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A Symmetric Porosity Conjecture of Zajíček

Section 1 - Introduction

If A is a subset of the real line $\mathbb{R}$ and $x \in \mathbb{R}$, then the *porosity of A at x* is defined to be

$$\limsup_{r \to 0+} \frac{\lambda(A, x, r)}{r},$$

where $\lambda(A, x, r)$ is the length of the longest open interval contained in either $(x, x+r) \cap A^c$ or $(x-r, x) \cap A^c$ and A^c denotes the complement of A. A set is said to be *porous at x* if it has positive porosity at x and is called a *porous set* if it is porous at each of its points. A set is said to be *strongly porous at x* if it has porosity one at x and is called a *strongly porous set* if it is strongly porous at each of its points. A set is called *σ-porous* if it is a countable union of porous sets. Right porosity and left porosity are defined in the obvious manner and a set is called *bilaterally porous at x* if it is both right and left porous at x. Likewise, a set is said to be *bilaterally strongly porous at x* if it has both left and right porosity one at x, and is called a *bilaterally strongly porous set* if it is bilaterally strongly porous at each of its points. Thomson's book [5] and the survey articles by Bullen [2] and Zajíček [6] are good sources for historical information on porosity and on some of its applications.

Recently, Zajíček [7] has found an application for a notion of porosity even stronger than that of bilateral porosity, a concept which he calls *symmetric porosity.* It is closely related to the "index" of Denjoy [3, 4], who utilized it in his study of second order symmetric differentiation. The *symmetric porosity of A at x* is defined as

$$\limsup_{r \to 0+} \frac{\gamma(A, x, r)}{r},$$

where $\gamma(A, x, r)$ is the supremum of all positive numbers h such that there is a positive number t with $t + h \leq r$ such that both of the intervals $(x - t - h, x - t)$ and $(x + t, x + t + h)$ lie in A^c. A set A is *symmetrically porous* if it has positive symmetric porosity at each of its points and is called *strongly symmetrically porous* if it has symmetric porosity one at each of its points. With *σ-symmetric porosity* given its natural meaning, Zajíček [7] extended a result in [1], by showing that a certain exceptional set, which was known to be σ-porous, is actually σ-symmetrically porous.

While it is clear that the notion of symmetric porosity is strictly stronger than that of porosity, it is not as obvious that σ-symmetric porosity is strictly stronger than σ-porosity. In [6] Zajíček conjectured that it is. Indeed, he asserted, "It seems to be probable that there exist strongly bilaterally porous sets which are not σ-symmetrically porous and that the analogue of Proposition 2.15 for symmetric porosity does not hold", where Proposition 2.15 in [6] reads

> *Let $c < 1$. Then any σ-porous set $A \subset \mathbb{R}$ may be expressed as the union of a sequence of sets $\{A_n\}$ such that the porosity of each $\{A_n\}$ at each of its points is at least c.*

A proof of this proposition can be found in [5]. The purpose of the next section is to verify that Zajíček's intuition was right on target. We accomplish this via two elementary examples, each of which involves a symmetric Cantor set. In Section 3 we undertake a somewhat deeper investigation of symmetric porosity for symmetric Cantor sets.

Section 2 - Two Specific Examples

Theorem 1 *There is a strongly bilaterally porous set of real numbers which is not σ-symmetrically porous.*

Proof: First, we shall define a certain symmetric Cantor set E. The set we seek will then be a certain residual subset of E. For readers familar with the notation commonly used for symmetric Cantor sets (e.g. see [6]), the set E will be the set $C(\alpha_n)$, where $\{\alpha_n\}$ is the sequence $\{1/(4n+3)\}$. Specifically, E is defined as the intersection of the closed sets E_n, where $E_0 \equiv [0,1]$, and for $n \geq 1$, E_n is the union of the 2^n disjoint closed intervals obtained by partitioning each of the 2^{n-1} disjoint closed intervals whose union is E_{n-1} into $4n+3$ equal subintervals and deleting the middle open interval. These 2^{n-1} open intervals removed from E_{n-1} to create E_n are denoted

$$I_{n,1}, I_{n,2}, \ldots, I_{n,2^{n-1}},$$

where the second subscript orders the intervals from left to right.

