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MONOTONICITY AND LOCAL SYSTEMS

Using the notion of a local system with some "intersection conditions", considered by Thomson in [11] and [12], we extend Theorems 6, 3 and 4 of Preiss ([8]). The main result of the paper is the monotonicity Theorem 4. In [3] the author extends Bruckner's reduction theorem (see Theorem 8), but we don't know if Theorem 4 of Preiss ([8]) and our Theorem 4 follow by Theorem 8 of [3]. For convenience, if P is a property for functions defined on a certain domain, we will also use P to denote the class of all functions having property P.

We need the following definitions and notations:

Definition 1. ([11], p.3 and [12], p. 280). The family $\mathcal{L} = \{\mathcal{L}(x) : x \in R\}$ is said to be a local system of sets provided it has the following properties: (i) $\{x\} \notin \mathcal{L}(x)$; (ii) if $S \in \mathcal{L}(x)$ then $x \in S$; (iii) if $S_1 \in \mathcal{L}(x)$ and $S_2 \supset S_1$ then $S_2 \in \mathcal{L}(x)$; (iv) if $S \in \mathcal{L}(x)$ and $\delta > 0$ then $S \cap (x-\delta, x+\delta) \in \mathcal{L}(x)$. The system $\mathcal{L}$ is bilateral (resp. bilaterally c-dense) provided every set $S \in \mathcal{L}(x)$ contains points on either side of x (resp. is bilaterally c-dense in itself).

Definition 2. ([11], p. 117). Let $\mathcal{L}$ be a local system. A function $f : [0,1] \to R$ is said to be $\mathcal{L}$-increasing at a point x provided $\sigma_x = \{y : y = x \text{ or } (f(y) - f(x))/(y-x) \geq 0\} \in \mathcal{L}(x)$. If "$\geq$" is replaced by "$>$" we say that f is strictly $\mathcal{L}$-increasing. Similarly we define the conditions $\mathcal{L}$-decreasing and strictly $\mathcal{L}$-decreasing. We denote by $\mathcal{L} - \underline{D}f(x) = \sup\{c \in R : \{x\} \cup \{y : (f(y) - f(x))/(y - x) > c\} \in \mathcal{L}(x)\}$. $\mathcal{L} - \overline{D}f(x)$ is defined similarly (see [12], p. 281).

An exact $\mathcal{L}$-derivative of f at x_0, if it exists, is any number c (including $\pm\infty$) such that, for any neighborhood U of c the set of points $\{y : y = x \text{ or } \frac{f(y)-f(x_0)}{y-x_0} \in U\}$ belongs to $\mathcal{L}(x_0)$. In this case we write $(\mathcal{L}) - Df(x_0) = c$, with the warning that the number c need not be unique, nor have an immediate relations with the two extreme $(\mathcal{L})$-derivates. The set of all $(\mathcal{L})$-derivates of a function f at a point x_0 will be denoted by $(\mathcal{L}) - \Delta(f, x_0)$ ([11], p. 140).

Definition 3. ([12], p. 292 and [2], p. 101). A local system $\mathcal{L} = \{\mathcal{L}(x) : x \in R\}$ will be said to satisfy the intersection conditions listed below if corresponding to any choice $\{\sigma_x : x \in R\}$ from $\mathcal{L}$ there must exist a positive function δ such that whenever $x, y \in R$ and $O < y - x < \min\{\delta(x), \delta(y)\}$ the two sets δ_x and δ_y must intersect in the asserted fashion:

(3.1) intersection condition (I.C.): $\delta_x \cap \delta_y \cap [x, y] \neq \emptyset$;

(3.2) external intersection condition (E.I.C.): $\delta_x \cap \delta_y \cap (y, 2y - x) \neq \emptyset$ and $\delta_x \cap \delta_y \cap (2x - y, x) \neq \emptyset$;

(3.3) external intersection condition, parameter mm (E.I.C.[m]): $\delta_x \cap \delta_y \cap (y, (m+1)y - mx) \neq \emptyset$ and $\delta_x \cap \delta_y \cap ((m+1)x - my, x) \neq \emptyset$;

(3.4) $\delta_x \cap \delta_y \cap (-\infty, x] \neq \emptyset$ and $\delta_x \cap \delta_y \cap [y, +\infty) \neq \emptyset$;

(3.5) $\delta_x \cap \delta_y \cap (-\infty, x] \neq \emptyset$

(3.6) $\delta_x \cap \delta_y \cap [y, +\infty) \neq \emptyset$.

Let $f : [0,1] \to \overline{R}$ be a function. We denote by $E_a(f) = \{x : f(x) > a\}$; $E^a(f) = \{x : f(x) < a\}$; $E_a^b(f) = \{x : a < f(x) < b\}$.

Definition 4. ([5],[7]). A measurable function $f : [0,1] \to \overline{R}$ is said to have the Denjoy-Clarkson property (D.C. - property) if for $-\infty < a < b < +\infty$, the set $E_a^b(f)$ has positive measure in every one-sided neighborhood of any of its points when $E_1^b(f) \neq \emptyset$.

Definition 5. ([5],[7]). A measurable function $f : [0,1] \to \overline{R}$ is m_2 (resp. $\overline{m}_2$) if $E_a(f)$ (resp. $E^a(f)$) for $a \in R$ has positive measure in any one sided neighborhood of any of its points when $E_a(f) \neq \emptyset$ (resp. $E^a(f) \neq \emptyset$).

Definition 6. (Baire conditions). Let $f : [0,1] \to \overline{R}$. Then $f \in \underline{B_1}$ (resp. $\overline{B}_1$) iff $E_a(f)$ (resp. $E^a(f)$) is F_σ. It follows that $B_1 = \overline{B}_1 \cap \underline{B_1}$.

Let $m_2 = \overline{m}_2 \cap \underline{m_2}$; $\underline{M_2} = \underline{B_1} \cap \underline{m_2}$; $\overline{M}_2 = \overline{B}_1 \cap \overline{m}_2$; $M_2 = m_2 \cap B_1 \subsetneqq DB_1$ (DB_1 = condition Darboux Baire one), see [13]).

Definition 7. ([5]). A function $f : [0,1] \to \overline{R}$ is wB_1 (wide B_1) if for $-\infty < a < b < +\infty$ and for every open interval I the sets $\{x : f(x) \leq a\}$ and $\{x : f(x) \geq b\}$ are not simultaneously dense in $I \cap \overline{E_a^b(f)}$ when $I \cap E_a^b(f) \neq \emptyset$. Clearly $B_1 \subsetneqq wB_1$ (see Theorem 1 of [8], p. 376).

Definition 8. ([8], Theorem 4, p. 378). Let $f : [0,1] \to \overline{R}$. If $\lim_{b \to 0_+} f(x - b) \leq f(x)$ for $x \in (0,1]$ and $\lim_{b \to 0_+} f(x + b) \geq f(x)$, for $x \in [0,1)$ (if these limits exist) then we say that f is uP. If $-f \in uP$ then we say that $f \in 1P$. Let $\mathcal{P} = 1P \cap uP$.

Definition 9. ([4], p. 424). A function $f : [0,1] \to R$ is uCM if f is increasing on the closed subinterval $[c,d] \subset [0,1]$ whenever it is so on the open interval (c,d). Let $1CM = \{f : -f \in uCM\}$ and let $CM = 1CM \cap uCM$. Let $sCM = \{f : f(x) + \lambda x \in CM, \text{ for each } \lambda \in R\}$.

Definition 10. ([2], p. 104). Let δ be a positive function and let X be a set of real numbers. By a δ–decomposition of X we shall mean a sequence of sets $\{X_n\}$ which is a relabelling of the countable collection $Y_{mj} = \{x \in X : \delta(x) > 1/m\} \cap [j/m, (j+1)/m]$, $m = 1, 2, \dots$ and $j = 0, \pm 1, \pm 2, \pm 3, \dots$.

Remark 1. ([2], p. 104, [11], p. 32-33). The key features of such a decomposition of the set X are: (i) $\bigcup_{n=1}^{\infty} X_n = X$; (ii) if x and y belong to the same set X_n then $|x-y| < \min\{\delta(x), \delta(y)\}$; (iii) if $x \in X \cap \overline{X_n}$ and $y \in (x-\delta(x),\ x+\delta(x)) \cap X_n$ then again one must have $|x-y| < \min\{\delta(x), \delta(y)\}$.

Let $f : [0,1] \to \overline{R}$ and let P be a subset of $[0,1]$, $a \in R$. Let $E_a(f;P) = \{x \in P : f(x) > a\}$; $E^a(f;P) = \{x \in P : f(x) < a\}$.

Theorem A. *Let $f : [0,1] \to \overline{R}$. The following assertions are equivalent:*

(A.1) *$f \in B_1$ (f is in Baire class one);*

(A.2) *for each closed subset P of $[0,1]$ and for any real numbers $a < b$ at most one of the sets $\{x \in P : f(x) \geq b\}$, $\{x \in P : f(x) \leq a\}$ is dense in P;*

(A.3) *for each closed subset P of $[0,1]$ there exists at most one real number p (depending on P) such that $\overline{E_p(f;P)} = \overline{E^p(f;P)} = P$;*

(A.4) *for each closed subset P of $[0,1]$ and for any real numbers $a < b$ at most one of the sets $E_b(f;P)$, $E^a(f;P)$ is dense in P.*

Proof. The equivalence of (A.1) and (A.2) follows by [8] (Theorem 1, p. 376). We show that (A.2) implies (A.3). Suppose that $f \in$ (A.2) and $f \notin$ (A.3). Then there exist a closed subset P of $[0,1]$ and real numbers $a < b$ such that $\overline{E_a(f;P)} = \overline{E^a(f;P)} = \overline{E_b(f;P)} = \overline{E^b(f;P)} = P$. Hence $\overline{\{x \in P : f(x) \leq a\}} = \overline{\{x \in P : f(x) \geq b\}} = P$. Therefore $f \notin$ (A.2). We show that (A.3) implies (A.4). Suppose that $f \in$ (A.3) and $f \notin$ (A.4). Then there exist a closed subset P of $[0,1]$ and real numbers $a < b$ such that $\overline{E_b(f;P)} = \overline{E^a(f;P)} = P$.

Since $E_b(f;P) \subset E_a(f;P)$ and $E^a(f;P) \subset E^b(f;P)$, it follows that $\overline{E^a(f;P)} = \overline{E_a(f;P)} = \overline{E^b(f;P)} = \overline{E_b(f;P)} = P$. Hence $f \notin$ (A.3). We show that (A.4) implies (A.2). Suppose that $f \in$ (A.4) and $f \notin$ (A.2). Then there exist a closed subset P of $[0,1]$ and real numbers $a < b$ such that $\overline{\{x \in P : f(x) \geq b\}} = \overline{\{x \in P : f(x) \leq a\}} = P$. Let $a < a_1 < b_1 < b$ then $\overline{E_{b_1}(f;P)} = \overline{E^{a_1}(f;P)} = P$. Hence $f \notin$ (A.4).

