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Set Functions, Finite Additivity and Joint Distribution Function Representations

Abstract

Suppose that F is a field of subsets of a set U, N is a positive integer, $\{\alpha_k\}_{k=1}^N$ is a sequence of functions from F into $\exp(\mathbf{R})$ and μ is a real, nonnegative - valued finitely additive function on F. Suppose that Ξ^N is the set of all N - dimensional subintervals of $\mathbf{R}^N$. It is shown that there is a nonnegative - valued function A from $\Xi^N \times F$ into $\mathbf{R}$ such that for each V in F, $A(\cdot, V)$ is finitely additive on Ξ^N, such that if for $k = 1, ..., N$, α_k is μ - summable (see "Fields of Sets, Set Functions, Set Function Integrals, and Finite Additivity", Internat. J. Math. & Math. Sci., Vol. 7 No. 2 (1984) pp 209 - 233) and g is a real - valued function on $\mathbf{R}^N$ satisfying certain continuity and boundedness conditions, then

$$\int \cdots \int_R [g \int_U A(\cdot, \cdot)] \to \sigma_\mu(g(\alpha_1, ..., \alpha_N))(U),$$

$$R = [H_1; K_1] \times \cdots \times [H_N; K_N],\ \min\{-H_1, ..., -H_N, K_1, ..., K_N\} \to \infty,$$

where σ_μ is the μ - summability operator and all integrals are refinement - wise limits of the appropriate sums.

1. Introduction.

We begin by stating a "classical" distribution function representation theorem:

Theorem C. Suppose that $\{\Omega, \sum, \mu\}$ is a measure space and $g$is a real-valued function defined on Ω, measurable and μ - summable. For each x in $\mathbf{R}$, let $F(x) = \mu(\{w | g(w) \leq x\})$. Then $\int_\Omega g d\mu = \int_{-\infty}^{+\infty} x dF(x)$.

Our purpose in this paper is to extend, from 1 to a given positive integer N, a previous generalization (see [6,8] and the Main Theorem below) of this theorem. We begin by once again (see [8,11]) considering a notion of distribution function for a set function, as before, in a finitely additive setting. We proceed as follows:

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Suppose that U is a set, F is a field of subsets of U, μ is a real nonnegative - valued finitely additive function on F. Suppose that α is a function from F into $\exp(\mathbf{R})$. As a slight modification of a previous definition (again, see [6,8]), we give the following:

Definition 1.1. $\beta(\alpha)$ denotes the function from $\mathbf{R} \times F$ into $\exp(\{0,1\})$ such that if t is in $\mathbf{R}$ and I is in F, then $\beta(\alpha)(t, I)$ contains 1 iff $x \leq t$ for some x in $\alpha(I)$, and contains 0 iff $t < x$ for some x in $\alpha(I)$.

Before proceeding, we refer the reader to section 2 of this paper for the notions of L and G, the sum supremum and sum infimum functional, respectively.

Let us note that if $a < b$, α is a function from F into $\exp(\mathbf{R})$ and t is in $\mathbf{R}$, then, even if the range union of α is bounded and $\int_U \alpha\mu$ exists, $\int_U \beta(\alpha)(t,\cdot)\mu$ does not necessarily exist. That observation is what caused us in [8] to consider L and G and define the following upper distribution and lower distribution functions,

$$u(\alpha)(t) = \int_U L(\beta(\alpha)(t,\cdot)\mu) \text{ and } v(\alpha)(t) = \int_U G(\beta(\alpha)(t,\cdot)\mu),\ t \text{ in } \mathbf{R},$$

respectively. In [8] we showed interrelations between the integrability of $\alpha\mu$ and the behavior of $u(\alpha)$ and $v(\alpha)$, given in the first five theorems immediately below and stated here with the minor changes mentioned above.

Theorem 1.A.1. If V is in F and α has bounded range union and $s < t$, then

$$\int_V L(\beta(\alpha)(s,\cdot)\mu) - \int_V G(\beta(\alpha)(t,\cdot)\mu) \leq (1/(t-s))[\int_V L(\alpha\mu) - \int_V G(\alpha\mu)].$$

Theorem 1.A.2. If V is in F, range union $\alpha \subseteq (a;b]$ and h is a function defined, continuous and nondecreasing on $[a;b]$, then

$$\int_a^b [h(x) d\int_V L(\beta(\alpha)(x,\cdot)\mu)] \leq \int_V G(h(\alpha)\mu) \leq \int_V L(h(\alpha)\mu) \leq$$

$$\int_a^b [h(x) d\int_V G(\beta(\alpha)(x,\cdot)\mu)].$$

Theorem 1.A.3. If V is in F and range union $\alpha \subseteq (a;b]$, then the following three statements are equivalent:

1) $\int_V \alpha\mu$ exists,

2) $u_V(\alpha)(x+) = v_V(\alpha)(x+)$ on $\mathbf{R}$, where for each t in $\mathbf{R}$,

$$u_V(\alpha)(t) = \int_V L(\beta(\alpha)(t,\cdot)\mu),\ v_V(\alpha)(t) = \int_V G(\beta(\alpha)(t,\cdot)\mu)$$

and

3)

$$\int_a^b [xd\int_V L(\beta(\alpha)(x,\cdot)\mu)] = \int_a^b [xd\int_V G(\beta(\alpha)(x,\cdot)\mu)].$$