For $x \in E_n$, let I_{n,j_x} denote that interval among $I_{n,1}, I_{n,2}, \ldots, I_{n,2^{n-1}}$, which is closest to x. Then, if the reflection of I_{n,j_x} about x has nonempty intersection with

$$\bigcup_{m=1}^{n-1} \bigcup_{j=1}^{2^{m-1}} I_{m,j},$$

then

$$\frac{|I_{n,j_x}|}{\operatorname{dist}(x, I_{n,j_x})} \leq \frac{|I_{n,j_x}|}{n|I_{n,j_x}|} = \frac{1}{n}.$$

From this, it easily follows that E has symmetric porosity 0 at each of its points.

Now, according to a theorem of Denjoy [4] (cf. [5], p. 188), for every perfect and nowhere dense set F of real numbers, the set of points in F at which F is bilaterally strongly porous is residual in F. Let S be the set of points in the set E at which E is bilaterally strongly porous.

Suppose that S were σ-symmetrically porous, say

$$S = \bigcup_{n=1}^{\infty} S_n,$$

where each S_n is symmetrically porous. Since S is residual in E, there would exist an open interval I and a natural number n_0 such that $I \cap E$ is nonempty and S_{n_0} is dense in $I \cap E$. However, this leads to a contradiction, for if $x \in S_{n_0} \cap I \cap E$, then S_{n_0} has positive symmetric porosity at x, while E has symmetric porosity 0 at x, an impossible situation due to the denseness of S_{n_0} in $E \cap I$.

This completes the proof that the set S has the properties we claimed it to have.

Theorem 2 *There is a set of real numbers which has symmetric porosity at least one-half at each of its points, but which cannot be expressed as a countable union of sets each having symmetric porosity greater than four-fifths at each of its points.*

Proof: This time, our set E will be the Cantor "middle halves" set; i.e., E is the symmetric Cantor set, $C(\alpha)$, where α is the constant sequence $\{\frac{1}{2}, \frac{1}{2}, \dots\}$. Specifically, E is defined as the intersection of the closed sets E_n, where $E_0 \equiv [0,1]$, and for $n \geq 1$, E_n is the union of the 2^n disjoint closed intervals obtained by deleting from each of the 2^{n-1} disjoint closed intervals, whose union is E_{n-1}, the central open interval having length one-half that of the closed interval. These 2^{n-1} open intervals removed from E_{n-1} to create E_n are denoted

$$I_{n,1}, I_{n,2}, \dots, I_{n,2^{n-1}},$$

where the second subscript orders the intervals from left to right.

For $x \in E_n$, let I_{n,j_x} denote that interval among $I_{n,1}, I_{n,2}, \dots, I_{n,2^{n-1}}$, which is closest to x. Let $L(x)$ denote the length of the interval obtained by intersecting the reflection of I_{n,j_x} about x with $\bigcup_{m=1}^{n-1} \bigcup_{j=1}^{2^{m-1}} I_{m,j}$, and let

$$f(x) = \frac{L(x)}{\mathrm{dist}(x, I_{n,j_x}) + |I_{n,j_x}|}.$$

Then it is an elementary calculus exercise to see that f has an absolute minimum value of $1/2$, attained at the points x which are endpoints of their respective I_{n,j_x}, and an absolute maximum value of $4/5$, which is attained at the midpoint of each of the 2^n closed intervals whose union is E_n. Therefore, it readily follows that the symmetric porosity of E at each of its points is at least $1/2$, but is at most $4/5$.

Now, suppose that E could be expressed as

$$E = \bigcup_{n=1}^{\infty} S_n,$$

where each S_n has symmetric porosity exceeding $4/5$ at each of its points. There would exist an open interval I and a natural number n_0 such that

$I \cap E$ is nonempty and S_{n_0} is dense in $I \cap E$. As in the previous example, this is an untenable situation because E has symmetric porosity at most 4/5 at each of its points.

Section 3 - Symmetric Cantor Set Results

Here, we undertake a more in depth study of the symmetric porosity characteristics of symmetric Cantor sets, in general. We begin by establishing some notation. Let Σ denote the set of all finite sequences of 0's and 1's. If $\sigma \in \Sigma$ we denote the length of σ by $|\sigma|$ and will write σ in expanded form as $\sigma(1)\sigma(2)\sigma(3)\ldots\sigma(n)$. If $0 < \alpha_n < 1$ for all $n \in \mathbf{N}$, then $\{\alpha_n\}$ determines a symmetric Cantor set, $C(\alpha_n)$, in $[0,1]$. We identify certain intervals which arise in the construction of this Cantor set using the subscripts from Σ in a natural way, i.e. $I_\emptyset = (\frac{1}{2} - \frac{\alpha_1}{2}, \frac{1}{2} + \frac{\alpha_1}{2})$, J_0 and J_1 are the right and left hand components of the complement of $I_\emptyset$ respectively; I_0 and I_1 are the open intervals of length $\alpha_2(1-\alpha_1)/2$ centered in J_0 and J_1 respectively, and so on. The Cantor set is then

