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# REAL NUMBERS WITH REDUNDANT REPRESENTATIONS

**Abstract**

Let $(P_j)_{j=1,2,3,\ldots}$ be a sequence of sets $P_j$ of real numbers such that each $P_j$ is countable and has more than one point. Let $A$ denote the set of all numbers $x$ that can be uniquely expressed $x = \sum_{j=1}^{\infty} a_j$ $(a_j \in P_j)$. Let $B$ denote the set of numbers $y$ that can be expressed in at least two ways: $y = \sum_{j=1}^{\infty} b_j = \sum_{j=1}^{\infty} c_j$ $(b_j \in P_j, c_j \in P_j)$ such that $b_j \neq c_j$ for at least one index $j$. Here we prove that if $A \cup B$ is a second category subset of $R$, then $A \cup B$ is a subset of the closure of $B$. In particular, if $A \cup B$ is a dense second category subset of $R$, then $B$ is dense in $R$; if $B$ is a nowhere dense subset of $R$, then $A$ is a first category subset of $R$. This unifies and generalizes results of M. Petkovsek [P] and of M. Starbird and T. Starbird [SS].

**1.** In this paper $(P_i)_{i=1,2,3,\ldots}$ will be a sequence of sets of real numbers and each $P_i$ will be a finite or denumerably infinite set containing at least two elements. Let $A$ denote the set of real numbers $x$ that can be uniquely expressed as $x = \sum_{i=1}^{\infty} a_i$ $(a_i \in P_i)$. Let $B$ denote the set of all real numbers $y$ that can be so expressed in at least two ways: to wit as $y = \sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} b_i$ where $a_i \in P_i$, $b_i \in P_i$ and $a_i \neq b_i$ for some index $i$. Throughout this paper we assume that $A \cup B$ is a nonvoid set. At least one number can be so expressed. We will generalize and unify two propositions suggested in [P] and [SS].

**Proposition 1.** *Let $A \cup B = R$. Then $B$ is a dense subset of $R$.*

**Proposition 2.** *Let $B$ be void. Then $R \setminus A$ is an uncountable dense subset of $R$.*

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Proposition 1 is proved in [SS]. An earlier version appeared in [P] in which each $P_i$ is assumed to be a finite set containing 0, and any series $\sum_{i=1}^{\infty} a_i$ $(a_i \in P_i)$ is assumed to converge. Proposition 2 appeared in [P] under this same hypothesis. Proposition 2 is not addressed in [SS].

We will prove:

**Theorem I.** *Let $A \cup B$ be a second category subset of $R$. Then $A$ is a subset of $B$ closure.*

Thus if $A \cup B$ is a dense second category subset of $R$, $B$ must be dense in $R$. This generalizes Proposition 1. If $B$ is void or a nowhere dense subset of $R$, then $A$ must be a first category subset of $R$. This generalizes Proposition 2. Thus Theorem I unifies Propositions 1 and 2 as well.

Next we consider results when each $P_i$ is assumed to be a finite set. We will prove:

**Theorem II.** *Let each $P_i$ be a finite set containing at least two elements. Then*

(1) *$A \cup B$ is the union of countably many closed sets,*

(2) *If $A \cup B$ is a second category subset of $R$, then there exist real numbers $r$ and $s$ such that any $x$ in the unit interval $[0,1]$ can be expressed $x = r + \sum_{i=1}^{\infty} sa_i$ $(a_i \in P_i)$ in at least one way.*

Of course part (2) is an immediate consequence of part (1).

**2.** We present some notation and definitions to be used.

For any integer $n$, let $T(n)$ denote the set of all sums of the form $\sum_{i=n}^{\infty} a_i$ $(a_i \in P_i)$. Hence $T(1) = A \cup B$.

For any integer $n$ and $\varepsilon > 0$, let $T(n, \varepsilon)$ denote the set of all numbers that can be represented in the form $\sum_{i=n}^{\infty} a_i$ $(a_i \in P_i)$ such that $|\sum_{i=n+j}^{n+j+k} a_i| < \varepsilon$ for all $j \geq 0$ and $k \geq 0$. Let $|T(n,\varepsilon)|$ denote the set $\{|u| : u \in T(n, \varepsilon)\}$.

