-1)+j'-1}{3n} h', y_0 + \frac{3(i'-1)+j'}{3n} h'\right] \subset E.$$

Since $\{0\} \times R \subset E$, the second condition is obvious for any $n \in N$, $\delta > 0$, $i' \in \{1, \dots, n\}$, $j' \in \{1, 2, 3\}$, $h' \in (0, \delta)$ and $y_0 \in [0, 1]$.

If $y_0 = 1$, since $[0, +\infty) \times [1, +\infty) \subset E$, the conditions above are true. Assume that $y_0 < 1$. Observe that if $y_0 = c_{n_0}$ for some $n_0 \in N$, then $(R \times \{y_0\}) \cap D = [a_{n_0}, b_{n_0}] \times \{c_{n_0}\}$. Let $0 < \delta_{(x_0, y_0)} < a_{n_0}$. It is obvious that if $c_n \neq y_0$ for each $n \in N$, then $(R \times \{y_0\}) \cap D = \emptyset$ and we put $\delta_{(x_0, y_0)} = 1$.

Now, let n be an arbitrary natural number. Put $\delta = \delta_{(x_0, y_0)}$. Let $(h, h') \in (0, \delta) \times (0, \delta)$ and $(i, i') \in \{1, \dots, n\} \times \{1, \dots, n\}$. If $\left[\frac{3(i-1)+2}{3n} h, \frac{i}{n} h\right] \cap \bigcup_{p=1}^{\infty} [a_p, b_p] = \emptyset$, then $\left(\left[\frac{3(i-1)+2}{3n}h, \frac{i}{n}h\right] \times R\right) \cap D = \emptyset$ and $\left[\frac{3(i-1)+2}{3n}h, \frac{i}{n}h\right] \times \left[y_0 + \frac{3(i'-1)+2}{3n}h', y_0 + \frac{i'}{n}h'\right] \subset E$.

Therefore we put $j = 3$ and $j' = 3$. If $\left[\frac{3(i-1)+2}{3n}h, \frac{i}{n}h\right] \cap \bigcup_{p=1}^{\infty}[a_p, b_p] \neq \emptyset$, for p_0 such that $\left[\frac{3(i-1)+2}{3n}h, \frac{i}{n}h\right] \cap [a_{p_0}, b_{p_0}] \neq \emptyset$ and $a_{p_0} \in \left(\frac{3(i-1)+2}{3n}h, \frac{i}{n}h\right]$, then we have $b_{p_0+1} = \frac{1}{2}a_{p_0} \leq \frac{1}{2n}h$. Therefore $\frac{3(i-1)+2}{3n}h - b_{p_0+1} \geq \frac{3(i-1)+2}{3n}h - \frac{i}{2n}h = \frac{h}{n}\frac{3i-2}{6} \geq \frac{h}{6n} > 0$ and $b_{p_0+1} < \frac{3(i-1)+2}{3n}h$. If $b_{p_0} \in \left[\frac{3(i-1)+2}{3n}h, \frac{i}{n}h\right)$, then $a_{p_0-1} = 2b_{p_0} \geq \frac{6(i-1)+4}{3n}h$. Therefore $a_{p_0-1} - \frac{i}{n}h \geq \frac{6(i-1)+4}{3n}h - \frac{i}{n}h = \frac{h}{n}\frac{3i-2}{3} \geq \frac{h}{3n} > 0$ and $a_{p_0-1} > \frac{i}{n}h$. If $a_{p_0} \leq \frac{3(i-1)+2}{3n}h$ and $b_{p_0} \geq \frac{i}{n}h$, then $[a_{p_0}, b_{p_0}] \supset \left[\frac{3(i-1)+2}{3n}h, \frac{i}{n}h\right]$. Thus for each $p \in N$, $p \neq p_0$, $[a_p, b_p] \cap \left[\frac{3(i-1)+2}{3n}h, \frac{i}{n}h\right] = \emptyset$.