We then obtained the following set function version, for the bounded case, of Theorem C :

Theorem 1.A.4. If V is in F, range union $\alpha \subseteq (a;b]$ and h is a real - valued function defined and continuous on $[a;b]$ and $\int_V \alpha\mu$ exists, then

$$\int_a^b [h(x)d\int_V L(\beta(\alpha)(x,\cdot)\mu)] = \int_V h(\alpha)\mu = \int_a^b [h(x)d\int_V G(\beta(\alpha)(x,\cdot)\mu)].$$

Finally, we obtained the following "summable" (see section 2) extension of Theorem 1.A.4:

Theorem 1.A.5. Suppose that α is μ - summable (see section 2 and [2,11]). Suppose that g is a function from $\mathbf{R}$ into $\mathbf{R}$, continuous and such that $\{g(x)/|x| : 1 \leq |x|\}$ is bounded. Then $g(\alpha)$ is μ - summable and if Q is L or G, then

$$\sigma_\mu(g(\alpha))(U) = \int_{-\infty}^{+\infty} [g(x)d\int_U Q(\beta(\alpha)(x,\cdot)\mu)],$$

where σ_μ is the μ - summability operator (see section 2 and [2,11]).

We pause here to point out a matter that is already apparent and which will become more so as our work progresses, namely the use of "zero - one set functions", *i. e.*, functions from F into $\exp(\{0,1\})$. These have found application in theorems (see [1,3,4,6,7,8,9,10,12,13,14,15,16]) about absolute continuity, set function integrability, closest approximations and their representations, function decompositions and, as in this paper, distribution function representations.

Now suppose that N is a positive integer. Let Ξ^N denote the set of all subintervals of $\mathbf{R}^N$. Our main task in this paper is to extend Theorem 1.A.5, first to a sequence $\{\alpha_k\}_{k=1}^N$ of functions from F into $\exp(\mathbf{R})$ with bounded range unions (see Theorem 3.1), then to a sequence $\{\alpha_k\}_{k=1}^N$ of functions from F into $\exp(\mathbf{R})$ that are μ - summable. With reference to the nonintegrability statements and the consequent use of L and G as discussed earlier in this introduction, we let, for each $k = 1, ..., N$, Q_k be L or G and define the following function on $\Xi^N \times F$:

$$B^{(N)}(R,V) = (\prod_{k=1}^N [[\int_V Q_k(\beta(\alpha_k)(y_k,\cdot)\mu) - \int_V Q_k(\beta(\alpha_k)(x_k,\cdot)\mu)])/[\mu(V)^{N-1}],$$

where $R = [x_1;y_1] \times \cdots \times [x_N;y_N]$.

We observe that for each V in F, $B^{(N)}(\cdot, V)$ is a real - valued function, finitely additive on Ξ^N.

The principal and final result of this paper is the following representation theorem, which we shall prove in section 5:

Main Theorem. Suppose that $\{\alpha_k\}_{k=1}^N$ is a sequence of functions from F into exp($\mathbf{R}$), μ - summable (see section 2). Suppose that g is a function from $\mathbf{R}^N$ into $\mathbf{R}$, continuous and such that

$$\{g(x_1, ..., x_N)/max\{|x_1|, ..., |x_N|\} : 1 \leq \max\{|x_1|, ..., |x_N|\}\}$$

is bounded. Then $g(\alpha_1, ..., \alpha_N)$ is μ - summable (see [5]) and

$$\int \cdots \int_R [g \int_U B^{(N)}(\cdot, \cdot)] \to \sigma_\mu(g(\alpha_1, ..., \alpha_N))(U),$$

$$R = [H_1; K_1] \times \cdots \times [H_N; K_N], \ \min\{-H_1, ..., -H_N, K_1, ..., K_N\} \to \infty.$$

2. Preliminary theorems and definitions.

In this paper the integrals discussed will either be integrals of functions defined on sets of elements of Ξ^N for some N, or integrals of functions defined on F. Briefly, and we shall expand on this further below, both types shall be limits, for subdivision refinement, of the appropriate sums. The notions of subdivision and refinement, and the integrals that arise therefrom, as they pertain to Ξ^N and F, respectively, are sufficiently similar that we shall make our more detailed remarks for F and assume that the reader can effect the necessary modifications for Ξ^N, e. g., "mutually exclusive" for elements of F vs. "nonoverlapping" for elements of Ξ^N.

We shall let $r(F)$ denote the set of all functions from F into exp($\mathbf{R}$).

We adopt the convention that if δ is a function from F into $\mathbf{R}$, then δ shall be regarded as "equivalent" to the following element of $r(F)$;

$$\{(V, \{\delta(V)\}) : V \text{ in } F\}.$$

If V is in F, then the statement that D is a subdivision of V means that D is a finite collection of mutually exclusive sets of F with union V. The statement that H is a refinement of E, denoted by $H << E$, means that for some W in F, each of H and E is a subdivison of W and each element of H is a subset of some element of E.