$$C(\alpha_n) = \bigcap_{n=1}^{\infty} \bigcup_{|\sigma|=n} J_\sigma$$

and the collection of the I_σ's forms the set of contiguous intervals to this Cantor set. Note that

$$|J_\sigma| = \prod_{n=1}^{|\sigma|} (\frac{1-\alpha_n}{2}) \text{ and } |I_\sigma| = \alpha_{|\sigma|+1}|J_\sigma|.$$

In the proof of Theorem 2 we noted that the symmetric porosity of $C(1/2)$ is bounded above by 4/5 at each of its points. This is a striking difference between porosity and symmetric porosity, because, as we noted earlier, any perfect nowhere dense set contains a residual subset at each point of which the set has porosity one. Our first theorem in this section shows that this behavior is not peculiar to $C(1/2)$.

Theorem 3 *Let $\{\alpha_n\}$ be a sequence of numbers such that $0 < \alpha_n < 1$ and let $\alpha = \limsup \alpha_n$. The symmetric porosity of $C(\alpha_n)$ at each of its points is bounded above by $\frac{4\alpha}{3\alpha+1}$.*

Proof. Let $\sigma \in \Sigma$ and suppose $|\sigma| = n$. For each $x \in \mathbf{R}$, lying to the left of I_σ, we let $s(x,\sigma) = \sup\{1-\delta : (x+\delta h, x+h) \subseteq I_\sigma$ and $(x-h, x-\delta h) \subseteq I_\tau$ for some τ such that $|\tau| \le |\sigma|\}$.

If p is the midpoint of $J_{\sigma 0}$, then

$$s(p,\sigma) \le \frac{|I_\sigma|}{\operatorname{dist}(p, I_\sigma) + |I_\sigma|} = \frac{|I_\sigma|}{|J_{\sigma 0}|/2 + |I_\sigma|} = \frac{4\alpha_{n+1}}{3\alpha_{n+1}+1}.$$

If $x < p$, then

$$s(x,\sigma) \le \frac{|I_\sigma|}{\operatorname{dist}(x, I_\sigma) + |I_\sigma|} \le \frac{|I_\sigma|}{\operatorname{dist}(p, I_\sigma) + |I_\sigma|} = \frac{4\alpha_{n+1}}{3\alpha_{n+1}+1}.$$

If $x > p$, then

$$s(x,\sigma) \le \frac{|I_\sigma| - 2(x-p)}{\operatorname{dist}(x, I_\sigma) + |I_\sigma|} \le \frac{|I_\sigma| - 2(x-p)}{\operatorname{dist}(p, I_\sigma) - (x-p) + |I_\sigma|} \le \frac{4\alpha_{n+1}}{3\alpha_{n+1}+1}.$$

Defining $s(x,\sigma)$ analogously for x to the right of I_σ, we obtain in a symmetric fashion that

$$s(x,\sigma) \le \frac{4\alpha_{n+1}}{3\alpha_{n+1}+1}.$$

Since this inequality holds for each $\sigma \in \Sigma$ and $\alpha = \limsup \alpha_n$, it readily follows that the symmetric porosity of $C(\alpha_n)$ at each of its points is at most $\frac{4\alpha}{3\alpha+1}$.

This theorem has the following result as an immediate consequence.

Corollary 1 *If $\lim_{n\to\infty} \alpha_n = 0$, then $C(\alpha_n)$ has symmetric porosity zero at each of its points.*

Note that this corollary permits the construction of many examples exhibiting the phenomena illustrated by Theorem 1 in the previous section; that is, if $\lim \alpha_n = 0$, then $C(\alpha_n)$ will contain a set S, which is bilaterally strongly porous, but is not σ−symmetrically porous.

If we next consider the case of a symmetric Cantor set determined by a constant sequence, then we can obtain an upper bound on the symmetric porosity of the set at each of its points, which is smaller than that guaranteed by Theorem 3.