Now we consider the closed interval $\left[y_0 + \frac{i'-1}{n}h', y_0 + \frac{i'}{n}h'\right]$. If $c_{p_0} \notin \left[y_0 + \frac{i'-1}{n}h', y_0 + \frac{i'}{n}h'\right]$, then we have $\left[\frac{3(i-1)+2}{3n}h, \frac{i}{n}h\right] \times \left[y_0 + \frac{3(i'-1)+2}{3n}h', y_0 + \frac{i'}{n}h'\right] \cap D = \emptyset$ and we put $j = 3$ and $j' = 3$. If $c_{p_0} \in \left[y_0 + \frac{i'-1}{n}h', y_0 + \frac{i}{n}h'\right]$, then there exists $j_0' \in \{1,2,3\}$ such that $c_{p_0} \notin \left[y_0 + \frac{3(i'-1)+j_0'-1}{3n}h', y_0 + \frac{3(i'-1)+j_0'}{3n}h'\right]$ and therefore $\left[\frac{3(i-1)+2}{3n}h, \frac{i}{n}h\right] \times \left[y_0 + \frac{3(i'-1)+j_0'-1}{3n}h', y_0 + \frac{3(i'-1)+j_0'}{3n}h'\right] \cap D = \emptyset$. Then we put $j = 3$ and $j' = j_0'$. Since $\delta = \delta_{(x_0,y_0)}$, the first and the second conditions are true.

Now we'll show that there exists a closed set $T \subset E$ such that (x_0, y_0) is a strong $\mathcal{I}$-density point of T, what is enough to conclude that E is open in the $\mathcal{T}_s$ topology.

In order to do that, for simplicity, assume that $(x_0, y_0) = (0,0)$. Let, for each $n \in N$, $\delta > 0$ such that for any $(h, h') \in (0, \delta) \times (0, \delta)$ and $i, i' \in \{1, \ldots, n\}$ there exists $(j, j') \in \{1,2,3\} \times \{1,2,3\}$ such that

$$\left[\frac{3(i-1)+j-1}{3n}h, \frac{3(i-1)+j}{3n}h\right] \times \left[\frac{3(i'-1)+j'-1}{3n}h', \frac{3(i'-1)+j'}{3n}h'\right] \subset E$$

$$\left[\frac{3(i-1)+j-1}{3n}h, \frac{3(i-1)+j}{3n}h\right] \times \{0\} \subset E,$$

$$\{0\} \times \left[\frac{3(i'-1)+j'-1}{3n}h', \frac{3(i'-1)+j'}{3n}h'\right] \subset E.$$

For fixed $n \in N$, $\delta > 0$ with the meaning above, $m, m' \in N$, $(j, j') \in \{1,2,3,\} \times \{1,2,3\}$, $(i, i') \in \{1, \ldots, n\} \times \{1, \ldots, n\}$, let $P_{mm'}^{nii'jj'}$ be the set of all points $(h, h') \in R^2$ such that the following conditions hold simultaneously:

$$\left(\frac{3(i-1)+j-1}{3(i-1)+j}\right)^m \delta \ \leq \ h < \left(\frac{3(i-1)+j-1}{3(i-1)+j}\right)^{m-1} \delta,$$

$$\left(\frac{3(i'-1)+j'-1}{3(i'-1)+j')}\right)^{m'}\delta \ \leq \ h' < \left(\frac{3(i'-1)+j'-1}{3(i'-1)+j'}\right)^{m'-1}\delta.$$

Then for any $(i,i') \in \{1,\dots,n\} \times \{1,\dots,n\}$ and $(j,j') \in \{1,2,3\} \times \{1,2,3\}$ $\bigcup_{m=1}^{\infty}\bigcup_{m'=1}^{\infty} P_{mm'}^{nii'jj'} = (0,\delta) \times (0,\delta)$.

For fixed $n \in N$, $(i,i') \in \{1,\dots,n\} \times \{1,\dots,n\}$, $(j,j') \in \{1,2,3\} \times \{1,2,3\}$, denote by $H^n_{ii'jj'}$ the closure of the set

$$\bigcup_{m=1}^{\infty}\bigcup_{m'=1}^{\infty}\bigcup_{(h,h')\in P_{mm'}^{nii'jj'}}\left[\frac{u-1}{3n}h+\frac{1}{9n}h,\frac{u}{3n}h-\frac{1}{9n}h\right]\times\left[\frac{u'-1}{3n}h'+\frac{1}{9n}h'\frac{u'}{3n}h'-\frac{1}{3n}h'\right],$$