For any integer $n$, $\varepsilon > 0$, and $a \in P_n$, let $S(n, \varepsilon, a)$ denote the set of all sums $\sum_{i=n}^{\infty} a_i$ $(a_i \in P_i)$ in $T(n, \varepsilon)$ such that $a_n = a$. Thus for any $\varepsilon > 0$, $T(n, \varepsilon)$ and $S(n, \varepsilon, a)$ are nonvoid for large enough $n$ because $A \cup B$ is a nonvoid set.

**Definition.** We say that the system $(P_i)$ is <u>regular</u> if there is an $\varepsilon > 0$ such that

$$\lim_{n \to \infty} \sup |T(n, \varepsilon)| = 0.$$

Otherwise we say that the system $(P_i)$ is <u>irregular</u>.

For example, $P_i = \{0, 1/i\}$ provides an irregular system; on the other hand $P_i = \{0, 1/2^i\}$ provides a regular system.

Note that if $(P_i)$ is irregular, then for any $\varepsilon > 0$, we have

$$\limsup_{n\to\infty} \sup\{x : x \in T(n,\varepsilon)\} > 0, \quad \text{or}$$

$$\limsup_{n\to\infty} \sup\{-x : x \in T(n,\varepsilon)\} > 0, \quad \text{or both.}$$

This is true because, if $(P_i)$ is not regular, then

$$\limsup_{n\to\infty} \sup\{|x| : x \in T(n,\varepsilon)\} > 0.$$

The plan is to dispose of Theorems I and II quickly for irregular systems $(P_i)$. Then we will concentrate on the regular systems.

**3.** Theorems I and II can be inferred from Lemma 1 when the system $(P_j)$ is irregular.

**Lemma 1.** *Let $(P_j)$ be an irregular system and let $A \cup B$ be nonvoid. Then $A \cup B$ is an unbounded interval of one of the forms $(u,\infty), [u,\infty), (-\infty,u), (-\infty,u]$, $(-\infty,\infty)$. Moreover the interior of $A \cup B$ is a subset of $B$.*

**Proof.** Let $x = \sum_{j=1}^{\infty} a_j$ $(a_j \in P_j)$ be a point in $A \cup B$. For any $\varepsilon > 0$, $\limsup_{n\to\infty} \sup |T(n,\varepsilon)| > 0$ because $(P_j)$ is irregular. Without loss of generality we let $\limsup_{n\to\infty} \sup T(n,\varepsilon) > 0$ for each $\varepsilon > 0$. (The proof for $\limsup_{n\to\infty} \sup -T(n,\varepsilon) > 0$ will be analogous.)

Fix any number $t > x$. The plan is to prove that $t \in B$, and hence $A \cup B$ will be an interval unbounded on the right. To this end, we will construct a sequence of points $\sum_{j=1}^{\infty} b_{ij}$ $(i = 0,1,2,3,\ldots)$ in $A \cup B$. Let $b_{0j} = a_j$ for all $j \geq 1$.

Select any positive integer $n_0$ and let $\varepsilon_1 = \frac{1}{2}(t-x) > 0$. Let $\delta_1 = \frac{1}{2}\limsup_{n\to\infty} \sup T(n,\varepsilon_1)$. Then $\varepsilon_1 > 0$ and $\delta_1 > 0$. Choose an index $n_1 > n_0$ and $\sum_{j=-n_1}^{\infty} c_j$ $(c_j \in P_j)$ in $T(n_1,\varepsilon_1)$ such that

$$\sum_{j=n_1}^{\infty} c_j > \frac{1}{2}\delta_1 \quad \text{and} \quad |\sum_{j=n_1+i}^{n_1+i+k} a_j| < \frac{1}{4}\delta_1 \quad \text{for all } i \geq 0,\ k \geq 0.$$

Let $b_{1j} = a_j = b_{0j}$ for $1 \leq j \leq n_1 - 1$, and $b_{1j} = c_j$ for $j \geq n_1$. Then $\frac{1}{2}\delta_1 < \sum_{j=n_1}^{\infty} c_j \leq \varepsilon_1$, $\varepsilon_1 - \frac{1}{4}\delta_1 > 0$, and