If S is a set, γ is a function with domain S and range a collection of sets and $T \subseteq S$, then the statement that b is a γ - function on T means that b is a function with domain T such that if x is in T, then $b(x)$ is in $\gamma(x)$.

Suppose that γ is in $r(F)$ and V is in F. The statement that K is an integral of γ on V means that K is in $\mathbf{R}$ and if $0 < c$, then there is $D << \{V\}$ such that if $E << D$ and b is a γ - function on E, then

$$|K - \sum_E b(I)| < c.$$

There is no more than one K' such that K' is an integral of γ on V; if, then, K is an integral of γ on V, then K is unique and shall be denoted, variously, by

$$\int_V \gamma(I), \int_V \gamma(J), \int_V \gamma, \text{ etc.,}$$

depending upon circumstances. We shall use the phrase "$\int_V \gamma(I)$ exists" to mean that there is K such that K is an integral of γ on V. Now, if $\int_U \gamma$ exists, then for each W in F, $\int_W \gamma$ exists and

$$\{(W, \int_W \gamma) : W \text{ in } F\},$$

which we shall denote by $\int \gamma$, is a real-valued finitely additive function on F.

Again, suppose that γ is in $r(F)$. If V is in F, then the statement that γ is $\sum$ -bounded on V with respect to D means that $D << \{U\}$ and

$$\{\sum_E b(J) : E << \{V\}, b \text{ a } \gamma \text{ - function on } E, E \subseteq H \text{ for some } H << D\}$$

is bounded. We have the following results:

Theorem 2.A.1 (see [11]). If γ is in $r(F)$ and is $\sum$ - bounded on U with respect to D, then the following statements are true:

1) If V is in F, then γ is $\sum$ - bounded on V with respect to D.

2) Suppose that $L_D(\gamma)$ and $G_D(\gamma)$ denote the functions with domain F given, for each I in F as, respectively, the sup and inf of

$$\{\sum_E b(J) : E << \{I\}, b \text{ a } \gamma \text{ - function on } E, E \subseteq H \text{ for some } H << D\}.$$

Then, if V is in F, $H_1 << \{V\}$, $H_2 << \{V\}$ and for $i = 1, 2$, $M << H_i$, then

$$\sum_{H_1} G_D(\gamma)(I) \leq \sum_M G_D(\gamma)(J') \leq \sum_M L_D(\gamma)(J') \leq \sum_{H_2} L_D(\gamma)(J).$$

3) If V is in F, then the following existence and inequality holds:

$$\int_V G_D(\gamma) \leq \int_V L_D(\gamma).$$

4) If V is in F, then $\int_V \gamma$ exists iff

$$\int_V G_D(\gamma) = \int_V L_D(\gamma),$$

in which case

$$\int_V G_D(\gamma) = \int_V \gamma = \int_V L_D(\gamma).$$

5) If V is in F, Q is L_D or G_D, $E << \{V\}$ and $0 < c$, then there is $H << E$ and a γ - function a on H such that

$$\sum_H |Q(\gamma)(J) - a(J)| < c.$$

We shall, when circumstances permit, in particular when the elements of $r(F)$ under discussion are $\sum$ - bounded with respect to $\{U\}$, write L and G without subscript.

Theorem 2.A.2 (see [11]). Suppose that each of γ and δ are in $r(F)$ and $\sum$ - bounded on U with respect to D. Then $\gamma + \delta$ is $\sum$ - bounded on U with respect to D and the following statements are true:

1) For each I in F,

$$G(\gamma)(I) + G(\delta)(I) \leq G(\gamma + \delta)(I) \leq L(\gamma + \delta)(I) \leq L(\gamma)(I) + L(\delta)(I).$$

2) If c is in $\mathbf{R}$, then $c\gamma$ is $\sum$ - bounded on U with respect to D and for each I in F,

$$G(c\gamma)(I) = cG(\gamma)(I) \text{ and } L(c\gamma)(I) = cL(\gamma)(I) \text{ if } 0 \leq c, \text{ and}$$

$$G(c\gamma)(I) = cL(\gamma)(I) \text{ and } L(c\gamma)(I) = cG(\gamma)(I) \text{ if } c \leq 0.$$

3) If $\int_U \gamma$ exists, and Q is L or G, then $\int Q(\gamma + \delta) = \int \gamma + \int Q(\delta)$.

We now state Kolmogoroff's differential equivalence theorem.

Theorem 2.K.1 (see [11]). If γ is in $r(F)$ and $\int_U \gamma$ exists, then for each I in F, $\int_I \gamma$ exists and the following existence and equality holds:

$$\int_U |\gamma(I) - \int_I \gamma| = 0,$$

i. e., if $0 < c$, then there is $D << \{U\}$ such that if $E << D$ and a is a γ - function on E, then

$$\sum_E |a(I) - \int_I \gamma| < c,$$

so that if $H \subseteq E$ and b is a γ - function on H, then

$$\sum_H |b(I) - \int_I \gamma| < c.$$

We refer the reader, again to [4], for certain of the more immediate consequences of Theorem 2.K.1; these consequences treat conditions under which, given an element γ of $r(F)$ such that $\int_U \gamma$ exists, $\int_I \gamma$ and $\gamma(I)$ can be interchanged. Throughout this paper there will be portions of arguments in which assertions of integral existence or integral equivalence follow from Theorem 2.K.1 and these consequences. In such cases we shall feel free to simply make these assertions and leave the details to the reader. Now, before we continue with some more specialized matters, we remark that we shall often assert and use, without preamble, certain simple inequality and linearity existence and equivalence properties of set function integrals.