Theorem B. (Theorem 1 of [5]). *Let $f : [0,1] \to \overline{R}$ be a Darboux function. Then f is wB_1 iff for $-\infty < a < b < +\infty$ and for each open interval with $I \cap E_a^b(f) \neq \emptyset$ there exists an open subinterval J of I with $J \cap E_a^b(f) \neq \emptyset$ such that either $J \subset E_a(f)$ or $J \subset E^b(f)$.*

Theorem C. *Let $f : [0,1] \to \overline{R}$. We have:* a) *If f is a Darboux function and $f \in wB_1 \cap m_2$ then $f \in D.C.$ (see the proof of Theorem 3 of* [5]; b) *If f is a Darboux function and $f \in D.C.$ then $f \in m_2$;* c) *If f is finite and $f \in D.C.$ then $f \in m_2$.*

Proof. Let $f : [0,1] \to \overline{R}$, $f \in D.C.$ We prove that $f \in \underline{m_2}$ (that $f \in \overline{m}_2$ follows analogously). Let $a \in R$ and let $x_0 \in E_a(f)$. If $f(x_0) < +\infty$ then there exists a natural number n such that $f(x_0) < n$. Let $\delta > 0$ and $\mathcal{T} = (x_0 - \delta, x_0)$ or $\mathcal{T} = (x_0, x_0 + \delta)$. Since $f \in D.C.$, $m(E_a^n(f) \cap \mathcal{T} > 0$, hence $m(E_a(f) \cap \mathcal{T} > 0$. If $f(x_0) = +\infty$, suppose that there exists $\delta > 0$ such that, for example, $m(E_a(f) \cap (x_0, x_0 + \delta)) = 0$. Let $x_1 \in (x_0, x_0 + \delta/2)$ such that $f(x_1) \leq a$. Since f is Darboux, there exists $x_2 \in (x_0, x_0 + \delta/2)$ such that $f(x_2) \in E_a^k(f)$ for some natural number k. Hence $m(E_a^k(f) \cap (x_0, x_0 + \delta)) > 0$, a contradiction. It follows that $f \in \underline{m_2}$.

Remark 2. There exists a function $f : [0,1] \to \overline{R}$ which is not Darboux such that $f \in D.C. \cap B_1$ and $f \notin m_2$. (Indeed, let $f(x) = 0$, $x \in [0,1] \setminus \{1/2\}$ and $f(1/2) = +\infty$.)

Corollary D. *Let $f : [0,1] \to \overline{R}$, $f \in B_1$. If $f \in m_2$ then f is Darboux and $f \in D.C.$*

Proof. Since $f \in B_1 \cap m_2 = M_2 \subsetneqq DB_1$ and $B_1 \subsetneqq wB_1$, by Theorem C,a), it follows that $f \in D.C.$

Remark 3. Corollary D was obtained before by Mukhopadhyay in [7] (Theorem 1, p. 280), but for f a finite function.

Proposition 1. *Let $f : [0,1] \to R$, $f \in uCM$ and let $h : [0,1] \to R$, h continuous and increasing on $[0,1]$. Then $f - h \in uCM$.*

Proof. Let $g(x) = f(x) - h(x)$ and let $(c,d) \subset [0,1]$ such that g is increasing on (c,d). It follows that $f(x) = g(x) + h(x)$ is increasing on (c,d). Since $f \in uCM$ it follows that f is increasing on $[c,d]$. Suppose that there exists $x_1 \in (c,d)$ such that $g(x_1) > g(d)$. Let $\varepsilon = g(x_1) - g(d)$. Since h is continuous it follows that there exists $d \in (0, d - x_1)$ such that $h(x) > h(d) - \varepsilon$, for each $x \in (d - \delta, d)$. Since $g(x) > g(x_1)$, for each $x \in (d - \delta, d)$, it follows that $f(x) = g(x) + h(x) > f(d)$. This contradicts the fact that f is increasing on $[c,d]$. Hence g is increasing on $[c,d]$ and $g \in uCM$.

Corollary 1. *Let $f : [0,1] \to R$. Then the following conditions are equivalent: a) $f \in sCM$; b) $f(x) + \lambda x$ and $\lambda x - f(x)$ are uCM for each $\lambda \geq 0$.*

Proof. a) $\Rightarrow$ b) is evident. We show that b) $\Rightarrow$ a). If $\lambda = 0$ then $f(x) \in CM$. If $\lambda < 0$ then by Proposition 1, $f(x) + \lambda x \in uCM$. By hypothesis, $-f(x) - \lambda x \in uCM$, hence $f(x) + \lambda x \in CM$. If $\lambda > 0$ then by hypothesis $f(x) + \lambda x \in uCM$. By Proposition 1, $-f(x) - \lambda x \in uCM$, hence $f(x) + \lambda x \in CM$.

Example 1. Let $F : [0,1] \to [-1,1]$, $F(x) = 1 - x$, $x \in [0,1)$ and $F(1) = -1$. Then we have:

a) $F \in CM \subset uCM$ on $[0,1]$.

b) $F(x) + \lambda x \notin uCM$ on $[0,1]$ if $\lambda \geq 1$.

c) $F \notin uP$ on $[0,1]$.

Example 2. Let $F : [0,1] \to [-1,1]$, $F(x) = x \sin\frac{2\pi}{x}$, $x \in (0,1]$, $F(0) = 1$. Then we have:

a) F is continuous on $(0,1]$

b) $F \notin uP$ on $[0,1]$

c) $F \in sCM$ on $[0,1]$

Proof. a) is evident; b) $F(0) = 1 \not< \lim_{x \to 0+} F(x) = 0$; c) we prove that $F(x) + \lambda x$ is uCM for each $\lambda \in R$. The case $F(x) + \lambda x$ is ℓCM is similar. Let (c,d) be a subinterval of $(0,1)$ such that $F(x) + \lambda x$ is increasing on (c,d). If $c \neq 0$, since F is continuous on $(0,1]$ it follows that $F(x)+\lambda x$ is increasing on $[c,d]$. Hence $F(x)+\lambda x$ is uCM. If $c=0$, we observe that $F(x)+\lambda x$ is monotone on no (c,d). q.e.d.

Proposition 2. *For function $F:[0,1]\to R$ we have:*

a) *$u\mathcal{P}\oplus\mathcal{C}=u\mathcal{P}$, where $\mathcal{C}$ the class of all continuous function defined on $[0,1]$ and $\mathcal{A}_1\oplus\mathcal{A}_2$ denotes the linear space generated by the classes of functions $\mathcal{A}_1$ and $\mathcal{A}_2$.*

b) *$u\mathcal{P}\subsetneqq uCM$.*

c) *$\mathcal{P}\subsetneqq sCM$.*

Proof. a) is evident; b) $u\mathcal{P}\subset uCM$. Let $F:[0,1]\to R$, $F\in u\mathcal{P}$ and let $(c,d)\subset(0,1)$ such that F is increasing on (c,d). Let $x_1\in(c,d)$. Then $F(c)\le\lim_{x\to c^+}F(x)\le F(x_1)\le\lim_{x\to d^-}F(x)$. (These limits exist since F is increasing on (c,d).) Hence F is increasing on $[c,d]$, and consequently $F\in uCM$ on $[0,1]$. That $u\mathcal{P}\subsetneqq uCM$ follows from Example 1, a) and c).

c) For $\mathcal{P}\subset sCM$ see Proposition 2, a) and b) and Definitions 8 and 9. That $\mathcal{P}\subsetneqq sCM$ follows from Example 2, b) and c).

Lemma 1. (Theorem 50.2, p. 117 of [11]). *Let $\mathcal{L}$ be a local system which satisfies intersection condition* I.C. *Let $f:[0,1]\to R$. If f is $\mathcal{L}$–increasing on $[0,1]$ then f is increasing on $[0,1]$.*

Proof. (based on a different idea than that in [11]). Let P be the collection of all x for which there exists no open interval containing x on which f is increasing. It is easy to show that the complement of P is an open set U. Further

(1) f is increasing on the closure of each component interval of U.

This implies that P is a perfect set. We prove that P is empty. Suppose that $P\ne\emptyset$. For $\sigma_x=\{y:y=x \text{ or } (f(y)-f(x))/(y-x)\ge 0\}\in\mathcal{L}(x)$, let $\delta(x)>0$, $x\in[0,1]$ given by condition I.C. Let $P_{nm}=\{x\in P: x\in[m/n,(m+1)/n],\ 1/n<\delta(x)<1/(n-1)\}$, $n=2,3,\dots$, $m=0,1,\dots,n-1$. By the Baire Category Theorem, there exists an open interval (a,b) such that $\emptyset\ne(a,b)\cap P\subset\overline{P}_{nm}$ for some n and m. Let $x_0<y_0$, x_0 a right accumulation point of $P\cap(a,b)$ and y_0 a left accumulation point of $P\cap(a,b)$. Let $x_1,y_1\in P_{nm}$, $x_0<x_1<y_1<y_0$, $x_1\in(x_0,x_0+\delta(x))$, $y_1\in(y_0-\delta(y_0),y_0)$. Then $\sigma_{x_0}\cap\sigma_{x_1}\ne\emptyset$; $\sigma_{x_1}\cap\sigma_{y_1}\ne\emptyset$; $\sigma_{y_1}\cap\sigma_{y_0}\ne\emptyset$, hence $f(x_0)\le f(y_0)$. Now by (1) it follows that f is increasing on (a,b), a contradiction. Hence $P=\emptyset$.

Corollary 2. *Let $\mathcal{L}$ be a local system which satisfies intersection condition* I.C. *Let $f : [0,1] \to R$. If $\mathcal{L} - \underline{D}f(x) \geq 0$* a.e. *and $\mathcal{L} - \underline{D}f(x) > -\infty$ everywhere then f is increasing on $[0,1]$. Moreover, suppose that for each point $x \in [0,1]$ there exists an exact $\mathcal{L}$-derivative of f at x, denoted by $(\mathcal{L}) - Df(x)$. If $(\mathcal{L}) - Df(x) \geq 0$* a.e. *and $(\mathcal{L}) - Df(x) > -\infty$ everywhere then f is increasing on $[0,1]$.*

Proof. Let $\varepsilon > 0$ and let $E = \{x : \mathcal{L} - \underline{D}f(x) < 0\}$. Then $|E| = 0$. By [1] (Lemma 1.2, p. 124) there exists an increasing function $g : [0,1] \to [0, \varepsilon)$ such that $g'(x) = +\infty$ on E, $g(0) = 0$ and $g'(x) > 0$ for all $x \in [0,1] \backslash E$. Then $f + g$ is strictly $\mathcal{L}$-increasing on $[0,1]$ and by Lemma 1, $f + g$ is increasing on $[0,1]$. Since ε was arbitrary, f is increasing on $[0,1]$. For the second part we see that there exists an exact $\mathcal{L}$-derivative of $f + g$ at x denoted by $(\mathcal{L}) - D[f + g](k)$ which is everywhere strictly greater than 0. Hence $f + g$ is strictly $\mathcal{L}$-increasing on $[0,1]$. Using again Lemma 1, it follows that f is increasing on $[0,1]$.