$$\begin{aligned} t &= x + 2\varepsilon_1 \geq \sum_{j=1}^{n_1-1} a_j - \frac{1}{4}\delta_1 + 2\varepsilon_1 \geq \sum_{j=1}^{n_1-1} a_j + \sum_{j=n_1}^{\infty} c_j + \varepsilon_1 - \frac{1}{4}\delta_1 \\ &= \sum_{j=1}^{\infty} b_{1j} + \varepsilon_1 - \frac{1}{4}\delta_1 > \sum_{j=1}^{\infty} b_{1j} \end{aligned}$$

and

$$\begin{aligned}\sum_{j=1}^{\infty} b_{1j} &= \sum_{j=1}^{n_1-1} a_j + \sum_{j=n_1}^{\infty} c_j \geq \sum_{j=1}^{\infty} a_j - \frac{1}{4}\delta_1 + \sum_{j=n_1}^{\infty} c_j \\ &> \sum_{j=1}^{\infty} a_j + \frac{1}{4}\delta_1 = \sum_{j=1}^{\infty} b_{0j} + \frac{1}{4}\delta_1.\end{aligned}$$

Put $\varepsilon_2 = \frac{1}{2}(t - \sum_{j=1}^{\infty} b_{1j})$ and $\delta_2 = \frac{1}{2}\limsup_{n\to\infty} \sup T(n, \varepsilon_2)$. As in the preceding argument there is a point $\sum_{j=1}^{\infty} b_{2j}$ ($b_{2j} \in P_j$) and an index $n_2 > n_1$ such that $\sum_{j=n_2}^{\infty} b_{2j}$ is in $T(n_2, \varepsilon_2)$, $b_{2j} = b_{1j}$ for $1 \leq j \leq n_2 - 1$, $t > \sum_{j=1}^{\infty} b_{2j}$ and $\sum_{j=1}^{\infty} b_{2j} > \sum_{j=1}^{\infty} b_{1j} + \frac{1}{4}\delta_2$.

By induction on $k$, we construct a sequence of points $\sum_{j=1}^{\infty} b_{kj}$ ($b_{kj} \in P_j$) and indices $n_0 < n_1 < n_2 < n_3 < \cdots < n_k < \cdots$ such that for each $k \geq 1$

(1) $t > \sum_{j=1}^{\infty} b_{kj}$,

(2) $\sum_{j=n_k}^{\infty} b_{kj}$ is in $T(n_k, \varepsilon_k)$ where $\varepsilon_k = \frac{1}{2}(t - \sum_{j=1}^{\infty} b_{k-1,j})$,

(3) $b_{kj} = b_{k-1,j}$ for $1 \leq j \leq n_k - 1$,

(4) $\sum_{j=1}^{\infty} b_{kj} > \sum_{j=1}^{\infty} b_{k-1,j} + \frac{1}{4}\delta_k$ where $\delta_k = \frac{1}{2}\limsup_{n\to\infty} \sup T(n, \varepsilon_k)$.

Now put $d_j = b_{kj}$ for $1 \leq j \leq n_k - 1$. By (3), $d_j$ is well defined, and indeed $d_j \in P_j$ for each $j \geq 1$. Let $x_k = \sum_{j=1}^{\infty} b_{kj}$. Then $x_0 < x_1 < x_2 < x_3 < \cdots < x_k < \cdots < t$ by (4) and (1). We claim that $\lim_{k\to\infty} x_k = t$.

To prove this claim, suppose to the contrary that $t' = \lim_{k\to\infty} x_k$ and $t' < t$. Then $\varepsilon_k \geq \frac{1}{2}(t - t')$, for each $k$. Put $\varepsilon = \frac{1}{2}(t-t')$ and $\delta = \frac{1}{2}\limsup_{n\to\infty} \sup T(n, \varepsilon)$. Then $\delta_{k+1} \geq \delta$ for each $k$ because $\varepsilon_k \geq \varepsilon$. Choose $k$ so large that $x_k > t' - \frac{1}{4}\delta$. By (4)

$$x_{k+1} > x_k + \frac{1}{4}\delta_{k+1} \geq x_k + \frac{1}{4}\delta > t',$$

which is impossible. This contradiction proves that $\lim_{k\to\infty} x_k = t$.