where $u=(i-1)3+j$, $u'=3(i'-1)+j'$. We shall show that $H^n_{ii'jj'} \subset E$. Let $(x_0,y_0) \in H^n_{ii'jj'}$. Then there are sequences $\{x_s\}_{s\in N}$ and $\{y_s\}_{s\in N}$ tending to x_0 and y_0 respectively such that, for each $s \in N$ there are $m_s, m'_s \in N$ and $(h_s,h'_s) \in P_{m,m'}^{nii'jj'}$ such that $(x_s,y_s) \in \left[\frac{u-1}{3n}h_s+\frac{1}{9n}h_s, \frac{u}{3n}h_s-\frac{1}{9n}h_s\right]\times\left[\frac{u'-1}{3n}h'_s+\frac{1}{9n}h'_s, \frac{u'}{3n}h'_s-\frac{1}{9n}h'_s\right]$.

If $x_0=0$ and $y_0=0$, it is obvious that $(x_0,y_0) \in E$.

If $x_0 \neq 0$ and $y_0 = 0$, we may assert that there is no subsequence $\{m_{s_r}\}_{r\in N}$ of $\{m_s\}_{s\in N}$ which is tending to infinity. If not, in fact, $0 \leq x_{s_r} \leq \frac{u}{3n}h_{s_r} - \frac{1}{9n}h_{s_r} < \frac{u}{3n}h_{s_r} < \frac{u}{3n}\left(\frac{u-1}{u}\right)^{m_{s_r}-1}\delta$, for all $r \in N$. So $\lim_r h_{s_r} = \lim_r x_{s_r} = 0$ and $x_0 = 0$, a contradiction.

Therefore $\{m_s\}_{s\in N}$ contains a constant subsequence. For simplicity assume that all terms are equal to m_{s_0}. Suppose that there exists a subsequence $\{h_{s_r}\}_{r\in N}$ of $\{h_s\}_{s\in N}$ such that $x_0 \notin \left[\frac{u-1}{3n}h_{s_r}, \frac{u}{3n}h_{s_r}\right]$ for all $r \in N$. Then either $x_0 < \frac{u-1}{3n}h_{s_r}$ for all $r \in N$ or $x_0 > \frac{u}{3n}h_{s_r}$ for all $r \in N$ (choosing a subsequence if necessary). Consider, for example the first situation. Then, since $x_0 < \frac{u-1}{3n}h_{s_r} < x_{s_r}$, for all $r \in N$, it follows that $\lim_r \frac{u-1}{3n}h_{s_r} = x_0$ and $\lim_r\left(\frac{u-1}{3n}h_{s_r} + \frac{1}{9n}h_{s_r}\right) \geq x_0 + \frac{1}{9n}\left(\frac{u-1}{u}\right)^{m_{s_0}}\delta > x_0$.

Since $x_{s_r} \geq \frac{u-1}{3n}h_{s_r} + \frac{1}{9n}h_{s_r}$ for each $r \in N$, we have that $\lim_r x_{s_r} \geq \lim_r\left(\frac{u-1}{3n}h_{s_r} + \frac{1}{9n}h_{s_r}\right) > x_0$, a contradiction. Thus, there exists s^* such that, for $s \geq s^*$, $x_0 \in \left[\frac{u-1}{3n}h_s, \frac{u}{3n}h_s\right]$ and $(x_0,0) \in \left[\frac{u-1}{3n}h_s, \frac{u}{3n}h_s\right] \times \{0\} \subset E$.

If $x_0 = 0$ and $y_0 \neq 0$, the proof is similar.

If $x_0 \neq 0$ and $y_0 \neq 0$, we may choose, as above, a subsequence $\{h_{s_r}\}_{r\in N}$ of $\{h_s\}_{s\in N}$, if necessary, such that there is $r^* \in N$ such that, for each $r \geq r^*$, $x_0 \in \left[\frac{u-1}{3n}h_{s_r}, \frac{u}{3n}h_{s_r}\right]$ and we may choose a subsequence $\{h'_{s_{r_p}}\}_{p\in N}$ of $\{h'_{s_r}\}_{r\in N}$ such that there exists $p^* \in N$ such that, for all $p \geq p^*$, $y_0 \in \left[\frac{u'-1}{3n}h'_{s_{r_p}}, \frac{u'}{3n}h'_{s_{r_p}}\right]$. Then choose $p \geq p^*$ such that $(x_0,y_0) \in \left[\frac{u-1}{3n}h_{s_{r_p}}, \frac{u}{3n}h_{s_{r_p}}\right] \times \left[\frac{u'-1}{3n}h'_{s_{r_p}}, \frac{u'}{3n}h'_{s_{r_p}}\right] \subset E$. We have $H^n_{ii'jj'} \subset E$.