Theorem 4 *If $0 < \alpha = \alpha_n < 1$ for all n, then the symmetric porosity of $C(\alpha)$ at each of its points is bounded above by $\frac{4\alpha}{1+4\alpha-\alpha^2}$.*

Proof. Let $\sigma \in \Sigma$. As in the proof of Theorem 3, for each $x \in \mathbb{R}$, lying to the left of I_σ, we let $s(x,\sigma) = \sup\{1-\delta : (x+\delta h, x+h) \subseteq I_\sigma$ and $(x-h, x-\delta h) \subseteq I_\tau$ for some τ such that $|\tau| \leq |\sigma|\}$.

If u is the right endpoint of $J_{\sigma 00}$, then

$$\begin{aligned} s(u,\sigma) &\leq \frac{|I_\sigma|}{\text{dist}(u,I_\sigma)+|I_\sigma|} = \frac{|I_\sigma|}{|I_{\sigma 0}|+|J_{\sigma 01}|+|I_\sigma|} \\ &= \frac{\alpha|J_\sigma|}{\alpha|J_{\sigma 0}|+\left(\frac{1-\alpha}{2}\right)|J_{\sigma 0}|+\alpha|J_\sigma|} \\ &= \frac{\alpha|J_\sigma|}{\alpha\left(\frac{1-\alpha}{2}\right)|J_\sigma|+\left(\frac{1-\alpha}{2}\right)^2|J_\sigma|+\alpha|J_\sigma|} \\ &= \frac{4\alpha}{1+4\alpha-\alpha^2}. \end{aligned}$$

If $x < u$, then

$$s(x,\sigma) \leq \frac{|I_\sigma|}{\text{dist}(x,I_\sigma)+|I_\sigma|} \leq \frac{|I_\sigma|}{\text{dist}(u,I_\sigma)+|I_\sigma|} = \frac{4\alpha}{1+4\alpha-\alpha^2}.$$

If v is the left endpoint of $J_{\sigma 01}$, then

$$\begin{aligned} s(v,\sigma) &\leq \frac{|I_\sigma|-|I_{\sigma 0}|}{\text{dist}(v,I_\sigma)+|I_\sigma|} = \frac{|I_\sigma|-|I_{\sigma 0}|}{|J_{\sigma 01}|+|I_\sigma|} \\ &= \frac{\alpha|J_\sigma|-\alpha|J_{\sigma 0}|}{\left(\frac{1-\alpha}{2}\right)|J_{\sigma 0}|+\alpha|J_\sigma|} = \frac{\alpha|J_\sigma|-\alpha\left(\frac{1-\alpha}{2}\right)|J_\sigma|}{\left(\frac{1-\alpha}{2}\right)^2|J_\sigma|+\alpha|J_\sigma|} \\ &= \frac{2\alpha}{1+\alpha} = \frac{4\alpha}{1+4\alpha-\alpha^2+(\alpha-1)^2} < \frac{4\alpha}{1+4\alpha-\alpha^2}. \end{aligned}$$

If $x > v$, then

$$\begin{aligned} s(x,\sigma) &\leq \frac{|I_\sigma|-|I_{\sigma 0}|-2(x-v)}{\text{dist}(x,I_\sigma)+|I_\sigma|} \leq \frac{|I_\sigma|-|I_{\sigma 0}|-2(x-v)}{\text{dist}(v,I_\sigma)-(x-v)+|I_\sigma|} \\ &< \frac{4\alpha}{1+4\alpha-\alpha^2}. \end{aligned}$$

Consequently, for any $x \in \mathbb{R} \cap C(\alpha)$, lying to the left of I_σ ,we have

$$s(x,\sigma) \leq \frac{4\alpha}{1+4\alpha-\alpha^2}.$$

Defining $s(x,\sigma)$ analogously for $x \in \mathbb{R}$ to the right of I_σ, we obtain in a symmetric fashion that

$$s(x,\sigma) \leq \frac{4\alpha}{1+4\alpha-\alpha^2}.$$

Since this inequality holds for each $\sigma \in \Sigma$, it readily follows that the symmetric porosity of $C(\alpha)$ at each of its points is at most $\frac{4\alpha}{1+4\alpha-\alpha^2}$.

We next turn our attention to computing lower bounds on the symmetric porosity of symmetric Cantor sets. In stating and proving these results we have found the following notation to be convenient. We consider reformulating the definition of the symmetric porosity of a set A at the point x in the following manner, which is obviously equivalent to the formulation followed in the introduction. For each positive number h we let the symmetric porosity ratio of A at x determined by h be

$$\mathrm{spr}(A,x,h) = \sup\{1-\delta : (x-h, x-\delta h) \cup (x+\delta h, x+h) \subseteq A^c\}.$$

Then the symmetric porosity of A at x is

$$\mathrm{sp}(A,x) = \limsup_{h\to 0+} \mathrm{sp}(A,x,h).$$

If the set A is understood, then we simply write $\mathrm{spr}(x,h)$ and $\mathrm{sp}(x)$.