But $\varepsilon_k = \frac{1}{2}(t - x_{k-1})$ by (2), so $\lim_{k\to\infty} \varepsilon_k = 0$. Thus $|x_k - \sum_{j=1}^{n_k-1} b_{kj}| \leq |\sum_{j=n_k}^{\infty} b_{kj}| \leq \varepsilon_k$ also by (2), and it follows that $\lim_{k\to\infty} \sum_{j=1}^{n_k-1} b_{kj} = t$. By the definition of $d_j$, it follows that $\lim_{k\to\infty} \sum_{j=1}^{n_k-1} d_j = t$. Moreover

$$|\sum_{j=n_k}^{n} d_j| = |\sum_{j=n_k}^{n} b_{kj}| \leq \varepsilon_k \text{ for } n_k \leq n < n_{k+1} \text{ by (2)}$$

and it follows that $\lim_{n\to\infty} \sum_{j=1}^{n} d_j = t$.

Thus $t$ is in $A \cup B$. Recall that $\sum_{j=1}^{\infty} b_{1j} > \sum_{j=1}^{\infty} b_{0j}$. Let $p$ be an index so large that $\sum_{j=1}^{p} b_{1j} > \sum_{j=1}^{p} b_{0j}$. We repeat the construction with $p$ in place of $n_0$ and $\sum_j b_{0j}$ or $\sum_j b_{1j}$ in place of $\sum_j a_j$ to express $t$ as the sum of two series, one with a partial sum $\sum_{j=1}^{p} b_{0j}$ and the other with a partial sum$\sum_{j=1}^{p} b_{1j}$. Clearly $t$ is in $B$.

Finally, let $u = \inf(A \cup B)$. The preceding arguments show that any number $t > u$ is in $B$. So $A \cup B$ is $(u, \infty)$ or $[u, \infty)$, and $(u, \infty) \subset B$. □

Our next lemma is much like [SS, Lemma 1], and regularity plays no role in it.

**Lemma 2.** *Let $T(1) = A \cup B$ be a second category subset of $R$. Then for each $\varepsilon > 0$, $T(n, \varepsilon)$ is a second category set for all but finitely many indices $n$.*

**Proof.** Evidently $A \cup B$ is a subset of the union

$$\bigcup_{n=2}^{\infty} \bigcup_{a_j \in P_j} ((\sum_{j=1}^{n-1} a_j) + T(n, \varepsilon)).$$

There are countably many sets in this union because each $P_j$ is countable. Then one of these sets, say $\sum_{j=1}^{N-1} a_j + T(N, \varepsilon)$ is a second category set. But it is only a translate of $T(N, \varepsilon)$, so $T(N, \varepsilon)$ is a second category set.

Now $T(N, \varepsilon) \subset \bigcup_{a \in P_N} (a + T(N + 1, \varepsilon))$, and there are countably many sets in this union because $P_N$ is countable. Thus some one of the $a + T(N + 1, \varepsilon)$ is a second category set. It follows that $T(N + 1, \varepsilon)$ is a second category set. Likewise we prove that $T(N + 2, \varepsilon), T(N + 3, \varepsilon), T(N + 4, \varepsilon), \ldots$ are second category sets. □

In Lemmas 3, 4 and 5 we will consider two different kinds of regular systems.

**Lemma 3.** *Let the system $(P_j)$ be regular. Let there be an $\alpha > 0$ such that for any $\varepsilon$ with $0 < \varepsilon < \alpha$, the set $\{a : S(n, \varepsilon, a)$ is nonvoid$\}$ is a finite set for all but finitely many indices $n$. Then there is a $\beta > 0$ such that for any $\varepsilon, 0 < \varepsilon < \beta$, the set $T(n, \varepsilon)$ is a closed set for all but finitely many indices $n$.*

**Proof.** Assume to the contrary that for any $\beta > 0$ there is an $\varepsilon > 0$, depending on $\beta$, such that $0 < \varepsilon < \beta$ and $T(n, \varepsilon)$ is a nonclosed set for infinitely many indices $n$. Choose $\beta$ such that $0 < \beta < \alpha$. Let $\varepsilon$ satisfy $0 < \varepsilon < \beta$, and $T(n, \varepsilon)$ is a nonclosed set for infinitely many indices $n$. Let $N$ be an index such that $\{a : S(n, \varepsilon, a)$ is nonvoid$\}$ is a finite set for $n \geq N$. Choose $k > N$ such that $T(k, \varepsilon)$ is not a closed set. Let $(x_i)$ be a sequence of points in $T(k, \varepsilon)$ converging to $x \notin T(k, \varepsilon)$. Say $x_i = \sum_{j=k}^{\infty} b_{ij}$ $(b_{ij} \in P_j)$.