Now let $D_m = \left[0, \frac{1}{m}\right]^2 - \left[0, \frac{1}{m+1}\right]^2$ for $m \in N$ and let $T^{++} = \bigcup_{m=1}^{\infty} (D_m \cap \bigcup_{n=1}^{m} \bigcup_{i=1}^{n} \bigcup_{i'=1}^{n} \bigcup_{j=1}^{3} \bigcup_{j'=1}^{3} H_{ii'jj'}^{n}) \cup \{(0,0)\}$. T^{++} is closed and included in E. We shall show that $(0,0) \in \varphi_s^{++}(T^{++})$. Let $n \in N$, $k = 9$, $\delta > 0$, $(h, h') \in (0,\delta)\times(0,\delta)$ and $i, i' \in \{1, \dots, n\}$. For some $j, j' \in \{1,2,3\}$ we have

$$\left[\frac{u-1}{3n}h, \frac{u}{3n}h\right] \times \left[\frac{u'-1}{3n}h', \frac{u'}{3n}h'\right] \subset E,$$

$$\text{where } u = 3(i-1) + j \text{ and } u' = 3(i'-1) + j'.$$

Then

$$\left[\frac{u-1}{3n}h + \frac{1}{9n}h, \frac{u}{3n}h - \frac{1}{9n}h\right] \times \left[\frac{u'-1}{3n}h' + \frac{1}{9n}h', \frac{u'}{3n}h' - \frac{1}{9n}h'\right] \subset T^{++}.$$

Moreover, if $j_1 = 3j - 1, j_1' = 3j' - 1$, then $j_1, j_1' \in \{1, \dots, 9\}$ and

$$\left[\frac{9(i-1) + j_1 - 1}{9n}h, \frac{9(i-1) + j_1}{9n}h\right] \times \left[\frac{9(i'-1) + j_1' - 1}{9n}h', \frac{9(i'-1) + j_1'}{9n}h'\right] \subset T^{++}.$$

In a similar way we can construct closed sets T^{+-}, T^{--}, T^{-+} in such a way that $(0,0) \in \varphi_s(T)$, where $T = T^{++} \cup T^{+-} \cup T^{--} \cup T^{-+}$. Now let's show that, if P is a perfect set such that $F \subset P \subset E$, there is a point $(x_0, y_0) \in F$ which is not a strong $\mathcal{I}$-density point of P. Suppose the contrary and let P a perfect set such that $F \subset P \subset E$ and every point $(x_0, y_0) \in F$ is a strong $\mathcal{I}$-density point of P.

Let $G = R^2 - P$. Then $D \subset G$ and, for each $n \in N$, there exists $\delta_n > 0$ such that $[a_n, b_n] \times \{c_n\} \subset [a_n, b_n] \times [c_n - \delta_n, c_n + \delta_n] \subset G$. We want to show that $F \subset \bar{D}$. Let $(x_0, y_0) \in F$ and let $\gamma > 0$. Since $\lim_n b_n = 0$, then there exists n_0 such that, for $n \geq n_0$ we have $[a_n, b_n] \subset (0, \gamma)$. Therefore $\bigcup_{n=n_0}^{\infty} [a_n, b_n] \times \{c_n\} \subset (0, \gamma) \times [0, 1]$. Let $n_1 > n_0$ such that $c_{n_1} \in (y_0 - \gamma, y_0 + \gamma)$. Then we have

$$[a_{n_1}, b_{n_1}] \times \{c_{n_1}\} \subset (0, \gamma) \times (y_0 - \gamma, y_0 + \gamma) \text{ and, } ((-\gamma, \gamma) \times (y_0 - \gamma, y_0 + \gamma)) \cap D \neq \emptyset$$

Thus $(x_0, y_0) \in \bar{D}$.