Theorem 1. (An extension of Theorem 6 of [8]). *Let $\mathcal{L}$ be a local system which satisfies intersection condition* I.C. *Let $f : [0,1] \to R$ be a function such that:* (i) *$f \in sCM$ on $[0,1]$;* (ii) *An exact $\mathcal{L}$-derivative $(\mathcal{L}) - Df(x)$ exists finite or infinite at every point $x \in [0,1]$;* (iii) *$(\mathcal{L}) - Df(x)$ is B_1 on $[0,1]$. Then:* a) *$(\mathcal{L}) - Df(x)$ is m_2;* b) *$(\mathcal{L}) - Df(x)$ is a Darboux function and satisfies the* D.C.*-property;* c) *f fulfills the Mean Value Theorem.*

Proof. Let $g(x) = (\mathcal{L}) - Df(x)$. a) We show that $g \in \overline{m}_2$. Suppose that $E^\lambda = \{x : g(x) < \lambda\} \neq \emptyset$ and that there exist a point $x_0 \in E^\lambda$ and $\delta_0 > 0$ such that $x_0 - \delta_0 > 0$ and, for example, $g(x) > \lambda$ a.e. on $(x_0 - \delta_0, x_0)$. Let $A = \{x \in (x_0 - \delta_0, x_0) : g \text{ is continuous at } x\}$. If $x \in A$ then $g(x) \geq \lambda$. (Indeed, if $g(x) < \lambda$ then there exists $\delta > 0$ with $(x - \delta, x + \delta) \subset (x_0 - \delta, x_0)$, such that $g(y) < \lambda$ for each $y \in (x - \delta, x + \delta)$, a contradiction.) Let $x \in A$ then there exists a closed interval $[c, d] \subset (x_0 - \delta_0, x_0)$ such that $x \in (c, d)$ and $g(y) > -\infty$ on $[c, d]$. By Corollary 2, $f(x) - \lambda x$ is increasing on $[c, d]$, hence there exist maximal open intervals (a_n, b_n) such that $f(x) - \lambda x$ is increasing on each (a_n, b_n). By (i) it follows that $f(x) - \lambda x$ is increasing on $[a_n, b_n]$. Hence the set $G = \cup(a_n, b_n)$ is dense in $(x_0 - \delta_0, x_0)$ and the set $P = \overline{(x_0 - \delta_0, x_0)} \backslash G$ is a perfect set. Suppose on the contrary that $P \neq \emptyset$. Let $x_1 \in (x_0 - \delta_0, x_0) \backslash G$ be a point of continuity for $g|_P$. Then $g(x_1) \geq \lambda$. (Indeed, if $g(x_1) < \lambda$ then by (iii) there exists $\delta_1 > 0$ such that $(x_1 - \delta_1, x_1 + \delta_1) \subset (x_0 + \delta_0, x_0)$ and $g(y) < \lambda$ for each $y \in (x_1 - \delta_1, x_1 + \delta_1) \cap P$. Let $(a_n, b_n) \subset (x_1 - \delta_1, x_1 + \delta_1)$ for some natural number n. Then $g(a_n) \geq \lambda$ and $g(b_n) \geq \lambda$, a contradiction.) It follows that there exists a closed interval $[c_1, d_1]$ such that $g(y) > \lambda - 1$ on $P \cap [c_1, d_1]$. By Corollary 2 we have that $f(x) - \lambda x$ is increasing on $[c_1, d_1]$, a contradiction. Hence $G = (x_0 - \delta_0, x_0)$. By (i) it follows that $f(x) - \lambda x$ is increasing on $[x_0 - \delta_0, x_0]$, hence $g(x_0) \geq \lambda$, a contradiction.

b) See a), (iii) and Corollary D.

c) For every a, b, $0 \leq a < b \leq 1$, let $\lambda = \frac{f(b)-f(a)}{b-a}$. Suppose that there is no $x_0 \in (a, b)$ such that $g(x_0) = \lambda$. Since g is a Darboux function, it follows that either $g(x) > \lambda$ or $g(x) < \lambda$ on (a, b). In the first situation, for example, it follows by Corollary 2 that $f(x) - \lambda x$ is increasing on $[a, b]$. Since $g(x) > \lambda$ on (a, b) it follows that $f(b) - \lambda > f(a)$, a contradiction.

Observation. In Theorem 6 of [8] condition (i) is replaced by the restrictive condition $F \in \mathcal{P}$ (see Proposition 2, c)). Also the function F from Example 2 satisfies the hypothesis of our Theorem 1, but not of Preiss' Theorem 6.

Lemma 2. *Let $f : [0,1] \to R$ and let $P \neq \emptyset$ be a G_δ subset of $(0,1)$. Let $\mathcal{L}$ be a local system with intersection condition (3.4.). Let $A \subseteq \{x \in P : f$ is $\mathcal{L}-$ increasing at $x\}$ and $B \subseteq \{x \in P : f$ is strictly $\mathcal{L} -$ decreasing at $x\}$. Suppose that $P = \overline{A}$ (resp. $P = \overline{B}$). We have:* a) *B (resp. A) is of first category with respect to P*; b) *If $E = P \backslash (A \cup B)$ is countable then B (resp. A) is nowhere dense in P.*

Proof. Let $\sigma_x = \{y : y = x$ or $(f(y) - f(x))/(y - x) \geq 0,\ y \neq x\} \cap (0,1)$ for $x \in A$ and let $\sigma_x = \{y : y = x$ or $(f(y) - f(x))/(y - x) < 0,\ y \neq x\} \cap (0,1)$ for $x \in B$. Let $\delta(x) \in (0,1)$ be the δ given by condition (3.4) for $x \in A \cup B$. Hence, if $x, y \in A \cup B$ such that $|x - y| < \min\{\delta(x), \delta(y)\}$ then $\sigma_x \cap \sigma_y \cap (-\infty, x) \neq \emptyset$ and $\sigma_x \cap \sigma_y \cap (y, +\infty) \neq \emptyset$. Suppose that $P = \bar{A}$ (the second part follows analogously).

a) Let $G_n = \bigcap_{x \in A}(x - \delta(x)/n, x + \delta(x)/n)$ and let $H = P \cap (\bigcap_{n=1}^{\infty} G_n)$.
Then H is a dense G_δ set in P, $A \subset H \subset P$. We prove that $B \cap H = \emptyset$. Suppose on the contrary that $B \cap H \neq \emptyset$. Let $y \in B \cap H$. Let n be a natural number such that $1/n < \delta(y)$. Then $y \in G_n$, hence there exists $x \in A$ such that $y \in (x - \delta(x)/n, x + \delta(x)/n)$. Since $\delta(x)/n < 1/n < \delta(y)$, it follows that $|y - x| < \min\{\delta(x), \delta(y)\}$. Suppose, for example, that $x < y$ (the case $y > x$ is similar). Then we have two situations: 1) $f(x) \leq f(y)$ and 2) $f(x) > f(y)$.

1) Let $z \in (-\infty, x) \cap \sigma_x \cap \sigma_y \neq \emptyset$ (see condition (3.5)). Then $f(z) \leq f(x)$ and $f(z) > f(y)$, a contradiction. 2) Let $z \in (y, +\infty) \cap \sigma_x \cap \sigma_y \neq \emptyset$ (see condition (3.6)). Then $f(z) \geq f(x)$ and $f(z) < f(y)$, a contradiction. It follows that $B \cap H = \emptyset$, hence $B \subset P \backslash H$ which is a set of first category with respect to P.

b) Suppose on the contrary that $\emptyset \neq (c, d) \cap P \subset \bar{B}$. Let $A_{mn} = \{x \in (c, d) \cap A \cap [m/n, (m + 1)/n] : \delta(x) \in (1/n, 1/(n - 1)]\}$ and let $B_{mn} = \{x \in (c, d) \cap B \cap [m/n, (m + 1)/n] : \delta(x) \in (1/n, 1/(n - 1)]\}$, where $n = 2, 3, \ldots$, $m = 0, 1, 2, \ldots, n - 1$. Then $(c, d) \cap P = \bigcup_{n,m}(A_{mn} \cap B_{mn}) \cap E$. By the Baire Category Theorem it follows that there exists an open interval $(a, b) \subset (c, d)$ such that either 1) $\emptyset \neq (a,b) \cap P \subset \bar{A}_{mn}$ or 2) $\emptyset \neq (a,b) \cap P \subset \bar{B}_{mn}$, for some n and m. 1) Let $y \in (a,b) \cap B$ and let $x \in (y - \delta(y), y + \delta(y)) \cap A_{mn} \cap (a,b)$. Then $|x-y| < \min\{\delta(x), \delta(y)\}$, a contradiction (as at a), 1) and 2)). 2) Let $x \in (a,b) \cap A$ and let $y \in (x - \delta(x), x + \delta(x)) \cap B_{mn} \cap (a,b)$. Then $|x-y| < \min\{\delta(x), \delta(y)\}$, a contradiction (as at a), 1) and 2)).

Theorem 2. (An extension of Theorem 3 of [8]). *Let $F : [0,1] \to R$ and let $\mathcal{L}$ be a local system with intersection condition* (3.4) *such that $\mathcal{L} - DF(x)$ exists (finite or infinite) at each point $x \in [0,1]$. Then $\mathcal{L} - DF(x)$ is B_1 on $[0,1]$.*

Proof. Let $f(x) = \mathcal{L} - DF(x)$. Suppose that $f \notin B_1$. By (A.3) (Theorem A), there exist a closed subset P of $[0,1]$ and real numbers $a < b$ such that $\overline{E_a(f;P)} = \overline{E_b(f;P)} = \overline{E^a(f;P)} = \overline{E^b(f;P)} = P$. Applying Lemma 2, a) to $F(x) - ax$ and $F(x) - bx$ it follows that $E_a(f;P)$, $E^a(f;P)$, $E_b(f;P)$, $E^b(f;P)$ are of first category with respect to P. Hence $\{x \in P : f(x) = a\}$ and $\{x \in P : f(x) = b\}$ are residual sets with respect to P, a contradiction.

Definition 11. ([4], p. 69 and [9], p. 236). A function $F : [0,1] \to R$ is said to be $\underline{AC}$ on a set $E \subset [0,1]$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\Sigma(F(b_i) - F(a_i)) > -\varepsilon$ for each finite set $\{[a_i, b_i]\}$ of nonoverlapping intervals with endpoints in E and $\Sigma(b_i - a_i) < \delta$. $F \in \overline{AC}$ on E if $-F \in \underline{AC}$ on E. $AC = \overline{AC} \cap \underline{AC}$.

Definition 12. Let $f : [0,1] \to R$ and let $E \subset [0,1]$. We say that $f \in \underline{L}$ on E if there exists $\lambda \in R$ such that $f(y) - f(x) > \lambda(y - x)$, $y > x$, $x, y \in E$. $f \in \overline{L}$ on E if $-f \in \underline{L}$ on E.

Let P be a closed subset of $[0,1]$. We denote by P^+ (resp. P^-) the set $\{x \in P : x$ is a right (resp. left) accumulation point of $P\}$.