Now $|b_{ij}| < \varepsilon$ for each $i$ and $j$. We select a subsequence $(x_i^{(k)})$ of $(x_i)$ where $x_i^{(k)} = \sum_{j=k}^{\infty} b_{ij}^{(k)}$ and $(b_{ik}^{(k)})_i$ converges, say to $d_k$. We select a subsequence $(x_i^{(k+1)})$

of $(x_i^{(k)})$ so that $(b_{ik}^{(k+1)})_i$ and $(b_{i,k+1}^{(k+1)})_i$ converge, say to $d_k$ and $d_{k+1}$. Again we select a subseqence $(x_i^{(k+2)})$ of $(x_i^{(k+1)})$ so that $(b_{ik}^{(k+2)})_i, (b_{i,k+1}^{(k+2)})_i$ and $(b_{i,k+2}^{(k+2)})_i$ converge, say to $d_k, d_{k+1}$ and $d_{k+2}$. We continue in this manner to find $d_j$ for all $j \geq k$.

Note that each $b_{ij}$ $(j \geq k)$ lies in a finite set $\{a : S(j, \varepsilon, a)$ is nonvoid$\}$, and this set also contains $d_j$. Hence $d_j \in P_j$ for all $j \geq k$. Because for fixed $v$, $\lim_{i\to\infty} b_{ij}^{(v)} = d_j$ $(v \geq j \geq k)$, we see that $b_{ij}^{(v)} = d_j$ for large enough $i$ $(v \geq j \geq k)$.

Now $\sum_{j=k} d_j$ cannot sum to $x$, for otherwise it is easy to see that $\sum_{j=k}^{\infty} d_j = x$ must lie in $T(k, \varepsilon)$. Let $\delta > 0$ be a number such that

$$|x - \sum_{j=k}^{p} d_j| \geq 2\delta$$

for infinitely many indices $p$.

Fix an index $q > k$. We find an index $p > q$ and an $x_i^{(p)}$ such that

(1) $b_{ij}^{(p)} = d_j$ for $k \leq j \leq p$, and

(2) $|\sum_{j=k}^{\infty} b_{ij}^{(p)} - \sum_{j=k}^{p} d_j| \geq \delta$.

From (1) and (2) we obtain $|\sum_{j=p+1}^{\infty} b_{ij}^{(p)}| \geq \delta$. Because $q$ is arbitrarily large, we get

$$\limsup_{p\to\infty} \sup T(p, \varepsilon) \geq \delta > 0.$$

But $\varepsilon < \beta$ and $\beta$ is arbitrarily small, so we see that $(P_j)$ is an irregular system, contrary to hypothesis. □

Next we find some numbers that are the sum of more than one of our series.

**Lemma 4.** *Let $(P_j)$ be a regular system satisfying all the hypotheses of Lemma 3. Furthermore let $A \cup B$ be a second category set. Then for each $\varepsilon > 0$ there is an index $N(\varepsilon)$ such that for any $n \geq N(\varepsilon)$, $T(n, \varepsilon)$ contains a point $x$ that can be represented in two different ways as a sum $\sum_{j=n}^{\infty} a_j$ $(a_j \in P_j)$.*

**Proof.** Take any $\varepsilon > 0$. By Lemmas 2 and 3 there is a $\lambda$ and an index $N(\varepsilon)$ such that $0 < \lambda < \varepsilon$ and such that $T(n, \lambda)$ is a closed second category set for all $n \geq N(\varepsilon)$. Fix $n \geq N(\varepsilon)$. Now $T(n, \lambda) \subset T(n, \varepsilon)$, so it suffices to prove that $T(n, \lambda)$ contains the desired point $x$.