Using this fact, we can define by induction a sequence $\{[\alpha_k, \beta_k]\}_{k \in N}$ of intervals such that for each $k \in N$ $[\alpha_{k+1}, \beta_{k+1}] \subset [\alpha_k, \beta_k]$, $\beta_{k+1} - \alpha_{k+1} \leq \frac{1}{2}(\beta_k - \alpha_k)$ and for each $k \in N$ there exists n_k such that $[\alpha_k, \beta_k] \subset [c_{n_k} - \delta_{n_k}, c_{n_k} + \delta_{n_k}]$.

Let $y_0 \in \bigcap_{k=1}^{\infty} [\alpha_{k+1}, \beta_{k+1}]$. We will show that $(0, y_0)$ is not a strong $\mathcal{I}$-density point of P. Since $y_0 \in [\alpha_k, \beta_k] \subset [c_{n_k} - \delta_{n_k}, c_{n_k} + \delta_{n_k}]$, for every $k \in N$, we have

$[a_{n_k}, b_{n_k}] \times \{y_0\} \subset [a_{n_k}, b_{n_k}] \times [c_{n_k} - \delta_{n_k}, c_{n_k} + \delta_{n_k}] \subset G$. Let $S = \bigcup_{k=1}^{\infty}(a_{n_k}, b_{n_k})$ and $t_k = \frac{1}{b_{n_k}}$. For each subsequence $\{t_{k_p}\}_{p\in N}$ of $\{t_k\}_{k\in N}$ we have $(\frac{1}{2}, 1) \subset \bigcup_{p=1}^{\infty} t_{k_p} \cdot S \cap [0, 1]$. Moreover

$$t_k \cdot S = t_k \cdot \bigcup_{r=1}^{\infty}(a_{n_r}, b_{n_r}) \supset t_k \cdot (a_{n_k}, b_{n_k}) = \frac{1}{b_{n_k}}(\frac{1}{2}b_{n_k}, b_{n_k}) = (\frac{1}{2}, 1).$$

Then 0 is not an $\mathcal{I}$-dispersion point of S. Thus $(0, y_0)$ is not a strong $\mathcal{I}$-dispersion point of G and $(0, y_0)$ cannot be a strong $\mathcal{I}$-density point of P.

Theorem 6: *$\mathcal{T}_s$ is not a completely regular topology on the plane.*

Proof: We'll show that there exist a set D closed in the $\mathcal{T}_s$ topology and a point $(x_0, y_0) \notin D$ such that there is no strongly $\mathcal{I}$-approximately continuous function $f : R^2 \to R$ such that $f(x_0, y_0) = 1$ and $f(D) \subset \{0\}$.

Let D be the set defined as in the proof of the theorem above. Let (x_0, y_0) be a point of $\{0\} \times [0, 1]$. Suppose that there is a strongly $\mathcal{I}$-approximately continuous function f such that $f(x_0, y_0) = 1$ and $f(D) \subset \{0\}$.

Let $A = \{(x, y) : f(x, y) > \frac{1}{2}\}$ and $B = \{(x, y) : f(x, y) < \frac{1}{2}\}$. Then $(x_0, y_0) \in A$ and $A \in \mathcal{T}_s$. Thus there exists a closed set $P \subset A$ such that (x_0, y) is a strongly $\mathcal{I}$-density point of P. Therefore there exists $[a, b] \subset [0, 1]$ such that $\{0\} \times [a, b] \subset A$.