We shall use the convention that if each of p and q is in $\mathbf{R}$, then p/q shall be 0 if $q = 0$ and have the usual meaning otherwise. It shall be further understood that 0^0 shall be 0 (refer, among other things, back to the Main Theorem in the introduction and Theorem 2.2 below). We now state a theorem concerning set function integral existence.

Theorem 2.A.3 [5]. Suppose that $S = [a_1; b_1] \times \cdots \times [a_N; b_N]$ is in Ξ^N, g is a real - valued function defined and continuous on S, $\{\alpha_k\}_{k=1}^N$ is a sequence of elements of $r(F)$ such that for each $k = 1, ..., N$, range union $\alpha_k \subseteq [a_k; b_k]$ and $\int_U \alpha_k \mu$ exists. Then $\int_U g(\alpha_1, ..., \alpha_N)\mu$ exists.

We now state Corollary 2.2 and Corollary 2.3, which are, respectively, immediate consequences of Theorem 2.A.4 and Corollary 2.2. We leave the easy proofs to the reader.

Corollary 2.2. Assume the hypothesis of Theorem 2.A.3, but without specific reference to g. Then for each V in F the following existence and equality holds:

$$\int_V (\prod_{k=1}^N \alpha_k)\mu = \int_V ([\prod_{k=1}^N (\alpha_k \mu)]/[\mu^{N-1}]) = \int_V ([\prod_{k=1}^N (\int \alpha_k \mu)]/[\mu^{N-1}]).$$

Corollary 2.3. Suppose that for $k = 1, ...N$, ξ_k is in $Lip(\mu)$. Then for each V in F the following existence and equality holds:

$$\int_V (\Pi_{k=1}^N (\xi_k/\mu))\mu = \int_V ((\Pi_{k=1}^N \xi_k)/\mu^{N-1});$$

furthermore, $\int((\Pi_{k=1}^{N}\xi_k)/\mu^{N-1})$ is in $Lip(\mu)$.

Corollary 2.3 immediately implies that for $B^{(N)}$ as defined for the Main Theorem, $R = [x_1; y_1] \times \cdots \times [x_N; y_N]$, and for $k = 1, ..., N$,

$$\xi_k = \int Q_k(\beta(\alpha_k)(y_k, \cdot)\mu) - \int Q_k(\beta(\alpha_k)(x_k, \cdot)\mu),$$

that for each V in F, $\int_V B^{(N)}(R, \cdot)$ exists.

Before we proceed we remark that for expressions involving integrals dealing with functions defined on subsets of $\Xi^N \times F$, we shall feel free, as in Corollary 2.3 above, when there is little risk of ambiguity, to either omit "variables" or put dots in the "variable" positions.

Suppose that α is an element of $r(F)$. We observe, much as before (see [8]), that if $s \leq t$ and V is in F, then

$$\int_V G(\beta(\alpha)(s, \cdot)\mu) \leq \int_V G(\beta(\alpha)(t, \cdot)\mu) \leq$$

$$\int_V L(\beta(\alpha)(t, \cdot)\mu) \geq \int_V L(\beta(\alpha)(s, \cdot)\mu).$$

In what follows, among the results discussed, we shall state three well - known theorems: 2.I.1, 2.I.2 and 2.I.3. They are from a large variety of well - known multiple integral facts. We give them in a form that most immediately applies to the matters of this paper.

Theorem 2.I.1. Suppose that $0 \leq M$, $W = [a_1; b_1] \times \cdots \times [a_N; b_N]$, $\{h_k\}_{k=1}^N$ is a sequence of real - valued functions such that if $k = 1, ..., N$, then h_k is defined and nondecreasing on $[a_k; b_k]$. Suppose that B is a function whose domain includes the subintervals of W such that if $[x_1; y_1] \times \cdots \times [x_N; y_N] = R \subseteq W$, then

$$B(R) = M \prod_{k=1}^{N}(h_k(y_k) - h_k(x_k)).$$

Then the contraction of B to the subintervals of W is nonnegative - valued and finitely additive.

Definition 2.1. If W is in Ξ^N, then $A(W)^+$ denotes the set of all real - nonnegative - valued functions defined and finitely additive on the subintervals of W.

We also note that if m is a positive integer, W is a subinterval of $\mathbf{R}^N$, $\{c_i\}_{i=1}^m$ is a sequence of nonnegative numbers and $\{B_i\}_{i=1}^m$ is a sequence of elements of $A(W)^+$, then $\sum_{i=1}^m c_i B_i$ is in $A(W)^+$.