Definition 13. Let $f : [0,1] \to R$ and let P be a perfect subset of $[0,1]$. We say that $f \in \underline{L}'$ (resp. $\underline{L}''$) on P if there exists $\lambda \in R$ such that $f(y) - f(x) > \lambda \cdot (y - x)$, $y > x$, $x \in P^+$, $y \in P^-$ (resp. $x \in P^-$, $y \in P^+$). $f \in \bar{L}'$ (resp. $\bar{L}''$) on P if $-f \in \underline{L}'$ (resp. $\underline{L}''$) on P.

Definition 14. Let $f : [0,1] \to R$ and let P be a perfect subset of $[0,1]$. We say that $f \in \underline{AC}'$ (resp. $\underline{AC}''$) on P if for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $I_k = [a_k, b_k]$, $k = 1, 2, \ldots$, is a sequence of nonoverlapping intervals with $a_k \in P^+$, $b_k \in P^-$ (resp. $a_k \in P^-$, $b_k \in P^+$) and $\Sigma(b_k - a_k) < \delta$ then $\Sigma(f(b_k) - f(a_k)) > -\varepsilon$. $f \in \overline{AC}'$ (resp. $\overline{AC}''$) on P if $-f \in \underline{AC}'$ (resp. $\underline{AC}''$) on

P.

Remark 4. a) $\underline{L} \subset \underline{AC}$; $\overline{L} \subset \overline{AC}$ for finite functions on a set $E \subset [0,1]$; b) For finite functions on a perfect set P we have: $\underline{L} \subset \underline{L}'$; $\underline{L} \subset \underline{L}''$; $\overline{L} \subset \overline{L}'$; $\overline{L} \subset \overline{L}''$; $\underline{AC} \subset \underline{AC}'$; $\overline{AC} \subset \overline{AC}''$; $\overline{AC} \subset \overline{AC}'$; $\underline{AC} \subset \underline{AC}''$; c) $\underline{L}' \subset \underline{L}$ *and* $\underline{L}'' \subset \underline{L}$ for finite functions on an interval (a,b).

Lemma 3. *Let P be a perfect subset of $[0,1]$ and let (a_n, b_n) be the intervals contiguous to P with respect to $(0,1)$. Let $F : [0,1] \to R$ and $f : [0,1] \to R$ be such that $f(x) = F(x)$, $x \in P$ and $f(x)$ is linear on each $[a_n, b_n]$. If $F \in \underline{AC}$ (resp. VB) on P then $f \in \underline{AC}$ (resp. VB) on $[0,1]$.* (For VB see [14], p. 221.)

Proof. Let $\varepsilon > 0$. For $\varepsilon/3$ we consider $\delta > 0$ given by the fact that $F \in \underline{AC}$ on P. Let $\mathcal{A}_- = \{n : F(b_n) < F(a_n)\}$, $\mathcal{A}_+ = \{n : F(b_n) \geq F(a_n)\}$. Let $\mathcal{A}'_-$ be a finite subset of $\mathcal{A}_-$ such that $\sum_{n \in \mathcal{A}_- \setminus \mathcal{A}'_-} (b_n - a_n) < \delta$. Let $m_\delta = \min\{(F(b_k) - F(a_k))/(b_k - a_k),\ k \in \mathcal{A}'_-\}$ and let $\eta > 0$ such that $m_\delta \cdot \eta > -\varepsilon/3$. Let $\delta_1 = \min\{\delta, \eta\}$. A closed interval $I = [a,b] \subset [0,1]$ is said of first kind of $a, b \in P$, and of second kind if $(a,b) \subset (0,1)\setminus P$. If $J = [c,d] \subset [0,1]$ is not of first or of second kind then $[c,d] \cap P \neq \emptyset$. Let $c_1 = \inf(P \cap [c,d])$ and $d_1 = \sup(P \cap [c,d])$. Then

2) $[c,d] = [c,c_1] \cup [c_1,d_1] \cup [d_1,d]$ and $F(d) - F(c) = F(d) - F(d_1) + F(d_1) - F(c_1) + F(c_1) - F(c)$.

Also $[c_1,d_1]$ is of first kind and $[c,c_1]$ and $[d_1,d]$ are of second kind. Let $J_i = [c_i,d_i]$, be a finite sequence of closed intervals such that $\Sigma(d_i,c_i) < \delta_1$. 1) If each J_i is of first kind, then clearly $\Sigma(f(d_i) - f(c_i)) = \Sigma(F(d_i) - F(c_i)) > -\varepsilon/3$. 2) If each J_i is of second kind then $\Sigma(f(d_i) - f(c_i)) = \sum_{i \in B_1}(f(d_i) - f(c_i)) + \sum_{i \in B_2}(f(d_i) - f(c_i)) + \sum_{i \in B_3}(f(d_i) - f(c_i)) \geq \sum_{n \in \mathcal{A}_- \setminus \mathcal{A}'_-}(F(b_n) - F(a_n)) + \sum_{i \in \mathcal{A}'_-} m_\delta \cdot |J_i| > -\varepsilon/3 - \varepsilon/3 = (-2/3)\varepsilon$, where $B_1 = \{i : (c_i,d_i) \subset \bigcup_{n \in \mathcal{A}_- \setminus \mathcal{A}'_-}(a_n,b_n)\}$; $B_2 = \{i : (c_i,d_i) \subset \bigcup_{n \in \mathcal{A}_-}(a_n,b_n)\}$; $B_3 = \{i : (c_i,d_i) \subset \bigcup_{n \in \mathcal{A}_+}(a_n,b_n)\}$. The general case follows by 1), 2) and (2). The second part follows similarly.

Remark 5. a) Let C be the Cantor ternary set and let (a_n,b_n), $n \geq 1$ be the intervals contiguous to C with respect to $(0,1)$. Let $f,g : [0,1] \to R$ such that $f(x) = g(x) = 0$, $x \in C\setminus(\cup\{a_n,b_n\})$; $f(a_n) = g(b_n) = 1$; $f(b_n) = g(a_n) = -1$, f,g are linear on each $[a_n,b_n]$. Then $f \in \underline{L}'$ on C; $g \in \underline{L}''$ on C; $f,g \notin [VBG]$ on C; f,g are Darboux on $[0,1]$; $f,g \notin B_1$ on C. Since $\underline{L}' \subset \underline{AC}'$ it follows that we can not replace $\underline{AC}$ by $\underline{AC}'$ in Lemma 3.

b) Lemma 3 is often used in [4] and [9] but without proof. Recall that a function F is VBG on a set X is X can be expressed as the union of a sequence of sets on each of which F is of bounded variation VB; if the sets in the sequence can be taken to be closed, F is said to be $[VBG]$.

Lemma 4. *Let $F : [0,1] \to R$. If $F \in \underline{AC}$ on $[0,1]$ then $F \in VB$ on $[0,1]$.*

Proof. For $\varepsilon = 1$ let $\delta > 0$ given by the fact that $F \in \underline{AC}$ on $[0,1]$. First we prove the following assertion:

(3) If $[a,b] \subset [0,1]$, $b - a < \delta$, $a = y_0 < y_1 < \cdots y_{k-1} < y_k = b$ then

$$\sum_{i=0}^{k-1} |F(y_{i+1}) - F(y_i)| < F(b) - F(a) + 2.$$

Let $\mathcal{A}_- = \{i : F(y_{i+1}) - F(y_i) < 0,\ i \in \{0,1,\ldots,k-1\}\}$ and $\mathcal{A}_+ = \{i : F(y_{i+1}) - F(y_i) \geq 0,\ i \in \{0,1,\ldots,k-1\}\}$. Since $F(b) - F(a) = \sum_{i=0}^{k-1}(F(y_{i+1}) - F(y_i))$ it follows that $\sum_{i=0}^{k-1} |F(y_{i+1}) - F(y_i)| = \sum_{i\in\mathcal{A}_+}(F(y_{i+1}) - F(y_i)) - \sum_{i\in\mathcal{A}_-}(F(y_{i+1}) - F(y_i)) = F(b) - F(a) - 2\sum_{i\in\mathcal{A}_-}(F(y_{i+1}) - F(y_i)) < F(b) - F(a) + 2$ and we have (3). Now we prove that $F \in VB$ on $[0,1]$. Let n be a natural number such that $(n-1)\cdot\delta \leq 1 < n\cdot\delta$. Let $0 = x_0 < x_1 < \cdots < x_m = 1$. Let j_i be such that $x_{j_i} \leq i/n < x_{j_i+1}$, $i = 1,2,\ldots,n-1$, $j_0 = 0$, $j_n = m$. By (3) we have $\sum_{j=0}^{m} |F(x_{j+1}) - F(x_j)| \leq \sum_{i=0}^{n-1}(|F(x_{j_i+1}) - F(i/n)| + |F(x_{j_i+2}) - F(x_{j_i+1})| + \cdots + |F((i+1)/n) - F(x_{j_i+1})|) < \sum_{i=0}^{n-1}(2 + F((i+1)/n) - F(i/n)) = 2n + F(1) - F(0)$, hence $F \in VB$ on $[0,1]$.

Remark 6. Let P, F, f be defined as in Lemma 3. If F is $\underline{AC}$ on P then F is VB on P. Indeed, if $F \in \underline{AC}$ on P then by Lemma 3, f is $\underline{AC}$ on $[0,1]$. By Lemma 4, $f \in VB$ on $[0,1]$, hence F is VB on P. This assertion is often used in [4] but without proof.

Lemma 5. *Let $F : [0,1] \to R$, $F \in \underline{AC}$. If $F'(x) \geq 0$ a.e. where $F'(x)$ exists then F is increasing on $[0,1]$.*

Proof. By Lemma 4 it follows that F is VB, hence F is derivable on a measurable set A, $|A| = 1$. By Vitali's covering theorem, applied to A and by the fact that $F \in \underline{AC}$, it follows that F is increasing on $[0,1]$.

Remark 7. Lemma 5 follows also by [9] (Theorem V, p. 237) and [10] (Lemma, p. 4).

Lemma 6. *Let $F : [0,1] \to R$. Let P be a perfect subset of $[0,1]$ and let (a_n, b_n) be the intervals contiguous to P with respect to $(0,1)$. If F is $\underline{AC}'$ on P and F is increasing on each interval $[a_n, b_n]$ then F is $\underline{AC}$ on $[0,1]$.*

Proof. Let $\varepsilon > 0$ and let $\delta > 0$ be given by the fact that F is $\underline{AC}'$ on P. Let $\{[c_i, d_i]\}_i$, be a sequence of closed subintervals of $[0,1]$ such that $\Sigma(d_i - c_i) < \delta$. Let $\mathcal{A} = \{i : (c_i, d_i) \cap P \neq \emptyset\}$. If $i \notin \mathcal{A}$ then $(c_i, d_i) \subset (0,1) \backslash P$, hence there exists n such that $[c_i, d_i] \subset [a_n, b_n]$. Since F is increasing on each $[a_n, b_n]$ it follows that $F(d_i) - F(c_i) \geq 0$. For $i \in \mathcal{A}$ let $c_i' = \inf(P \cap (c_i, d_i))$ and $d_i' = \sup(P \cap (c_i, d_i))$. Then $c_i' \in P^+$ and $d_i' \in P^-$. Clearly $(c_i, c_i') \subset (0,1) \backslash P$ and $(d_i', d_i) \subset (0,1) \backslash P$, hence $F(d_i) - F(d_i') \geq 0$ and $F(c_i') - F(c_i) \geq 0$. Then $\sum_{i=1}^{\infty}(F(d_i) - F(c_i)) \geq \sum_{i \in \mathcal{A}}(F(d_i) - F(c_i)) = \sum_{i \in \mathcal{A}}(F(d_i) - F(d_i') + F(d_i') - F(c_i') + F(c_i') - F(c_i)) \geq \sum_{i \in \mathcal{A}}(F(d_i') - F(c_i')) > -\varepsilon$, hence $F \in \underline{AC}$ on $[0,1]$.