Let n be a natural number. If $(x, y) \in [a_n, b_n] \times \{c_n\}$, (where $[a_n, b_n] \times \{c_n\}$ is such that $D = \bigcup_{n=1}^{\infty}[a_n, b_n] \times \{c_n\}$), then $(x, y) \in B$ and (x, y) is a strong $\mathcal{I}$-density point of $\mathring{B}$. Thus there exist $\delta_{x,y}$ such that, for every $h, h' \in (0, \delta_{x,y})$ and for every $i, i' \in \{1, \ldots, n\}$

$$\left(x + \frac{i-1}{n}h, x + \frac{i}{n}h\right) \times \left(y + \frac{i'-1}{n}h', y + \frac{i'}{n}h'\right) \cap \mathring{B} \neq \emptyset,$$

$$\left(x + \frac{i-1}{n}h, x + \frac{i}{n}h\right) \times \left(y - \frac{i'-1}{n}h', y - \frac{i'}{n}h'\right) \cap \mathring{B} \neq \emptyset,$$

$$\left(x - \frac{i-1}{n}h, x - \frac{i}{n}h\right) \times \left(y + \frac{i'-1}{n}h', y + \frac{i'}{n}h'\right) \cap \mathring{B} \neq \emptyset,$$

$$\left(x - \frac{i-1}{n}h, x - \frac{i}{n}h\right) \times \left(y - \frac{i'-1}{n}h', y - \frac{i'}{n}h'\right) \cap \mathring{B} \neq \emptyset,$$

Now we define by induction sequences of sets $\{[a_{n_p}, b_{n_p}] \times \{c_{n_p}\}\}_{p\in N}$, $\{[\alpha_p, \beta_p]\}_{p\in N}$ and of points $\{(x_1^p, \ldots, x_{2^p}^p)\}_{p\in N}$ such that, for every $p \in N$ and $i \in \{1, \ldots, 2^p\}$ we have

i) $(\alpha_1, \beta_1) \subset (a, b)$

ii) $\beta_{p+1} - \alpha_{p+1} < \frac{1}{2}(\beta_p - \alpha_p)$

iii) $[\alpha_{p+1}, \beta_{p+1}] \subset [\alpha_p, \beta_p]$

iv) $[\alpha_p, \beta_p] = [c_{n_p} - \delta, c_{n_p}]$, where $\delta = \min\{\delta_{x_1^p, c_{n_p}}, \ldots, \delta_{x_{2^p}^p, c_{n_p}}\}$

v) $[a_{n_{p+1}}, b_{n_{p+1}}] \times \{c_{n_p}\} \subset (0, a_{n_p}) \times (\alpha_p, \beta_p)$

vi) $[a_{n_1}, b_{n_1}] \times \{c_{n_1}\} \subset (0, +\infty) \times (a, b)$

vii) $x_i^p \in \left(a_{n_p} + \frac{i-1}{2^p}(b_{n_p} - a_{n_p}), a_{n_p} + \frac{i}{2^p}(b_{n_p} - a_{n_p})\right)$.

For $p = 1$, let $n_1 \in N$ such that $[a_{n_1}, b_{n_1}] \times \{c_{n_1}\} \subset (0, +\infty) \times (a, b)$ and choose $x_1^1 \in (a_{n_1}, a_{n_1} + \frac{1}{2}(b_{n_1} - a_{n_1}))$ and $x_2^1 \in (a_{n_1} + \frac{1}{2}(b_{n_1} - a_{n_1}), b_{n_1})$.

Put $[\alpha_1, \beta_1] = [c_{n_1} - \delta, c_{n_1}]$, where $\delta = \min\{\delta_{x_1^1, c_{n_1}}, \delta_{x_2^1, c_{n_1}}\}$. Assume, by induction, that there exist $[a_{n_k}, b_{n_k}], c_{n_k}, [\alpha_k, \beta_k]$ and $x_1^p, \ldots, x_2^p$, satisfying i), ii), iii), iv), v), vi), vii), for all $p < p_0$. We want to show that there exist $[a_{n_{p_0}}, b_{n_{p_0}}]$, $c_{n_{p_0}}, [\alpha_{n_{p_0}}, \beta_{n_{p_0}}]$ and $x_1^{p_0}, \ldots, x_{2^{p_0}}^{p_0}$ with properties i), ii), iii), iv), v), vi) and vii).

For this purpose, let $[a_{n_{p_0}}, b_{n_{p_0}}]$ and $c_{n_{p_0}}$ such that $[a_{n_{p_0}}, b_{n_{p_0}}] \times \{c_{n_{p_0}}\} \subset (0, a_{n_{p_0-1}}) \times (\alpha_{p_0-1}, \beta_{p_0-1})$ and let, for each $i \in \{1, \ldots, 2^{p_0}\}$, $x_i^{p_0} \in (a_{n_{p_0}} + \frac{i-1}{2^{p_0}}(b_{n_{p_0}} - a_{n_{p_0}}), a_{n_{p_0}} + \frac{i}{2^{p_0}}(b_{n_{p_0}} - a_{n_{p_0}}))$.