The next theorem is a Helly - type convergence result, whose corollary we shall use in the sections 3 and 5. The argument is carried out by well - known uniform continuity, consequent uniform convergence of integral approximation sums and "point - wise" convergence observations, and we therefore leave it to the reader.

Theorem 2.3. Suppose that R is in Ξ^N, g is a real - valued function defined and continuous on R, $(S, \leq^*)$ is a partially ordered and directed system and Z is a function with domain S such that for each y is S, $Z(y)$ is in $A(R)^+$. Suppose that H is a real - valued function defined on the subintervals of R such that if Y is a subinterval of R, then

$$Z(y)(Y) \to H(Y), \ \leq^* .$$

Then H is in $A(R)^+$and

$$\int \cdots \int_R gZ(y) \to \int \cdots \int_R gH, \ \leq^* .$$

Corollary 2.4. Suppose that W is in Ξ^N and T is a real - nonnegative - valued function defined on $\{(R, I)|R \text{ a subinterval of } W, \text{ I in } F\}$. Suppose that if R is a subinterval of W, then $\int_U T(R, \cdot)$ exists, and if V is in F, then $T(\cdot, V)$ is in $A(W)^+$. Suppose that g is a real - valued function defined and continuous on R. Then, for each $V in\ F$, $\int_V T(\cdot, \cdot)$ is in $A(W)^+$ and the following existence and equality assertion holds:

$$\int_V [\int \cdots \int_W gT(\cdot, \cdot)] = \int \cdots \int_W [g \int_V T(\cdot, \cdot)].$$

Proof: Let $S =$ the set of all subdivisions of V and $\leq^* = <<$. Let H, and for each D in S, $Z(D)$ be functions defined on the subintervals of W respectively by

$$Z(D)(R) = \sum_D T(R, I), \text{ and } \ H(R) = \int_V T(R, \cdot). \tag{2.4.1}$$

Clearly, for each D in S, $Z(D)$ is in $A(W)^+$ and by hypothesis, for each subinterval R of W, $Z(D)(R) \to H(R)$, $\leq^*$. Therefore, by Theorem 2.3 and the definitions of$\leq^*$, Z and H in this argument, it follows that for $\leq^*$ convergence,

$$\sum_D \int \cdots \int_W gT(\cdot, I) = \int \cdots \int_W g[\sum_D T(\cdot, I)] =$$

$$\int \cdots \int_W gZ(D) \to \int \cdots \int_W gH = \int \cdots \int_W g[\int_V T(\cdot, \cdot)], \tag{2.4.2}$$

so that $\int_V[\int\cdots\int_W gT(\cdot,\cdot)]$ exists and is $\int\cdots\int_W[g\int_V T(\cdot,\cdot)]$.

Theorem 2.I.2. Suppose that $R'=[a_1;b_1]\times\cdots\times[a_N;b_N]$ and $R=[a_1;b_1]\times\cdots\times[a_{N+1};b_{N+1}]$. Suppose that $B^{(N)}$ is in $A(R')^+$ and h is a real - valued nondecreasing function defined on $[a_{N+1};b_{N+1}]$. For each subinterval $[p_1;q_1]\times\cdots\times[p_N;q_N]\times[p_{N+1};q_{N+1}]$ of R, let

$$B^{(N+1)}([p_1;q_1]\times\cdots\times[p_N;q_N]\times[p_{N+1};q_{N+1}])=$$

$$B^{(N)}([p_1;q_1]\times\cdots\times[p_N;q_N])\cdot(h(q_{N+1})-h(p_{N+1})).$$

Then $B^{(N+1)}$ is in $A(R)^+$ and, if g is a function defined and continuous on R, then

$$\int\cdots\int_R gB^{(N+1)}=\int_{a_{N+1}}^{b_{N+1}}[\int\cdots\int_{R'}g(\cdot,...,\cdot,x_{N+1})B^{(N)}]dh(x_{N+1})$$

$$=\int\cdots\int_{R'}[\int_{a_{N+1}}^{b_{N+1}}g(x_1,...,x_N,\cdot)dh]B^{(N)}.$$

Theorem 2.I.3. Suppose that $R=[a_1;b_1]\times\cdots\times[a_N;b_N]$, for each $k=1,...,N$, h_k is a real - valued function defined and nondecreasing on $[a_k;b_k]$. Suppose that v is in $\{1,...,N\}$ and f is a real - valued function defined and continuous on $[a_v;b_v]$. Suppose that B is a function defined on the subintervals of R such that for each subinterval $W=[p_1;q_1]\times\cdots\times[p_N;q_N]$ of R, $B(W)=\prod_{k=1}^N(h_k(q_k)-h_k(p_k))$. Then

$$\int\cdots\int_R fB=(\int_{a_v}^{b_v}fdh_v)\prod_{k\neq v}^N(h_k(b_k)-h_k(a_k)).$$

3. A representation theorem for an integral set function with bounded range union.

In this section we prove a bounded version of the Main Theorem. Certain of our calculations in a later part of the proof, in addition to using Theorem 2.I.1 and Corollary 2.3, use, in conjunction with differential equivalence, the following lemma, whose proof we leave to the reader. It is more specialized than is generally necessary, but its form points to its specific application.