There is a compact interval $J \subset T(n, \lambda)$ because $T(n, \lambda)$ is a closed second category set. Let $u \in J$, $v \in J$ and $u > v$. Say $u = \sum_{j=n}^{\infty} b_j$ and $v = \sum_{j=n}^{\infty} c_j$ $(b_j \in P_j, c_j \in P_j)$. Select an index $p$ so that $\sum_{j=n}^{p-1} b_j > \sum_{j=n}^{p-1} c_j$. Then $u \in \sum_{j=1}^{p-1} b_j +$

$T(p,\lambda)$ and $v \in \sum_{j=n}^{p-1} c_j + T(p,\lambda)$. Each set $\sum_{j=1}^{p-1} a_j + T(p,\lambda)$ $(a_j \in P_j)$ is a closed set and

$$J \subset T(n,\lambda) \subset \bigcup_{a_j \in P_j} ((\sum_{j=n}^{p-1} a_j) + T(p,\lambda)).$$

Moreover, there are only countably many of the closed sets in the union, and two different ones contain $u$ and $v$ respectively. By [E] there is an $x \in J$ that lies in two distinct such sets, and has two distinct series representations, $x = \sum_{j=n}^{\infty} a_j$ $(a_j \in P_j)$. Finally, $x \in T(n,\lambda) \subset T(n,\varepsilon)$. □

The next lemma is reminiscent of [SS, Proposition 1], and regularity plays no role in it.

**Lemma 5.** *For each $\alpha > 0$, let there be an $\varepsilon$, $0 < \varepsilon < \alpha$, such that the set $\{a : S(n,\varepsilon,a)$ is nonvoid$\}$ is an infinite set for infinitely many indices $n$. Let $A \cup B$ be a second category set. Then for any $\beta > 0$ there are finitely many indices $n$ for which $T(n,\beta)$ contains a point $x$ that can be represented in two different ways as a sum $\sum_{j=n}^{\infty} a_j$ $(a_j \in P_j)$.*

**Proof.** Choose $\beta > 0$. Choose $\varepsilon$ so that $0 < \varepsilon < \beta$, and use Lemma 2 to choose an index $N$, such that the set $\{a : S(n,\frac{1}{4}\varepsilon,a)$ is nonvoid$\}$ is infinite for infinitely many $n$ and $T(n,\frac{1}{4}\varepsilon)$ is a second category set for $n \geq N$. Fix $n \geq N$ satisfying these conditions. Now $\{a : S(n,\frac{1}{4}\varepsilon,a)$ is nonvoid$\}$ is an infinite subset of the interval $[-\varepsilon,\varepsilon]$ and must have an accumulation point. Also $T(n+1,\frac{1}{4}\varepsilon)$ is a second category set whose closure contains a compact interval $I$. Choose points $a_n, b_n \in P_n$ such that $0 < |a_n - b_n| < \frac{1}{2}m(I)$. Then the intersection of the intervals $a_n + I$ and $b_n + I$ contains a compact interval $J$. Thus $J$ is a subset of the closures of $a_n + T(n+1,\frac{1}{4}\varepsilon)$ and of $b_n + T(n+1,\frac{1}{4}\varepsilon)$. Put $\delta = \min(\varepsilon, m(J))/4$.

Let $\mu > 0$ and let $c_1,\dots,c_m$ be numbers. We say that the sequence $c_1,\dots, c_m,\dots,c_k$ *$\mu$-extends* $c_1,\dots,c_m$ if $|\sum_{j=p}^{q} c_j| < \mu$ for $m+1 \leq p \leq q \leq k$.

Again by Lemma 2, $T(k,\delta)$ is a second category set for large enough $k$. We truncate an appropriate sum in $T(n+1,\frac{1}{4}\varepsilon)$ to $\varepsilon/4$-extend $a_n$ to $a_n, a_{n+1},\dots,a_{k(1)-1}$ such that $\sum_{j=n}^{k(1)-1} a_j$ is in the middle third of $J$ and $T(k(1),\delta)$ is a second category set. Let $J_1$ be a compact interval lying in the closure of $(\sum_{j=n}^{k(1)-1} a_j) + T(k(1),\delta)$. It follows that $J_1$ is in the closure of $(\sum_{j=n}^{k(1)-1} a_j) + T(k(1),\frac{1}{4}\varepsilon)$ and $J_1 \subset J$. Likewise we use a member of $T(n+1,\varepsilon/4)$ to $\varepsilon/4$-extend $b_n$ to $b_n, b_{n+1},\dots,b_{k(2)-1}$ $(k(2) > k(1))$ and find a compact interval $J_2 \subset J_1$ such that $J_2$ lies in the closure of $(\sum_{j=n}^{k(2)-1} b_j) + T(k(2),\varepsilon/8)$. (This time put $\delta = \min(\varepsilon, m(J_1))/8$.)