Let $\delta = \min\{\delta_{x_1^{p_0}, c_{n_{p_0}}}, \ldots, \delta_{x_{2^{p_0}}^{p_0}, c_{n_{p_0}}}\}$ and put $[\alpha_{p_0}, \beta_{p_0}] = [c_{n_{p_0}} - \delta^*, c_{n_{p_0}}]$, where δ^* is a positive number less than δ and such that ii) and iii) are satisfied.

If y^* is such that $\{y^*\} = \bigcap_{p=1}^{\infty} [\alpha_p, \beta_p]$, we have $y^* \in (a, b), (0, y^*) \in \{0\} \times (a, b) \subset A$. Then $(0, y^*) \in A$.

Let k be an arbitrary natural number. Then there exists p_0 such that, for each $p > p_0, b_{n_p} < \frac{1}{k}$ and $p > 2k$. Moreover, there exists p_1 such that $c_{n_p} \in \left(y^*, y^* + \frac{1}{k}\right)$ for every $p > p_1$.

Let $p > \max\{p_1, p_0\}$. Put $h = b_{n_p}, h' = c_{n_p} - y^*$. Then $(h, h') \in \left(0, \frac{1}{k}\right) \times (0, \frac{1}{k})$. Put $i = i' = 2$ and fix $j, j' \in \{1, \ldots, k\}$. Since $p > 2k$, there exists $\ell \in \{1, \ldots, 2^p\}$ such that $x_1^p \in \left(a_{n_p} + \frac{j-1}{2k} b_{n_p}, a_{n_p} + \frac{j}{2k} b_{n_p}\right)$ and $\delta_{x_1^p, c_{n_p}} > c_{n_p} - y^*$.

Let t such that $0 < t < \delta_{x_1^p, c_{n_p}}$ and $\delta_{x_1^p, c_{n_p}} + t < a_{n_p} + \frac{j}{2k} b_{n_p}$. Put $t' = \frac{1}{2}(c_{n_p} - y^*)$. Then $t' < \delta_{x_1^p, c_{n_p}}$.

Thus, for each $m, m' \in \{1, \ldots, n_p\}$, the set

$$I = \left(x_1^p + \frac{m-1}{n_p} t, x_1^p + \frac{m}{n_p} t\right) \times \left(c_{n_p} - \frac{m'}{n_p} t', c_{n_p} - \frac{m'-1}{n_p} t'\right) \cap \overset{\circ}{B} \neq \emptyset.$$

Consequently, the intersection

$$\left(a_{n_p} + \frac{j-1}{2k} b_{n_p}, a_{n_p} + \frac{j}{2k} b_{n_p}\right) \times$$
$$\left(y^* + \left(\frac{1}{2} + \frac{j'-1}{2k}\right)(c_{n_p} - y^*), y^* + \left(\frac{1}{2} + \frac{j'}{2k}\right)(c_{n_p} - y^*)\right) \cap B$$

includes I and then it's not empty.

In conclusion, we may assert that for each $k \in N$ there exist $h, h' \in \left(0, \frac{1}{k}\right)$ such that for each $j, j' \in \{1, \ldots, k\}$ we have

$$\left(\left(\frac{1}{2} + \frac{j-1}{2k}\right) h, \left(\frac{1}{2} + \frac{j}{2k}\right) h\right) \times$$
$$\left(y^* + \left(\frac{1}{2} + \frac{j'-1}{2k}\right) h', y^* + \left(\frac{1}{2} + \frac{j'}{2k}\right) h'\right) \cap (-A) \neq \emptyset.$$

Then $(0, y^*)$ is not a strong $\mathcal{I}$-density point of the set A and that is in contradiction with the fact that $(0, y^*) \in A$.

Remark 2: It is obvious that D is closed in the $\tilde{d}$-topology (see [3]). Then an analogous proposition for strong (ordinary) density holds, i.e. the Lusin-Menchoff theorem for $\tilde{d}$-topology is not true.

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