__Lemma 7__. *Let $F : [0,1] \to R$, $F \in uCM$. Let P be the collection of all x for which there exists no open interval containing x on which F is increasing. If there exists a portion $(a,b) \cap P$ such that $F'(x) \geq 0$ a.e. where F is derivable and (i) $F \in \underline{AC}'$ on $(a,b) \cap P$ or (ii) $F \in \bar{L}''$ with constant $\lambda \in (-\infty, 0)$ on $(a,b) \cap P$, then $P = \emptyset$, hence F is increasing on $[0,1]$.*

Proof. It is easy to show that the complement of P is an open set U and F is increasing on each component of U. Since F is uCM it follows that F is increasing on the closure of each component interval of U, which implies that P is a perfect set. Suppose on the contrary that P is nonempty. By hypothesis there exists a portion $(a,b) \cap P \neq \emptyset$ such that we have (i) or (ii). (i) Since F is increasing on the closure of each component interval of U, by Lemma 6, it follows that F is $\underline{AC}$ on (a,b). Since $F'(x) \geq 0$ a.e. on (a,b) it follows that F is increasing, a contradiction. (ii) Suppose that there exists $(c,d) \subset (a,b) \cap P$. By Remark 4, c), F is $\bar{L}$ with constant λ on (a,b) and $F'(x) \leq \lambda < 0$ a.e. on (c,d), a contradiction. Hence $(a,b) \cap P$ is nowhere dense. Let $(r,s) \subset (a,b)$ be a component of U. Then $F(s) - F(r) < \lambda \cdot (s - r) < 0$, a contradiction (since F is increasing on $[r,s]$). It follows that $P = \emptyset$, hence F is increasing on $[0,1]$.

__Lemma 8__. *Let $\mathcal{L}$ be a local system with intersection condition (I.C.). Let $F : [0,1] \to R$, $A = \{x : \mathcal{L} - \underline{D}F(x) > -\infty\}$, such that $E = [0,1] \backslash A$ is at most countable and for each $x \in E$ there exists a bilateral set $E_x \in \mathcal{L}(x)$ such that*

$$\varlimsup_{\substack{y \nearrow x \\ y \in E_x}} F(y) \leq F(x) \leq \varliminf_{\substack{y \searrow x \\ y \in E_x}} F(y).$$

If $\mathcal{L} - \underline{D}F(x) \geq 0$ a.e. then F is increasing on $[0,1]$.

Proof. Clearly F is uCM on $[0,1]$. Let P and $U = \cup(a_n, b_n)$ be defined as in Lemma 7. Suppose that $P \neq \emptyset$. Since $F \in uCM$ it follows that F is increasing on each $[a_n, b_n]$, hence P is a perfect subset of $[0,1]$. Let $f : A \to R$, $-\infty < f(x) < \mathcal{L} - \underline{D}F(x)$ for each $x \in A$. Let $\sigma_x = \{y : y = x \text{ or } (F(y) - F(x))/(y - x) > f(x)\} \in \mathcal{L}(x)$ for $x \in A$ and $\sigma_x = E_x$ for $x \in E$. Let $\delta(x)$, $x \in [0,1]$, be a positive function such that whenever $0 < y - x < \min\{\delta(x), \delta(y)\}$ then $\sigma_x \cap \sigma_y \cap [x,y] \neq \emptyset$. Let $A_n = \{x \in A : f(x) > -n\}$. Let A_{nj} be a δ–decomposition of A_n. Since $P \subset E \cap (\cup_{n,j} A_{nj})$, by the Baire Category theorem, it follows that there exists an open interval (a,b) such that $(a,b) \cap P \neq \emptyset$ and $(a,b) \cap P \subset \overline{A}_{nj}$ for some n and j. We prove that F is $\underline{L}'$ with constant $-n$ on $(a,b) \cap P$, hence F is $\underline{AC}'$ on $(a,b) \cap P$.

1) Let $x < y$, $x, y \in A_{nj} \cap (a,b)$. Then for $t \in \sigma_x \cap \sigma_y \cap [x,y] \neq \emptyset$ we have $F(t) - F(x) \geq -n(t - x)$ and $F(y) - F(t) \geq -n(y - t)$. Hence $F(y) - F(x) \geq -n(y - x)$.

2) Let $x < y$, $x \in A \cap P^+ \cap (a,b)$, $y \in A_{nj} \cap (a,b)$. Let $x_k \searrow x$, $x_k \in A_{nj} \cap (x, x + \delta(x))$, $x_k < y$. By Remark 1, (ii), let $z_k \in \sigma_x \cap \sigma_{x_k} \cap [x, x_k] \neq \emptyset$. Hence $F(y) - F(x) = F(y) - F(x_k) + F(x_k) - F(z_k) + F(z_k) - f(x) > -n(y - x_k) - n(x_k - z_k) + f(x)(z_k - x) = -n(y - z_k) + f(x)(z_k - x)$. If $k \to \infty$ then $F(y) - F(x) \geq -n(y - x)$.

3) Let $x \in P^+ \cap E \cap (a,b)$, $y \in A_{nj} \cap (a,b)$, $x < y$. Let $x_k \searrow x$, $x_k \in A_{nj}$, $x_k \in (x, x + \delta(x))$, $x_k < y$. Let $z_k \in E_x \cap \sigma_{x_k} \cap [x, x_k] \neq \emptyset$. Then $F(y) - F(x) > -n(y - x_k)$; $F(x_k) - F(z_k) > -n(x_k - z_k)$; $\underline{\lim}_{k \to +\infty} F(z_k) \geq F(x)$. Hence $F(y) - F(x) \geq F(y) - \underline{\lim}_{k \to \infty} F(z_k) \geq -n(y - x)$. By Lemma 7, it follows that $P = \emptyset$, a contradiction.

Lemma 9. *Let $\mathcal{L}$ be a bilateral local system with intersection conditions* I.C. *and* E.I.C. [m]. *Let $F : [0,1] \to R$ and let $A = \{x \in [0,1] : \mathcal{L} - \underline{D}F(x) > -\infty\}$ such that $E = [0,1] \backslash A$ is at most countable and for each $x \in E$, $\varepsilon > 0$ the sets $\{z \in (x - \varepsilon, x) : F(z) < f(x) + \varepsilon\}$ and $\{z \in (x, x + \varepsilon) : F(z) > F(x) - \varepsilon\}$ are uncountable. If $\mathcal{L} - \underline{D}F(x) \geq 0$* a.e. *then F is increasing on* $[0,1]$.

Proof. We observe that $F \in uCM$ and $F'(x) \geq 0$ a.e. where F is derivable. Let P and $U = \cup(a_n, b_n)$ be the sets defined in Lemma 7 and suppose that P is nonempty. Since $F \in uCM$ it follows that F is increasing on each $[a_n, b_n]$, hence P is a perfect subset of $[0,1]$. Let $f : A \to R$ be a finite function such that $-\infty < f(x) < \mathcal{L} - \underline{D}F(x)$. Let $\sigma_x = \{y : y = x \text{ or } (F(y) - F(x))/(y - x) > f(x)\} \in \mathcal{L}(x)$ for $x \in A$. For each $x \in E$ let σ_x be a fixed set of $\mathcal{L}(x)$. Let $A_n = \{x \in A : f(x) > -n\}$, $n = 1, 2, \ldots$. Let $\delta(x)$, $x \in [0,1]$ be a positive function such that whenever $0 < y - x < \min\{\delta(x), \delta(y)\}$ then $\sigma_x \cap \sigma_y \cap [x,y] \neq \emptyset$; $\sigma_x \cap \sigma_y \cap (y, y + m(y - x) \neq \emptyset$; $\sigma_x \cap \sigma_y \cap (x - m(y - x), x) \neq \emptyset$. Let $\{A_{nj}\}$, $j \geq 1$, be a d–decomposition of A_n. By the Baire Category Theorem there exists an open interval (a,b) such that $\emptyset \neq (a,b) \cap P \subset \bar{A}_{nj}$ for some n and j. We prove that $F \in \underline{L}'$ with constant $-n$ on $(a,b) \cap P$. 1) If $x, y \in A_{nj}$, $x < y$

then $F(y) - F(x) > -n(y - x)$ (see Remark 1, (i), condition I.C. and 1) of the proof of Lemma 8). 2) If $x \in A \cap (a,b) \cap P^+$ and $y \in A_{nj} \cap (a,b)$, $x < y$ then $F(y) - F(x) \geq -n(y - x)$ (see 1), Remark 1, (ii) and 2) of the proof of Lemma 8). 3) Let $x \in P^+ \cap A \cap (a,b)$, $y \in P^- \cap E \cap (a,b)$, $x < y$ (the cases $x \in P^+ \cap E \cap (a,b)$, $y \in P^- \cap A \cap (a,b)$ and $x \in P^+ \cap E \cap (a,b)$, $y \in P^- \cap E \cap (a,b)$ are similar). Then $F(y) - F(x) \geq -n(y - x)$. Indeed, let $G(x) = F(x) + nx$. Suppose on the contrary that $G(x) > G(y)$. Let $\varepsilon < \min\{(y-x)/2, (G(x)-G(y))/2\}$. Since $y \in E$ it follows that $\{z \in (y - \varepsilon, y) : G(z) < G(y) \in \varepsilon\}$ is uncountable. We have two situations: (i) there exists $z \in (y - \varepsilon, y) \cap A \cap P_0$ such that $G(z) < G(y) + \varepsilon$, where $P_0 = \{x \in P : x \text{ is a bilateral accumulation point of } P\}$. Then by 2), $G(x) \leq G(z) < G(y) + \varepsilon < G(x)$, a contradiction. (ii) there exists $z \in (a_1, b_1) \subset (y - \varepsilon, y)$ for some i such that $G(z) < G(y) + \varepsilon$. Since F is increasing on $[a_i, b_i]$ it follows that G is strictly increasing on $[a_i, b_i]$ and $G(u) < G(y) + \varepsilon$, for each $u \in [a_i, z]$. Let $t \in A_{nj}$, $t < a_i$, $m(a_i - t) < z - a_i$, $a_i - t < \min\{\delta(a_i), \delta(t)\}$ and $v \in \sigma_t \cap \sigma_{a_i} \cap (a_i, a_i + m(a_i - t)) \subset (a_i, z)$ (see E.I.C. (m)). Then by 2), $G(x) < G(v) < G(y) + \varepsilon < G(x)$, a contradiction. By Lemma 7 it follows that P is empty.