We use $2^{-q}\varepsilon$-extensions to find an increasing sequence of indices $n+1 = k(0) < k(1) < k(2) < k(3) < \cdots$ and a contracting sequence of compact intervals $J_1 \supset$

$J_2 \supset J_3 \supset J_4 \supset \cdots$ and series $\sum_{j=n}^{\infty} a_j$ and $\sum_{j=n}^{\infty} b_j$ such that $J_q$ is in the closure of

$$(\sum_{j=n}^{k(q)-1} a_j) + T(k(q), 2^{-q-1}\varepsilon) \text{ for } q = 1, 3, 5, 7, \cdots,$$

$J_q$ is in the closure of

$$(\sum_{j=n}^{k(q)-1} b_j) + T(k(q), 2^{-q-1}\varepsilon) \text{ for } q = 2, 4, 6, 8, \ldots, \text{ and}$$

$$a_n, \ldots, a_{k(q)}, \ldots, a_{k(q+2)-1} 2^{-q-1}\varepsilon - \text{extends } a_n, \ldots, a_{k(q)-1} \text{ for } q = 1, 3, 5, 7, \ldots,$$

$$b_n, \ldots, b_{k(q)}, \ldots, b_{k(q+2)-1} 2^{-q-1}\varepsilon - \text{extends } b_n, \ldots, b_{k(q)-1} \text{ for } q = 2, 4, 6, 8, \ldots,$$

It follows from this and $|a_n| \leq \frac{1}{4}\varepsilon$, $|b_n| \leq \frac{1}{4}\varepsilon$, that the series $\sum_{j=n}^{\infty} a_j$ and $\sum_{j=n}^{\infty} b_j$ converge, and indeed their sums are in $T(n, \varepsilon)$.

On the other hand, the diameter of $J_q$ cannot exceed the diameter of $T(k(q), 2^{-q-1}\varepsilon)$, so $m(J_q) \leq 2^{-q}\varepsilon$. Thus $\bigcap_q J_q$ is a singleton; say $\bigcap_q J_q = \{x\}$.

Moreover $x$ lies in the closure of $(\sum_{j=n}^{k(q)-1} a_j) + T(k(q), 2^{-q-1}\varepsilon)$ and $|(\sum_{j=n}^{k(q)-1} a_j) - x| \leq 2^{-q}\varepsilon$ because $x \in J_q$. It follows that $x = \sum_{j=n}^{\infty} a_j$. Likewise $x = \sum_{j=n}^{\infty} b_j$. But $a_n \neq b_n$, so $x$ is the desired point. Recall that $\varepsilon < \beta$, so $x \in T(n, \varepsilon) \subset T(n, \beta)$. □

**Lemma 6.** *Let $(P_j)$ be a regular system. Let $A \cup B$ be a second category set. Then for any $\varepsilon > 0$, there are infinitely many integers $n > 0$ for which the set $T(n, \varepsilon)$ contains a point $x$ that can be represented in two different ways as a sum $x = \sum_{j=n}^{\infty} a_j$ $(a_j \in P_j)$.*

**Proof.** Lemmas 4 and 5. □

**Proof of Theorem I.** Take any $\varepsilon > 0$ and any $y \in A \cup B$. In view of Lemma 1 we can (and do) assume that $(P_j)$ is regular. Let $y = \sum_{j=1}^{\infty} c_j$ $(c_j \in P_j)$. Then there is an index $p$ such that $|\sum_{j=1}^{p-1} c_j - y| < \varepsilon$ and $T(p, \varepsilon)$ contains a point $x$ as described in Lemma 6. Let $w = x + \sum_{j=1}^{p-1} c_j$. It follows that $|w - y| \leq 2\varepsilon$, and $w \in B$. Hence $y \in B$ closure, and $A \cup B \subset B$ closure. □

**Proof of Theorem II.** We deduce from Lemma 3 that $A \cup B$ is the union of countably many translates of closed sets of the form $T(n, \varepsilon)$. This proves part (1). In part (2), at least one of these closed sets is a second category set and contains an interval. We omit the rest of the proof of part (2). □

## References

[E] R. Engelking, *General Topology*, Polish Scientific Publishers, Warsaw, 1977, Theorem 6.1.27, page 440.

[P] M. Petkovsek, *Ambiguous Numbers are Dense,* American Mathematical Monthly 97 (1990) pages 408-411.

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