The next result shows that if $\limsup(\alpha_n) = 1$, then the obvious upper bound of 1 for the symmetric porosity of $C(\alpha_n)$ at each of its points is also the lower bound.

Theorem 5 *If* $\limsup_{n\to\infty}(\alpha_n) = 1$, *then* $C(\alpha_n)$ *is strongly symmetrically porous.*

Proof: Suppose $\{\alpha_{n_k}\} \to 1$. Let $\epsilon > 0$ and σ be such that $|\sigma| = n_k - 1$ where $\frac{1-\alpha_{n_k}}{2\alpha_{n_k}} < \epsilon$. Note that $|J_{\sigma 0}| = \frac{1-\alpha_{n_k}}{2\alpha_{n_k}}|I_\sigma| < \epsilon|I_\sigma|$. Let I_τ ($|\tau| < |\sigma|$) be contiguous to J_σ. For notational convenience we assume I_τ is left of J_σ. We consider two cases depending on the relative lengths of I_τ and $J_{\sigma 0}$

1. Case $|I_\tau| \geq \frac{1}{\sqrt{\epsilon}}|J_{\sigma 0}|$. Then for $x \in J_{\sigma 0}$,

$$\operatorname{spr}(x,h) \geq \frac{L - 2|J_{\sigma 0}|}{L + |J_{\sigma 0}|}$$

where $L = \inf\{|I_\tau|, |I_\sigma|\}$ and h is measured from x to the most distant endpoint of the shorter of I_τ and I_σ. As $|J_{\sigma 0}| \leq \min\{\sqrt{\epsilon}|I_\tau|, \epsilon|I_\sigma|\}$ it follows that

$$\begin{aligned} \operatorname{spr}(x,h) &\geq \min\left\{\frac{|I_\tau| - 2\sqrt{\epsilon}|I_\tau|}{|I_\tau| + \sqrt{\epsilon}|I_\tau|}, \frac{|I_\sigma| - 2\epsilon|I_\sigma|}{|I_\sigma| + \epsilon|I_\sigma|}\right\} \\ &= \min\left\{\frac{1 - 2\sqrt{\epsilon}}{1 + \sqrt{\epsilon}}, \frac{1 - 2\epsilon}{1 + \epsilon}\right\}. \end{aligned}$$

2. Case$|I_\tau| < \frac{1}{\sqrt{\epsilon}}|J_{\sigma 0}|$. In this case we consider the interval $J_{\sigma'}$ to be the J interval which abuts I_τ on the left and is of the same stage as I_σ (ie. $|\sigma| = |\sigma'|$). If $x \in J_{\sigma 0}$ we use the intervals I_σ and $I_{\sigma'}$ to compute a symmetric porosity ratio. Let h be measured to the right endpoint of I_σ. Then,

$$\begin{aligned} \operatorname{spr}(x,h) &\geq \frac{|I_\sigma| - |J_{\sigma 0}| - |I_\tau| - |J_{\sigma' 1}|}{|I_\sigma| + |J_{\sigma 0}|} = \frac{|I_\sigma| - 2|J_{\sigma 0}| - |I_\tau|}{|I_\sigma| + |J_{\sigma 0}|} \\ &\geq \frac{|I_\sigma| - 2|J_{\sigma 0}| - \frac{1}{\sqrt{\epsilon}}|J_{\sigma 0}|}{|I_\sigma| + \epsilon|I_\sigma|} \geq \frac{|I_\sigma| - (2 + \frac{1}{\sqrt{\epsilon}})\epsilon|I_\sigma|}{|I_\sigma| + \epsilon|I_\sigma|} \\ &= \frac{1 - 2\epsilon - \sqrt{\epsilon}}{1 + \epsilon}. \end{aligned}$$

Thus, in either case we are led to the conclusion that $\operatorname{sp}(x) = 1$ and the theorem is proved.

Having seen that a symmetric Cantor set need not be strongly symmetrically porous at a residual subset of points, we can ask just how symmetrically porous such a set must be at a residual subset of points. The following technical theorem not only adds insight into this behavior, but also permits the construction of examples that help pin down the possibilities.