<u>**Theorem 3**</u>. *Let $\mathcal{L}$ be a bilateral system with intersection conditions* I.E. *and* E.I.C. [m]. *Let $F : [0,1] \to R$, $F \in uCM$. Let $A = \{x : \mathcal{L} - \underline{D}F(x) > -\infty\}$, $B = \{x : \mathcal{L} - \underline{D}F(x) = -\infty$ and $\mathcal{L} - \overline{D}F(x) < 0\}$ such that $E = [0,1] \setminus (A \cup B)$ is at most countable and for each $x \in E$ there exists a bilateral set $E_x \in \mathcal{L}(x)$ with*

$$\varlimsup_{\substack{y \nearrow x \\ y < x,\, y \in E_x}} F(y) \;\leq\; F(x) \;\leq\; \varliminf_{\substack{y \to x \\ y > x,\, y \in E_x}} F(y).$$

If $\mathcal{L} - \underline{D}F(x) \geq 0$ a.e. *on $[0,1]$ then F is increasing on $[0,1]$.*

Proof. Let P and $U = \cup(a_n, b_n)$ be the sets defined in Lemma 7. Suppose that P is nonempty. Let $F : A \cup B \to R$ be a finite function such that $-\infty < f(x) < \mathcal{L} - \underline{D}F(x)$ if $x \in A$ and $\mathcal{L} - \overline{D}F(x) < f(x) < 0$ if $x \in B$. Let $\sigma_x = \{y : y = x \text{ or } (F(y) - F(x))/(y - x) > f(x)\} \in \mathcal{L}(x)$ if $x \in A$, $\sigma_x = E_x$ if $x \in E$ and $\sigma_x = \{y : y = x \text{ or } ((F(y) - F(x))/(y - x) < f(x)\} \in \mathcal{L}(x)$ if $x \in B$. Let $\delta(x)$, $x \in [0,1]$ be a positive function such that whenever $0 < |y - x| < \min\{\delta(x), \delta(y)\}$ then $\sigma_x \cap \sigma_y \cap [x,y] \neq \emptyset$, $\sigma_x \cap \sigma_y \cap (y, y + m(y - x)) \neq \emptyset$ and $\sigma_x \cap \sigma_y \cap (x - m(y - x), x) \neq \emptyset$. Let $A_n = \{x \in A : f(x) > -n\}$ and $B_n = \{x \in B : f(x) < -1/n\}$. Let $\{A_{nj}\}$, $j \geq 1$ be a δ-partition of A_n and $\{B_{nj}\}$, $j \geq 1$ a δ-partition of B_n. Since $P \subset \bigcup_{n,j}(A_{nj} \cup B_{nj} \cup E)$. By the Baire Category Theorem it follows that there exists an open interval (a,b) such that $(a,b) \cap P \neq \emptyset$ and (i) F is $\underline{L}'$ with constant $-n$ on $(a,b) \cap P \subset \overline{A}_{nj}$ for some n and j or (ii) F is $\overline{L}''$ with constant $-1/n$ on

$(a,b)\cap P\subset \overline{B}_{nj}$ for some n and j.

(i) We have four situations: a) If $x<y$, $x,y\in A_{nj}\cap(a,b)$ then $F(y)-F(x)\geq -n(y-x)$ (see Remark 1, (i) and condition I.C.). b) If $x<y$, $x\in A\cap P^+\cap(a,b)$, $y\in A_{nj}\cap(a,b)$ then $F(y)-F(x)\geq -n(y-x)$. Indeed, let $x_k\in(x,x+\delta(x))\cap A_{nj}$, $x_k\searrow x$, $x_k<y$, $k=1,2,\dots$ and let $z_k\in\sigma_k\cap\sigma_{x_k}\cap[x,x_k]\neq\emptyset$ (see Remark 1, (ii) and condition I.C.). Then $F(z_k)-F(x)>f(x)(z_k-x)$; $F(x_k)-F(z_k)>-n(x_k-z_k)$ and by a), $F(y)-F(x_k)>-n(y-x_k)$. It follows that $F(y)-F(x)>-n(y-z_k)+f(x)(z_k-x)$. If $k\to+\infty$ then $F(y)-F(x)\geq -n(y-x)$. c) If $x<y$, $x\in B\cap P^+\cap(a,b)$, $y\in A_{nj}\cap(a,b)$ then $F(y)-F(x)\geq -n(y-x)$. Indeed, let $x_k\searrow x$, $x_k\in(x,x+\delta(x))\cap A_{nj}\cap(a,b)$, $x_k<y$ and let $z_k\in\sigma_x\cap\sigma_{x_k}\cap(x-m(x_k-x),x)$ (see Remark 1, (ii) and condition E.I.C. [m]). Then $F(z_k)-F(x)\geq -f(x)(x-z_k)$, $F(x_k)-F(z_k)\geq -n(x_k-z_k)$ and by a), $F(y)-F(x_k)>-n(y-x_k)$. Hence $F(y)-F(x)\geq -n(y-z_k)-f(x)(x-z_k)$. If $k\to\infty$ then $z_k\nearrow x$, hence $F(y)-F(x)\geq -n(y-x)$. d) If $x<y$, $x\in E\cap P^+\cap(a,b)$, $y\in A_{nj}\cap(a,b)$ then $F(y)-F(x)\geq -n(y-x)$. Indeed, let $x_k\in(x,x+\delta(x))\cap A_{nj}\cap(a,b)$, $x_k\searrow x$, $x_k<y$. Let $z_k\in E_x\cap\sigma_{x_k}\cap[x,x_k]\neq\emptyset$ (see Remark 1, (ii) and condition I.C.). Then $F(x_k)-F(z_k)\geq -n(x_k-z_k)$ and by a), $F(y)-F(x_k)\geq -n(y-x_k)$. Since $F(x)\leq \underline{lim}_{k\to\infty}F(z_k)$ it follows that $F(y)-F(x)\geq -n(y-x)$.

(ii) Let $K_0=\{x\in P\cap(a,b): x$ is a bilateral accumulation point for $P\cap(a,b)\}$. We have four situations: A) If $x<y$, $x,y\in B_{nj}\cap(a,b)$ then $F(y)-F(x)\leq(-1/n)(y-x)$ (see Remark 1, (i) and condition I.C.). B) If $x<y$, $x\in A\cap P^-\cap(a,b)$, $y\in B_{nj}\cap(a,b)$ then $F(y)-F(x)\leq(-1/n)(y-x)$. Indeed, let $x_k\in B_{nj}$, $x_k\nearrow x$, $x_k\in(x-\delta(x),x)$ and let $z_k\in\sigma_{x_k}\cap\sigma_x\cap(x_k-m(x-x_k),x_k)$. If $k\to\infty$ then $z_k\nearrow x$, $z_k\in\sigma_x$. We have $F(x_k)-F(z_k)\leq(-1/n)(x_k-z_k)$, $F(y)-F(x_k)<(-1/n)(y-x_k)$ and $F(x)-F(z_k)>(x-z_k)f(x)$, hence $F(y)-F(z_k)+F(z_k)-F(x)<(-1/n)(y-z_k)+(z_k-x)f(x)$. If $k\to\infty$ it follows that $F(y)-F(x)\leq(-1/n)(y-x)$. C) Since F is increasing on each $[a_n,b_n]$ and $\mathcal{L}(x)$ is bilateral it follows that $B\cap P\cap(a,b)\subset K_0$. If $x<y$, $x\in B\cap P^+\cap(a,b)$, $y\in B_{nj}\cap(a,b)$ then $F(y)-F(x)\leq(-1/n)(y-x)$. Indeed, let $x_k\in(x,x+\delta(x))\cap B_{nj}$, $x_k\searrow x$, $x_k<y$, $k=1,2,\dots$ and let $z_k\in\sigma_x\cap\sigma_{x_k}\cap[x,x_k]\neq\emptyset$ (see Remark 1, (ii) and condition I.C.). Then $F(z_k)-F(x)<f(x)(z_k-x)$, $F(x_k)-F(z_k)<(-1/n)(x_k-z_k)$ and by A), $F(y)-F(x_k)<(-1/n)(y-x_k)$. It follows that $F(y)-F(x)<(-1/n)(y-z_k)+f(x)(z_k-x)$. If $k\to\infty$ then $F(y)-F(x)\leq(-1/n)(y-x)$. D) If $x\in E\cap P^-\cap(a,b)$, $y\in B_{nj}\cap(a,b)$, $x<y$ then $F(y)-F(x)\leq(-1/n)(y-x)$. Indeed, let $x_k\in B_{nj}$, $x_k\nearrow x$, $x_k\in(x-\delta(x),x)$ and let $z_k\subset\sigma_{x_k}\cap E_x\cap(x_k-m(x-x_k),x)$. Then $z_k\nearrow x$, $z_k\in E_x$ and $F(x_k)-F(z_k)<(-1/n)(x_k-z_k)$, $F(y)-F(x_k)<(-1/n)(y-x_k)$, $\overline{\lim}_{k\to\infty}F(z_k)\leq F(x)$. Hence $F(y)-F(x)\leq\overline{\lim}_{k\to\infty}(F(y)-F(z_k))\leq\overline{\lim}_{k\to\infty}(-1/n)(y-z_k)\leq(-1/n)(y-x)$. By Lemma 7 it follows that P is empty, a contradiction.

Theorem 4. (An extension of Theorem 4 of [8], p. 378). *Let $\mathcal{L}$ be a bilateral c–dense system which satisfies intersection conditions* I.C. *and* E.I.C. [m]. *Let $F : [0,1] \to R$, $F \in uCM$, and let E be a subset of $[0,1]$ such that if $x \notin E$ and $\mathcal{L} - \underline{D}F(x) = -\infty$ then $\mathcal{L} - \overline{D}F(x) < 0$. If* (i) $\mathcal{L} - \underline{D}F(x) \geq 0$ a.e. *on* $[0,1]$; (ii) *E is countable*; (iii) *F is $\overline{B}_1$ on $\overline{E}$*; (iv) *for each $x \in E$ and $\varepsilon > 0$ the sets $\{z \in (x-\varepsilon, x) : F(z) < F(x)+\varepsilon\}$ and $\{z \in (x, x+\varepsilon) : F(z) > F(x)-\varepsilon\}$ are uncountable; then F is increasing on* $[0,1]$.

Proof. Let $A = \{x : \mathcal{L} - \underline{D}F(x) > -\infty\}$ and $B = \{x : \mathcal{L} - \underline{D}F(x) = -\infty$ and $\mathcal{L} - \overline{D}F(x) < 0\}$. Then we observe that $[0,1] = A \cup B \cup E$. First we prove

(4) For each $x \in A$ and $\varepsilon > 0$, the sets $\{z : F(z) > F(x) - \varepsilon\} \cap (x, x+\varepsilon)$ and $\{z : F(z) < F(x) + \varepsilon\} \cap (x-\varepsilon, x)$ are uncountable.