Lemma 3.1. Suppose that $a<b$, v is a real - valued function defined on $[a;b]$, $|v(x)|\leq M$ for all x in $[a;b]$, $0\leq K$, I is in F, $W(\cdot,I)$ is a function defined and nondecreasing on $[a;b]$ and $|W(x,I)|\leq K\mu(I)$ for all x in $[a;b]$. Suppose that $\int_a^b vdW(\cdot,I)$ exists. Then $|\int_a^b vdW(\cdot,I)|\leq M2K\mu(I)$, so that $|\int_a^b vdW(\cdot,I)|/\mu(I)\leq M2K$.

Theorem 3.1. Suppose that μ is in $A(\mathbf{R})(F)^+$. Suppose that $R = [a_1; b_1] \times \cdots \times [a_N; b_N]$ and $\{\alpha_k\}_{k=1}^N$ is a sequence of elements of $r(F)$ such that if $k = 1, ..., N$, then range union of $\alpha_k \subseteq (a_k; b_k]$, $\int_U \alpha_k \mu$ exists and Q_k is L or G. Suppose that $B^{(N)}$ is a function defined on $\Xi^N \times F$ as in the Main Theorem by

$$B^{(N)}(W, I) = (\prod_{k=1}^{N} [\int_I Q_k(\beta(\alpha_k)(y_k, \cdot)\mu) - \int_I Q_k(\beta(\alpha_k)(x_k, \cdot)\mu)])/[\mu(I)^{N-1}],$$

where $W = [x_1; y_1] \times \cdots \times [x_N; y_N]$.

Then, if g is a function defined and continuous on R and V is in F, then

$$\int_V g(\alpha_1, ..., \alpha_N)\mu = \int \cdots \int_R [g \int_V B^{(N)}(\cdot, \cdot)].$$

Proof: We use induction. Consider $N = 1$. By Theorem 1.A.4, if V is in F, then

$$\int_V g(\alpha_1)\mu = \int_{a_1}^{b_1} [g(x_1) d \int_V Q_1(\beta(\alpha_1)(x_1, \cdot)\mu)], \tag{3.1.1}$$

which is the desired equation for

$$B^{(1)}(W, I) = (\int_I Q_1(\beta(\alpha_1)(y_1, \cdot)\mu) - \int_I Q_1(\beta(\alpha_1)(x_1, \cdot)\mu))/[\mu(V)^{1-1}], \tag{3.1.2}$$

where $W = [x_1; y_1]$. Now suppose that N is a positive integer such that for all positive integers $\leq N$ and all V in F the statement of the theorem holds. Suppose that the hypothesis for $N + 1$ holds. Suppose that g is a real - valued function defined and continuous on $R = [a_1; b_1] \times \cdots \times [a_{N+1}; b_{N+1}]$. Let $R' = [a_1; b_1] \times \cdots \times [a_N; b_N]$ and $B^{(N)}$ be a function defined on $\Xi^N \times F$ by

$$B^{(N)}(W, I) = (\prod_{k=1}^{N} [(\int_I Q_k(\beta(\alpha_k)(y_k, \cdot)\mu) - \int_I Q_k(\beta(\alpha_k)(x_k, \cdot)\mu)])/[\mu(I)^{N-1}], \tag{3.1.3}$$

where $W = [x_1; y_1] \times \cdots \times [x_N; y_N]$. Note that if (T, I) is in $\Xi^{N+1} \times F$ and $T = [x_1; y_1] \times \cdots \times [x_{N+1}; y_{N+1}]$ and $T' = [x_1; y_1] \times \cdots \times [x_N; y_N]$, then

$$B^{(N+1)}(T, I) =$$

$$B^{(N)}(T', I) \frac{\int_I Q_{N+1}(\beta(\alpha_{N+1})(y_{N+1}, \cdot)\mu) - \int_I Q_{N+1}(\beta(\alpha_{N+1})(x_{N+1}, \cdot)\mu}{\mu(I)}. \tag{3.1.4}$$

Let $Q = Q_{N+1}$. Suppose that D is subdivision of $[a_{N+1}; b_{N+1}]$. Let h denote a function defined on R such that for each $x_1, ..., x_{N+1}$ in R,

$$h(x_1, ..., x_{N+1}) = g(x_1, ..., x_N, a_{N+1}) +$$

$$\sum_D [(g(x_1, ..., x_N, q) - g(x_1, ..., x_N, p))/(q-p)] \max\{\min\{x_{N+1} - p, q-p\}, 0\}. \tag{3.1.5}$$