Theorem 6 *Let $\{\alpha_{n_k}\}$ be a subsequence of $\{\alpha_n\}$ that converges to α, and assume that $\{\alpha_{n_k+1}\}$ converges as well, say to β. Then there exists a residual subset E of $C(\alpha_n)$ such that*

$$\mathrm{sp}(C(\alpha_n), x) \geq \frac{4\alpha}{1 + 3\alpha + \beta(1-\alpha)}$$

for all $x \in E$.

Proof: Let $\sigma \in \Sigma$, then there is a shortest τ of the form $\tau = \sigma 011 \cdots 1$ such that $|J_\tau| < |I_\sigma|$ and $1 + |\tau| \in \{\alpha_{n_k}\}$. Let $K_\sigma = [A, B)$ where A is the righthand endpoint of $I_{\tau 1}$ and $B = A + \frac{|J_\tau|}{|\sigma|}$. If $x \in K_\sigma$ choose $h \equiv h(x, \sigma)$ and $\delta \equiv \delta(x, \sigma)$ so that $x - h$ is the lefthand endpoint of I_τ and $x - \delta h$ is the righthand endpoint of I_τ. Then

$$(x - h, x - \delta h) \cup (x + \delta h, x + h) \subseteq [0, 1] \setminus C(\alpha_n).$$

Then

$$\mathrm{spr}(x, h) = \frac{(1-\delta)h}{h} = \frac{|I_\tau|}{x - (x - h)} \geq \frac{|I_\tau|}{B - (x - h)} =$$

$$\frac{|I_\tau|}{|K_\sigma| + |I_{\tau 1}| + |J_{\tau 10}| + |I_\tau|} =$$

$$\frac{\frac{|I_\tau|}{|J_\tau|}}{\frac{|K_\sigma|}{|J_\tau|} + \frac{|I_{\tau 1}|}{|J_\tau|} + \frac{|J_{\tau 10}|}{|J_\tau|} + \frac{|I_\tau|}{|J_\tau|}}.$$

Since $|J_\tau| = \prod_{n=1}^{|\tau|} (\frac{1-\alpha_n}{2})$ and $|I_\tau| = \alpha_{|\tau|+1} |J_\tau|$, this equals

$$\frac{4\alpha_{n_k}}{\frac{4}{|\sigma|} + 2\alpha_{n_k+1} - 2\alpha_{n_k}\alpha_{n_k+1} + 1 - \alpha_{n_k+1} - \alpha_{n_k} + \alpha_{n_k}\alpha_{n_k+1} + 4\alpha_{n_k}}$$

$$= \frac{4\alpha_{n_k}}{\frac{4}{|\sigma|} + 1 + 3\alpha_{n_k} + \alpha_{n_k+1}(1 - \alpha_{n_k})}.$$

Let

$$E = \{x \in C(\alpha_n) : x \in K_\sigma \text{ for infinitely many } \sigma\}.$$

Then E is a dense G_δ and hence is residual. For $x \in E$, there exists a sequence $\{\sigma_m\}$ such that $x = \cap_{m=1}^{\infty} K_{\sigma_m}$. Then

$$\begin{aligned}
\mathrm{sp}(x) &\geq \lim_{m\to\infty} \mathrm{spr}(x, h(x,\sigma_m)) \\
&\geq \lim_{k\to\infty} \frac{4\alpha_{n_k}}{\frac{4}{|\sigma_m|} + 1 + 3\alpha_{n_k} + \alpha_{n_k+1}(1-\alpha_{n_k})} \\
&= \frac{4\alpha}{1+3\alpha+\beta(1-\alpha)},
\end{aligned}$$

completing the proof.

Corollary 2 *If $\{\alpha_n\} \not\to 0$, then there is a residual subset A of $C(\alpha_n)$ such that $C(\alpha_n)$ is symmetrically porous at every point of A.*

Corollary 3 *With hypotheses as in the theorem, if $\beta = 0$ (respectively, $\beta = 1$), then there is a residual subset E of $C(\alpha_n)$ such that $\mathrm{sp}(C(\alpha_n), x) \geq \frac{4\alpha}{1+3\alpha}$ (respectively, $\mathrm{sp}(C(\alpha_n), x) \geq \frac{2\alpha}{1+\alpha}$) for all $x \in E$.*

Before stating the next corollary, we define an equivalence relation on the collection of Cantor sets $C(\alpha_n)$ by considering two such sets, $C(\alpha_n)$ and $C(\lambda_n)$, to be equivalent if and only if

$$\limsup_{n\to\infty} \alpha_n = \limsup_{n\to\infty} \lambda_n.$$

We let $\mathcal{C}_\alpha$ denote the equivalence class consisting of those $C(\alpha_n)$ such that $\limsup_{n\to\infty} \alpha_n = \alpha$. Then we have the following result.