Let $x \in A$, $\varepsilon > 0$ and let $p \geq 1$ be a natural number such that $\mathcal{L} - \underline{D}F(x) > -p$. Then $S_x = \{y : y = x \text{ or } (F(y)-F(x))/(y-x) > -p\} \in \mathcal{L}(x)$ is bilaterally c–dense in itself. If $z \in [x, x+\varepsilon/p) \cap S_x$ then $F(z) > F(x) - p(z-x) > F(x) - p\varepsilon/p = F(x) - \varepsilon$. Similarly, if $z \in (x-\varepsilon/p, x]$ then $F(z) < F(x) + p(x-z) < F(x)+\varepsilon$. It follows that the sets $\{z : F(z) > F(x)-\varepsilon\} \cap (x, x+\varepsilon)$ and $\{z : F(z) < F(x)+\varepsilon\} \cap (x-\varepsilon, x)$ are uncountable, hence we have (4). Let P and $U = \cup(a_n, b_n)$ be the sets defined in Lemma 7 and suppose that P is nonempty. By Theorem 3 it follows that F is increasing on each component interval of $(0,1)\backslash\overline{E}$, hence $\overline{E} \supset P$. But clearly $E \subset P$, hence $P = \overline{E}$. By (iii) F is $\overline{B}_1$ on P. Let $P_0 = P \backslash (\cup\{a_n, b_n\} \cup E)$. Since $F \in uCM$ it follows that F is increasing on each $[a_n, b_n]$, hence P is a perfect subset of $[0,1]$. In what follows we prove

(5) If $x \in P^+ \cap E$ (resp. $x \in P^- \cap E$) and $\varepsilon > 0$ then the set $\{z \in (x, x+\varepsilon) \cap P_0 : F(z) > F(x) - \varepsilon\}$ (resp. $\{z \in (x-\varepsilon, x) \cap P_0 : F(z) < F(x)+\varepsilon\}$) is nonempty.

Suppose on the contrary that there exists $x_0 \in P^+ \cap E$ and $\varepsilon_0 > 0$ such that the set $A_0 = \{z \in (x_0, x_0+\varepsilon_0) \cap P_0 : F(z) > F(x_0) - \varepsilon_0\}$ is empty. Let $B_0 = \{b_k \in (x_0, x_0 + \varepsilon_0) : F(b_k) > F(x_0) - \varepsilon_0/2\}$. For $a \in (x_0, x_0+\varepsilon_0)$ let $\mathcal{A}_a = \{n : (a_n, b_n) \subset (x_0, a)\}$. Then $\mathcal{A}_a$ is infinite. Indeed, suppose on the contrary that $\mathcal{A}_a$ has p elements, i.e., $a_1 < a_2 < \ldots < a_p < a$. Then $x_0 < a$ (since $x_0 \in P^+$) and $[x_0, a_1] \subset P$, a contradiction (see (iv) and the fact that A_0 is empty). We prove that B_0 is nonempty and contains no islated points. Let $\varepsilon < \varepsilon_0/2$. By (iv), since A_0 is empty, it follows that there exists $z \in (a_k, b_k) \subset (x_0, x_0+\varepsilon) \subset (x_0, x_0+\varepsilon_0)$ for some natural number $k \in \mathcal{A}_{\varepsilon+x_0}$ such that $F(z) > f(x_0) - \varepsilon$. Since F is increasing on $[a_k, b_k]$ it follows that $F(b_k) \geq F(z) > F(x_0) - \varepsilon > F(x_0) - \varepsilon_0/2$. Hence $b_k \in B_0$ and B_0 is nonempty. Suppose on the contrary that B_0 contains an isolated point b_n. Then there exists $0 < \delta < \min\{x_0 + \varepsilon_0 - b_n : F(b_n) - F(x_0) + \varepsilon_0/2\}$ such that

$(b_n, b_n + \delta) \cap \{z : F(z) > F(x_0) - \varepsilon_0/2\} \cap (\bigcup_{i=1}^{\infty}[a_i, b_i] = \emptyset$. Since $A_0 = \emptyset$ it follows that $(b_n, b_n + \delta) \cap \{z : F(z) > F(x_0) - \varepsilon_0/2\} \cap P_0 = \emptyset$. Hence $(b_n, b_n + \delta) \cap \{z : F(z) > F(b_n) - \delta\}$ is at most countable (since $F(b_n) - \delta > F(x_0) - \varepsilon_0/n$) but this contradicts (4).

Since $\mathcal{L}$ is bilateral and F is increasing on $[a_n, b_n]$ it follows that $b_n \in A \cup E$. Hence $\overline{B}_0$ is a nonempty perfect subset of P. Since F is $\overline{B}_1$ on P it follows that there exists a sequence of sets Q_n, $n \geq 1$, $Q_n = \overline{Q}_n \subset P$, such that $\{x \in \overline{B}_0 : F(x) < F(x_0) - \varepsilon_0/2\} = \cup Q_n$. Since $A_0 = \emptyset$ it follows that $D = \{x \in \overline{B}_0 : F(x) \geq F(x_0) - \varepsilon_0/2\} \subset E \cup (\cup\{a_n, b_n\})$ is countable. Since $\overline{B}_0 = D \cup (\cup Q_n)$, by the Baire Category Theorem, there exists an open interval (a, b) such that $\emptyset \neq (a, b) \cap \overline{B}_0 \subset Q_n$ for some natural number n. Let $b_j \in (a, b) \cap B_0$. Then $F(b_j) > F(x_0) - \varepsilon_0/2$, a contradiction. It follows that A_0 is nonempty and we have (5). Let $f : A \cup B \to R$ be a finite function such that $-\infty < f(x) < \mathcal{L} - \underline{D}F(x)$ if $x \in A$ and $\mathcal{L} - \overline{D}F(x) < f(x) < 0$ if $x \in B$. Let $\sigma_x = \{y : y = x \text{ or } (F(y) - F(x))/(y - x) > f(x)\} \in \mathcal{L}(x)$ if $x \in A$, $\sigma_x = \{y : y = x \text{ or } (F(y) - F(x))/(y - x) < f(x)\} \in \mathcal{L}(x)$ if $x \in B$ and let $\sigma_x \in \mathcal{L}(x)$ be a fixed set if $x \in E$. Let $\delta(x)$, $x \in [0, 1]$ be a positive function such that whenever $0 < y - x < \min\{\delta(x), \delta(y)\}$ then $\sigma_x \cap \sigma_y \cap [x, y] \neq \emptyset$, $\sigma_x \cap \sigma_y \cap (y, y + m(y - x)) \neq \emptyset$ and $\sigma_x \cap \sigma_y \cap (x - m(y - x), x) \neq \emptyset$. Let $A_n = \{x \in A : f(x) > -n\}$ and $B_n = \{x \in B : f(x) < -1/n\}$. Let $\{A_{nj}\}$, $j \geq 1$, be a δ–partition of A_n and $\{B_{nj}\}$, $j \geq 1$ a δ–partition of B_n. By the Baire Category Theorem it follows that there exists an open interval $(a, b) \cap P \neq \emptyset$ such that (i) $(a, b) \cap P \subset \overline{A}_{nj}$ for some n and j or (ii) $(a, b) \cap P \subset \overline{B}_{nj}$ for some n and j.

(i) We prove that F is $\underline{L}'$ with constant $-n$ on $P \cap (a, b)$.

a) If $x < y$, $x, y \in A_{nj}$ then $F(y) - F(x) \geq -n(y - x)$.

b) If $x < y$, $x \in A \cap P^+ \cap (a, b)$, $y \in A_{nj} \cap (a, b)$ then $F(y) - F(x) \geq -h(y - x)$.

c) If $x < y$, $x \in B \cap P^+ \cap (a, b)$, $y \in A_{nj} \cap (a, b)$ then $F(y) - F(x) \geq -n(y - x)$. (For the proof of a), b), c) see the proof of Theorem 3.)

d) If $x < y$, $x \in E \cap P_0$ such that $F(z) > F(x) - \varepsilon_0$. By b) and c), $F(y) - F(z) \geq -n(y - z)$, hence $F(y) - F(x) + \varepsilon \geq -n(y - x) - n(x - z)$. Since $|x - z| < \varepsilon$ and ε is arbitrary, it follows that $F(y) - F(x) \geq -n(y - x)$.

(ii) We prove that F is $\overline{L}''$ with constant $-1/n$ on $P \cap (a, b)$.

A) If $x < y$, $x, y \in B_{nj}$ then $F(y) - F(x) \leq (-1/n)(y - x)$.

B) If $x < y$, $x \in A \cap P^- \cap (a, b)$, $y \in B_{nj} \cap (a, b)$ then $F(y) - F(x) \leq (-1/n)(y - x)$.

C) If $x < y$, $x \in B \cap P^+ \cap (a,b)$, $y \in B_{nj} \cap (a,b)$ then $F(y) - F(x) \leq (-1/n)(y - x)$.

D) If $x < y$, $x \in E \cap P^- \cap (a,b)$, $y \in B_{nj} \cap (a,b)$ then $F(y) - F(x) \leq (-1/n)(y - x)$. Let $\varepsilon > 0$, $x - \varepsilon > a$. By (5) it follows that there exists $z \in (x - \varepsilon, x)$ such that $F(z) < F(x) + \varepsilon$. By B) and C), $F(y) - F(z) \leq (-1/n)(y - z)$, hence $F(y) - F(x) - \varepsilon < F(y) - F(z) < (-1/n)(y - x + x - z)$. Since $|x - z| < \varepsilon$ and ε is arbitary it follows that $F(y) - F(x) \leq (-1/n)(y - x)$.

By Lemma 7 it follows that P is empty, a contradiction.

Remark 7. A local system $\mathcal{L} = \{\mathcal{L}(x) : x \in R\}$ will be said to be:

a) of ordinary type if $\mathcal{L}(x) = \{S : S$ contains an open interval about the point $x\}$ (see [2], p. 99 or [11], p. 4);

b) of $(1,1)$ density type if $\mathcal{L}(x) = \{S : S$ has density 1 at $x\}$ (see [2], p. 99 or [11], Definition 12.1, p. 22);

c) of (ρ, λ) density type if $\mathcal{L}(x) = \{S : S$ has right lower density exceeding ρ and left lower density exceeding λ at $x\}$ (see [2], p. 99);

d) of qualitative type if $\mathcal{L}(x) = \{S : S$ is residual in a neighborhood of $x\}$ (see [2], p. 99).

By [11] (Lemma 15.6, p. 34 and Lemma 15.7, p. 35) or by [2] (the proof of Theorem 3.5, p. 102), the ordinary; the (ρ, λ) density, $\rho > 1/2$, $\lambda > 1/2$ and the qualitative type systems are bilaterally c–dense and satisfy conditions I.C. and E.I.C [m].

If $\mathcal{L}$ is of ordinary type we obtain the ordinary lower derivative $\underline{D}F(x)$; if $\mathcal{L}$ is of $(1,1)$ density type we obtain the approximately lower derivative $\underline{D}_{ap}F(x)$; if $\mathcal{L}$ is of (ρ, λ) density type we obtain the $ap_{(\rho,\lambda)} - \underline{D}F(x)$ (see [12], part I, p. 75). For $\rho = \lambda = 1/2$ we obtain the lower preponderant Denjoy derivative $\underline{D}_{pr}F(x)$; if $\mathcal{L}$ is of qualitative type we obtain the lower qualitative Marcus derivatives $\underline{D}_qF(x)$ (see [1], p. 166).