We note that for each w in $[a_{N+1}; b_{N+1}]$, the function defined on $R' = [a_1; b_1] \times \cdots \times [a_N; b_N]$ by $k(x_1, ..., x_N) = g(x_1, ..., x_N, w)$ is continuous in R', so that for each $[p; q]$ in D the function m defined on R by

$$m(x_1, ..., x_{N+1}) = \frac{g(x_1, ..., x_N, q) - g(x_1, ..., x_N, p)}{(q-p)} \max\{\min\{x_{N+1} - p, q-p\}, 0\}$$

is continuous on R. Therefore h is continuous on R. In what follows, the successive changes in the form or our expressions arise from Theorem 2.I.1, Corollary 2.4, and, of course differential equivalence. Now

$$\int_V h(\alpha_1, ..., \alpha_{N+1})\mu = \int_V g(\alpha_1, ..., \alpha_N, a_{N+1})\mu +$$

$$\sum_D \int_V \frac{g(\alpha_1, ..., \alpha_N, q) - g(\alpha_1, ..., \alpha_N, p)}{(q-p)} \max\{\min\{\alpha_{N+1} - p, q-p\}, 0\}\mu =$$

$$\int_V ([(\int_I g(\alpha_1, ..., \alpha_N, a_{N+1})\mu)(\mu(I))]/\mu(I)) +$$

$$\sum_D \int_V [(\int_I [(g(\alpha_1, ..., \alpha_N, q) - g(\alpha_1, ..., \alpha_N, p))/(q-p)]\mu) \cdot$$

$$(\int_I \max\{\min\{\alpha_{N+1} - p, q-p\}, 0\}\mu)/\mu(I)] =$$

$$\int_V ([\int \cdots \int_{R'} [g(x_1, ..., x_N, a_{N+1}) \int_I B^{(N)}(\cdot, \cdot)]] \cdot$$

$$[\int_{a_{N+1}}^{b_{N+1}} 1 d \int_I Q(\beta(\alpha_{N+1})(x_{N+1}, \cdot)\mu)]/\mu(I)) +$$

$$\sum_D \int_V ([\int \cdots \int_{R'} [[(g(x_1, ..., x_N, q) - g(x_1, ..., x_N, p))/(q-p)] \int_I B^{(N)}(\cdot, \cdot)]] \cdot$$

$$[\int_{a_{N+1}}^{b_{N+1}} [\max\{\min\{x_{N+1} - p, q-p\}, 0\}] d \int_I Q(\beta(\alpha_{N+1})(x_{N+1}, \cdot)\mu)]/\mu(I)) =$$

$$\int_V (\int_I [\int \cdots \int_{R'} [g(x_1, ..., x_N, a_{N+1}) B^{(N)}(\cdot, \cdot)]] \cdot$$

$$[\int_{a_{N+1}}^{b_{N+1}} 1 d \int_I Q(\beta(\alpha_{N+1})(x_{N+1}, \cdot)\mu)] / \mu(I))$$

$$+ \sum_D \int_V (\int_I [\int \cdots \int_{R'} [[(g(x_1, ..., x_N, q) - g(x_1, ..., x_N, p))/(q-p)] B^{(N)}(\cdot, \cdot)]] \cdot$$

$$[\int_{a_{N+1}}^{b_{N+1}} [\max\{\min\{x_{N+1} - p, q-p\}, 0\}] d \int_I Q(\beta(\alpha_{N+1})(x_{N+1}, \cdot)\mu)] / \mu(I)) =$$

$$\int_V ([\int \cdots \int_{R'} [g(x_1, ..., x_N, a_{N+1}) B^{(N)}(\cdot, I)]] \frac{\int_{a_{N+1}}^{b_{N+1}} 1 d \int_I Q(\beta(\alpha_{N+1})(x_{N+1}, \cdot)\mu)]}{\mu(I)})$$

$$+ \sum_D \int_V ([\int \cdots \int_{R'} [[(g(x_1, ..., x_N, q) - g(x_1, ..., x_N, p))/(q-p)] B^{(N)}(\cdot, I)]] \cdot$$

$$[\int_{a_{N+1}}^{b_{N+1}} [\max\{\min\{x_{N+1} - p, q-p\}, 0\}] d \int_I Q(\beta(\alpha_{N+1})(x_{N+1}, \cdot)\mu)] / \mu(I)) =$$

$$\int_V ([\int \cdots \int_{R'} [g(x_1, ..., x_N, a_{N+1}) \frac{\int_{a_{N+1}}^{b_{N+1}} 1 d \int_I Q(\beta(\alpha_{N+1})(x_{N+1}, \cdot)\mu)}{\mu(I)} B^{(N)}(\cdot, I)]$$

$$+ \sum_D \int_V ([\int \cdots \int_{R'} [[(g(x_1, ..., x_N, q) - g(x_1, ..., x_N, p))/(q-p)] \cdot$$

$$\frac{\int_{a_{N+1}}^{b_{N+1}} [\max\{\min\{x_{N+1} - p, q-p\}, 0\}] d \int_I Q(\beta(\alpha_{N+1})(x_{N+1}, \cdot)\mu)}{\mu(I)} \cdot B^{(N)}(\cdot, I)]] =$$

$$\int_V ([\int \cdots \int_{R'} [\frac{\int_{a_{N+1}}^{b_{N+1}} g(x_1, ..., x_N, a_{N+1}) d \int_I Q(\beta(\alpha_{N+1})(x_{N+1}, \cdot)\mu)}{\mu(I)} B^{(N)}(\cdot, I)] +$$

$$\sum_D \int_V ([\int \cdots \int_{R'} [([\int_{a_{N+1}}^{b_{N+1}} [(g(x_1, ..., x_N, q) - g(x_1, ..., x_N, p))/(q-p)] \cdot$$