Corollary 4 *Let $\alpha \in [0,1]$. For each $C(\lambda_n) \in \mathcal{C}_\alpha$ there exists a residual subset $E(\lambda_n)$ of $C(\lambda_n)$ such that*

$$\mathrm{sp}(C(\lambda_n), x) \geq \frac{4\alpha}{1+4\alpha-\alpha^2}$$

for all $x \in E(\lambda_n)$. Further, $\frac{4\alpha}{1+4\alpha-\alpha^2}$ is the best (i.e., the largest) such lower bound for the equivalence class.

Proof: We assume that $\{\alpha_{n_k}\}$ converges to α. Then $\{\alpha_{n_k+1}\}$ is a bounded sequence and there is no loss of generality in assuming that it converges as well, say to limit β. Then $\beta \le \alpha$ and so

$$\frac{4\alpha}{1+3\alpha+\beta(1-\alpha)} \ge \frac{4\alpha}{1+4\alpha-\alpha^2},$$

and so $\frac{4\alpha}{1+4\alpha-\alpha^2}$ is a lower bound as stated. That it is the largest such lower bound for the equivalence class follows from the fact that the symmetric Cantor set $C(\alpha)$ determined by the constant sequence α was shown to have symmetric porosity at most $\frac{4\alpha}{1+4\alpha-\alpha^2}$ at each of its points in Theorem 4.

It is interesting to note that within a given $\mathcal{C}_\alpha$ there are sets $C(\alpha_n)$ where the universal upper bound $\frac{4\alpha}{1+3\alpha}$ on the symmetric porosity guaranteed by Theorem 3 is actually achieved at a residual subset of points of $C(\alpha_n)$. As pointed out in Corollary 3, this can be done by arranging $\beta = 0$. Further, note that in the case of a constant sequence $\{\alpha\}$, there is a residual subset of $C(\alpha)$ at each point of which the symmetric porosity of $C(\alpha)$ is exactly $\frac{4\alpha}{1+4\alpha-\alpha^2}$. In this latter setting, we can actually obtain a positive lower bound on the symmetric porosity of $C(\alpha)$ at each of its points. Toward this end, for $0 < \alpha < 1$, define an integer $\xi(\alpha)$ to be the smallest integer such that

$$\frac{\alpha}{2} \ge \left(\frac{1-\alpha}{2}\right)^{\xi(\alpha)}.$$

Theorem 7 *If $0 < \alpha = \alpha_n < 1$ for all n, then $C(\alpha)$ is symmetrically porous with symmetric porosity at least $\frac{2\alpha}{1+\alpha}(\frac{1-\alpha}{2})^{\xi(\alpha)}$ at each of its points.*

Proof: Let $\sigma \in \Sigma$ be given and define $k(\sigma) = \sigma 011...1$ where $|k(\sigma)| - |\sigma| = \xi(\alpha)$. As $\frac{\alpha}{2} \ge \left(\frac{1-\alpha}{2}\right)^{\xi(\alpha)}$ it follows that $2|J_{k(\sigma)}| \le |I_\sigma|$. Now, let I be any interval of length $|I_\sigma|$. We show that $I \setminus C(\alpha)$ contains an interval of length $|I_{k(\sigma)}|$. Let τ be such that $|\tau| = |k(\sigma)| + 1$. Then

$$|J_\tau| = \left(\frac{1-\alpha}{2}\right)^{|k(\sigma)|+1} = \left(\frac{1-\alpha}{2}\right)|J_{k(\sigma)}| < \frac{|I_\sigma|}{2}.$$