Systems of $(1/2, 1/2)$ density type do not satisfy in general an E.I.C.[m] but all the theorems of the present paper can be extended to them by decomposing the line into a sequence of sets $\{X_n\}_{n=3}^{\infty}$ so that for $x \in X_n$, the density of each $S \in \mathcal{L}(x)$ exceeds $(n+2)/(2n)$, and then the E.I.C.[m] can be used to yields results on each set of the sequence. Thus, those theorems that use the E.I.C.[m] apply to preponderant derivative, but with some technical modifications (see [2], p. 103).

Using Definition 8, the Preiss Theorem can be written in the following way:

Theorem 4 (Preiss). *Let $f : (a,b) \to R, F \in uP$ and let E be a subset of (a,b) such that if $x \notin E$ and $\underline{f}'_{ap}(x) = -\infty$ then $f'_{ap}(x) = -\infty$. If*

(i) $\underline{f}'_{ap}(x) \geq 0$ a.e. *on* (a,b).

(ii) *E is countable.*

(iii) *F is B_1 with respect to the set $\overline{E}$.*

(iv) *for each $x \in E$ and $\varepsilon > 0$ the sets $\{z \in (x-\varepsilon, x) : f(z) < f(x) + \varepsilon\}$, $\{z \in (x, x+\varepsilon) : f(z) > f(x) - \varepsilon\}$ are uncountable; then f is increasing on (a,b).*

Our Theorem 4 is a real extension of Preiss Theorem since:

a) $uP \subsetneqq uCM$ (see Proposition 2) b).

b) The Preiss conditions on the set E are stronger than ours.

c) Preiss assumed that "F is B_1 with respect to the set $\overline{E}$" and we suppose only "F is $\overline{B}_1$ with respect to the set $\overline{E}$". We think that this is the most important improvement of the Preiss Theorem.

d) Our Theorem 4 relates to several kinds of derivatives.

Example 3. Let C be the Cantor ternary set and let (a_i, b_i), $i \geq 1$ be the intervals contiguous to C. There exists a function $F : [0,1] \to [0,1]$ such that:

a) $F(0) = 0$; $F(1) = 1$

b) F is increasing on $[0,1]$

c) $F'(x) = +\infty$, for each $x \in C$

d) F is constant on each $(a_i, b_i), i \geq 1$

e) $F \notin \ell CM$ and $F \in uCM$, hence $F \notin CM$.

Proof. By [1] (Lemma 1.2, p. 124) there exists a function $G : [0,1] \to [0,1]$ such that:

(i) $G(0) = 0$ and $G(1) = 1$

(ii) G is continuous and strictly increasing on $[0,1]$

(iii) $G'(x) = +\infty$ for each $x \in \mathbf{C}$

Let $F(x) = \begin{cases} G(x), & x \in \mathbf{C} \\ G(c_i), & x \in (a_i, b_i), \text{ where } c_i = (a_i + b_i)/2,\ i \geq 1. \end{cases}$

a),b),d),e) are evident. c) Let $x \in \mathbf{C}$. Since $G'(x_0) = +\infty$ it follows that for $\alpha > 0$ there exists $\delta > 0$ such that

$$G(x) - G(x_0) > \alpha(x - x_0), \text{ for each } x \in [x_0, x_0 + \delta) \tag{6}$$

We have three situations:

1) If $x \in \mathbf{C} \cap (x_0, x_0 + \delta)$ then by (6), $F(x) - F(x_0) > \alpha(x - x_0)$

2) If $x \in (a_i, c_i) \cap [x_0, x_0 + \delta)$ for some i, then by (6)

$$F(x) - F(x_0) = G(c_i) - G(x_0) \geq G(x) - G(x_0) > \alpha(x - x_0)$$

3) If $x \in [c_i, b_i) \cap [x_0, x_0 + \delta)$ then by (6)

$$F(x) - F(x_0) = G(c_i) - G(x_0) > \alpha(c_i - x_0) > \frac{\alpha}{2}(x - x_0).$$

It follows that $G'^{+}(x_0) = +\infty$. Similarly $G'^{-}(x_0) = +\infty$, hence we have c).

Remark. Using the property of function G from Example 3, Preiss defines in [8] (p. 374) a function f_1 which has the same properties as our function F, but in contrast with the proof in [8], our proof is elementary.

Example 4 (Preiss). Let $F : [0,1] \to [0,1]$ be the function defined in Example 3. Let $G : [0,1] \to R$ be defined as follows:

$$G(x) = \begin{cases} 1 - F(x), & x \in C \setminus \left(\bigcup_{i=1}^{n} \{a_i, b_i\}\right) \\ 1 - F(x), & x \in \left[a_i + \frac{b_i - a_i}{2^i + 1},\ b_i - \frac{b_i - a_i}{2^i + 1}\right],\ i \geq 1 \\ 0, & x \in \{a_1, a_2, \ldots\} \\ 1, & x \in \{b_1, b_2, \ldots\} \end{cases}$$

On $(a_i, a_i + \frac{b_i - a_i}{2^i + 1})$ and $(b_i - \frac{b_i - a_i}{2^i + 1},\ b_i)$ we define $G(x)$ such that G is continuous and increasing on each $[a_i, b_i]$ and $G'(x)$ exists on (a_i, b_i), for each $i \geq 1$. Then we have:

a) G satisfies Darboux condition on $[0,1]$

b) $G \notin \overline{B}_1$, $G \notin \underline{B}_1$ on $[0,1]$

c) $G'_{ap}(x)$ exists (finite or infinite) n.e. on $(0,1)$ and $G'_{ap}(x) \geq 0$ a.e. on $[0,1]$.

Proof. For a) and c) see [8], p. 375. b) The set $\{x : G(x) > 0\} = (\bigcup_{i=1}^{\infty}(a_i, b_i)) \cup (C - \{a_1, a_2, \ldots\})$ which is not of F_σ–type. Indeed if $\{x : G(x) > 0\}$ is of F_σ–type then $\{x : G(x) > 0\} \cap C = \mathrm{C} - \cup\{a_1, a_2, \ldots\}$ is of F_σ–type. Suppose that there exists a sequence of closed sets $\{K_j\}_{j\geq 1}$ such that $C - \cup\{a_1, a_2, \ldots\} = \bigcup_{j\geq 1} K_j$. Then by Baire Category Theorem, there exist $\alpha, \beta \in [0,1]$ such that $\emptyset \neq [\alpha, \beta] \cap (C \backslash \{a_1, a_2, \ldots\})$ is not closed. Hence $F \notin \underline{B}_1$. Similarly we prove that the set $\{x : F(x) < 1\}$ is not of F_σ–type, hence $F \notin \overline{B}_1$.

__Remark__. Example 4 shows that in Theorem 4 we can not omit condition (iii).

__**Example 5 (Preiss)**__. Let $F : [0,1] \to [0,1]$ be the function defined in Example 3. Let $H : [0,1] \to R$ be defined as follows:

$$\begin{cases} 1 - F(x), & x \in C \backslash \left(\bigcup_{i=1}^{\infty}\{a_i, b_i\}\right) \\ 1 - F(x), & x \in \left[a_i + \frac{b_i - a_i}{2^{i+1}},\ b_i - \frac{b_i - a_i}{2^{i+1}}\right],\ i \geq 1 \\ 0, & x \in \{a_1, a_2, \ldots\} \\ 1, & x \in \{b_1, b_2, \ldots\} \\ -1, & x \in \bigcup_{i=1}^{\infty}\{a_i + \frac{b_i - a_i}{2^{i+2}}\} \\ 2, & x \in \bigcup_{i=1}^{\infty}\{b_i - \frac{b_i - a_i}{2^{i+2}}\} \end{cases}$$

On the intervals $\left(a_i, a_i + \frac{b_i - a_i}{2^{i+2}}\right)$; $\left(a_i + \frac{b_i - a_i}{2^{i+2}},\ a_i + \frac{b_i - a_i}{2^{i+1}}\right)$; $\left(b_i + \frac{b_i - a_i}{2^{i+1}},\ b_i + \frac{b_i - a_i}{2^{i+2}}\right)$; $\left(b_i + \frac{b_i - a_i}{2^{i+2}},\ b_i\right)$ we define H such that

(i) H is continuous on $[a_i, b_i]$, $i \geq 1$

(ii) H' exists on (a_i, b_i), $i \geq 1$

(iii) $H'^{+}(a_i) = -\infty$ and $H'^{-}(b_i) = -\infty$, $i \geq 1$

(iv) H is increasing on $\left[a_i + \frac{b_i - a_i}{2^{i+2}},\ b_i - \frac{b_i - a_i}{2^{i+2}}\right]$, $i \geq 1$

(v) H is decreasing on each $\left[a_i, a_i + \frac{b_i - a_i}{2^{i+2}}\right]$ and $\left[b_i + \frac{b_i - a_i}{2^{i+2}},\ b_i\right]$, $i \geq 1$

Then we have

a) H satisfies the Darboux property on $[0,1]$

b) $H \notin \overline{B}_1$, $H \notin \underline{B}_1$ on $[0,1]$

c) $H'_{ap}(x)$ exists (finite or infinite) for each $x \in (0,1)$.

Proof. For a) and c) see [8], p. 375 and for b) see the proof of Example 4) b).

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References

[1] A.M. Bruckner: *Differentiation of Real Functions*, Lecture Notes in Mathematics, 659, Springer-Verlag, New York (1978).

[2] A.M. Bruckner, R.J. O'Malley, B.S. Thomson: *Path derivatives: a unified view of certain generalized derivatives*, Trans. Amer. Math. Soc., 283(1984), 97-125.

[3] V. Ene: *Semicontinuity conditions and monotonicity*, Real Analysis Exchange, 14(1988-89), 393-412.

[4] C.M. Lee: *An analogue of the theorem Hake-Alexandroff-Looman*, Fund. Math., C (1978), 69-74.

[5] R.J. O'Malley: *Selective Derivatives and the M_2 or Denjoy-Clarkson Properties*, Acta Math. Acad. Sci. Hungar., 36(1980), 195-199.

[6] S. Marcus: *On a theorem of Denjoy and on approximate derivatives*, Monatsh. Math., 66(1962), 435-440.

[7] S.N. Mukhopadhyay: *On a certain property of the derivative*, Fund. Math., 67(1970), 279-284.

[8] D. Preiss: *Approximate derivatives and Baire classes*, Czech. Math. J., 21(96), (1971), 373-382.

[9] J. Ridder: *Uber den Perronschen Integralbegriff und seine Beziehung zu den R–, L– und D– Integralen*, Math. Z., 34(1932), 234-269.

[10] J. Ridder: *Uber approximative stetige Denjoy-Integrale*, Fund. Math., 21(1933), 1-10.

[11] B.S. Thomson: *Real Functions*, Lecture Notes in Mathematics, 1170, Springer-Verlag, New York (1985).

[12] B.S. Thomson: *Derivation Bases on the Real Line I - II*, Real Analysis Exchange, 8(1982-83).

[13] Z. Zahorski: *Sur la première derivée*, Trans. Amer. Math. Soc., 69(1950), 1-54.

[14] S. Saks: *Theory of the integral*, 2nd. rev. ed., Monografie Matematyczne, PWN, Warsaw (1937).

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