If I intersects J_τ and $J_{\tau'}$, where $\tau \ne \tau'$ and $|\tau| = |\tau'| = |k(\sigma)| + 1$, then I contains a contiguous interval from a stage before the $|k(\sigma)|^{th}$ stage and the conclusion is verified. If I intersects at most one such J_τ then $I \setminus C(\alpha)$ contains an interval of length at least $\frac{|I|-|J_\tau|}{2}$. But,

$$\frac{|I|-|J_\tau|}{2} = \frac{|I_\sigma|-|J_\tau|}{2}$$

$$= \frac{1}{2}\left[\alpha\left(\frac{1-\alpha}{2}\right)^{|\sigma|} - \left(\frac{1-\alpha}{2}\right)^{|k(\sigma)|+1}\right]$$

$$= \frac{1}{2}\alpha\left(\frac{1-\alpha}{2}\right)^{|k(\sigma)|}\left[\left(\frac{1-\alpha}{2}\right)^{|\sigma|-|k(\sigma)|} - \frac{1-\alpha}{2\alpha}\right]$$

$$\geq \frac{1}{2}|I_{k(\sigma)}|\left(\frac{2}{\alpha} - \frac{1-\alpha}{2\alpha}\right) \geq |I_{k(\sigma)}|.$$

This completes the verification that $I \setminus C(\alpha)$ contains an interval of length $|I_{k(\sigma)}|$. Let $x \in J_{\sigma 0}$ and consider the interval $I_x = 2x - I_\sigma$. As $|I_x| = |I_\sigma|$ it follows that $I_x \cap (\bigcup_{|\tau| \leq |k(\sigma)|} I_\tau)$ contains an interval whose length exceeds $|I_{k(\sigma)}|$. That is, there is an $h \equiv h(x,\sigma)$ and a $t \equiv t(x,\sigma)$ such that

$$(x+h, x+h+t) \cup (x-h-t, x-h) \subset [0,1] \backslash C(\alpha),$$

and,

$$\frac{|(x+h, x+h+t)|}{|(x, x+h+t)|} \geq \frac{|I_{k(\sigma)}|}{|J_{\sigma 1}| + |I_\sigma|}$$

$$= \frac{2\alpha(\frac{1-\alpha}{2})^{|k(\sigma)|-|\sigma|}}{1+\alpha}$$

$$= \frac{2\alpha}{\alpha+1}\left(\frac{1-\alpha}{2}\right)^{\xi(\alpha)}$$

The conclusion now follows.

By Theorems 4 and 7 we can now say more about the type of example constructed in Theorem 2. Namely, we can give specific bounds for the symmetric porosity at x in a Cantor set determined by constant sequence $\{\alpha\}$. For example, $\xi(\alpha)$ can be taken to be 1 if and only if $\alpha \geq \frac{1}{2}$ and in this case

$$\frac{2\alpha}{\alpha+1}\left(\frac{1-\alpha}{2}\right) \leq \mathrm{sp}(x) \leq \frac{4\alpha}{1+4\alpha-\alpha^2}$$

for all $x \in C(\alpha)$. Likewise, $\xi(\alpha)$ can be taken to be 2 if and only if $\alpha \geq 2-\sqrt{3}$ and in this case

$$\frac{2\alpha}{\alpha+1}\left(\frac{1-\alpha}{2}\right)^2 \leq \mathrm{sp}(x) \leq \frac{4\alpha}{1+4\alpha-\alpha^2}$$

for all $x \in C(\alpha)$. In the context of Theorem 2, we now have intervals of numbers that can play the role of the interval $[1/2, 4/5]$ in that theorem. For instance, we can observe that for any $\alpha \geq \frac{1}{2}$, the set $C(\alpha)$ has symmetric porosity at least $\frac{2\alpha}{\alpha+1}(\frac{1-\alpha}{2})$ at each of its points, yet cannot be written as a countable union of sets each of which has symmetric porosity greater than $\frac{4\alpha}{1+4\alpha-\alpha^2}$ at each of its points.

References

[1] C. L. Belna, M. J. Evans, and P. D. Humke, "Symmetric and ordinary differentiation," *Proc. Amer. Math. Soc.* 72 (1978), 261–267.

[2] P. S. Bullen, "Denjoy's index and porosity," *Real Anal. Exch.* 10 (1984-85), 85-144.

[3] A. Denjoy, "Sur une propriété des séries trigonométriques," *Verlag v.d.G.V. der Wis-en Natuur. Afd*, 30 Oct. 1920.

[4] A. Denjoy, "Lecons sur le calcul des coefficients d'une série trigonometrique, Part II, Métrique et topologie d'ensembles parfaits et de fonctions," Gauthier-Villars, Paris 1941.

[5] B. S. Thomson, "Real Functions", Lecture Notes in Mathematics 1170, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo 1985.

[6] L. Zajíček, "Porosity and σ-porosity," *Real Anal. Exch.* 13 (1987-88), 314–350.

[7] L. Zajíček, "On the symmetric and ordinary derivative," preprint.

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