$$[\max\{\min\{x_{N+1} - p, q-p\}, 0\}] d \int_I Q(\beta(\alpha_{N+1})(x_{N+1}, \cdot)\mu)] / \mu(I)) \cdot B^{(N)}(\cdot, I)]] =$$

$$\int_V ([\int \cdots \int_R [([g(x_1, ..., x_N, a_{N+1}) B^{(N+1)}(\cdot, \cdot)] +$$

$$\sum_D \int_V ([\int \cdots \int_R [([[(g(x_1, ..., x_N, q) -$$

$$g(x_1,...,x_N,p))/(q-p)] \cdot [\max\{\min\{x_{N+1}-p,q-p\},0\}]B^{(N+1)}(\cdot,\cdot)]] =$$

$$[\int \cdots \int_R [g(x_1,...,x_N,a_{N+1}) \int_V B^{(N+1)}(\cdot,\cdot)]+$$

$$\sum_D \int \cdots \int_R \frac{g(x_1,...,x_N,q)-g(x_1,...,x_N,p)}{/(q-p)} \cdot [\max\{\min\{x_{N+1}-p,q-p\},0\}]\cdot$$

$$\int_V B^{(N+1)}(\cdot,\cdot)] =$$

$$\int \cdots \int_R [g(x_1,...,x_N,a_{N+1}) \int_V B^{(N+1)}(\cdot,\cdot)]+$$

$$\int \cdots \int_R [\sum_D \frac{g(x_1,...,x_N,q)-g(x_1,...,x_N,p)}{(q-p)} \cdot [\max\{\min\{x_{N+1}-p,q-p\},0\}]]\cdot$$

$$\int_V B^{(N+1)}(\cdot,\cdot)] =$$

$$\int \cdots \int_R [g(x_1,...,x_N,a_{N+1}) + \sum_D [(g(x_1,...,x_N,q)-$$

$$g(x_1,...,x_N,p))/(q-p)]\cdot$$

$$[\max\{\min\{x_{N+1}-p,q-p\},0\}]] \int_V B^{(N+1)}(\cdot,\cdot)] =$$

$$\int \cdots \int_R [h(x_1,...,x_{N+1}) \int_V B^{(N+1)}(\cdot,\cdot)]. \tag{3.1.6}$$

Now suppose that $0 < c$. Let $T = 1 + \mu(V) + \int_V B^{(N+1)}(R,\cdot)$. From the uniform continuity of g on R it follows that there is a subdivision D of $[a_{N+1}, b_{N+1}]$ such that if $(x_1,...,x_{N+1})$ is in R, then, for h given for g and D as above,

$$|g(x_1,...,x_{N+1}) - h(x_1,...,x_{N+1})|| < c/3T, \tag{3.1.7}$$

so that

$$|\int_V g(\alpha_1,...,\alpha_{N+1})\mu - \int \cdots \int_R [g(x_1,...,x_{N+1}) \int_V B^{(N+1)}(\cdot,\cdot)]| \leq$$

$$|\int_V g(\alpha_1,...,\alpha_{N+1})\mu - \int_V h(\alpha_1,...,\alpha_{N+1})\mu| + |\int_V h(\alpha_1,...,\alpha_{N+1})\mu -$$

$$\int \cdots \int_R [h(x_1,...,x_{N+1}) \int_V B^{(N+1)}(\cdot,\cdot)]|+$$

$$|\int \cdots \int_R [h(x_1, ..., x_{N+1}) \int_V B^{(N+1)}(\cdot, \cdot)] -$$

$$\int \cdots \int_R [g(x_1, ..., x_{N+1}) \int_V B^{(N+1)}(\cdot, \cdot)]| \leq$$

$$c\mu(V)/3T + 0 + (c \int_V B^{(N+1)}(R, \cdot))/3T < 2c/3 < c. \tag{3.1.8}$$

Therefore

$$\int_V g(\alpha_1, ..., \alpha_{n+1})\mu = \int \cdots \int_R [g(x_1, ..., x_{N+1}) \int_V B^{(N+1)}(\cdot, \cdot)].$$

4. Distribution functions and summability.

In this section we discuss the notion of summability (see [11]), which is an extension of the integrability in sections 2 and 3 to the unbounded case. Thus, if α is in $r(F)$, μ is in $A(\mathbf{R})(F)^+$, the range union of α is bounded and $\int_U \alpha\mu$ exists, then the integral $\int_U \alpha\mu$ is a set function extension of the Lebesgue integral for the bounded case, and if the range union of α is not necessarily bounded and α is "μ - summable", then, as we see in [2,5,11] and below, the "μ - summability operator" value $\sigma_\mu(\alpha)(U)$ is a set function extension of the Lebesgue integral for the "unbounded" case; if the range union of α is bounded, then

$$\sigma_\mu(\alpha)(U) = \int_U \alpha\mu.$$

We now state some definitions and basic consequences.

Definition 4.1. Suppose that μ is in $A(\mathbf{R})(F)^+$. S_μ denotes the set to which α belongs iff α is an element of $r(F)$ such that for some M_1 and M_2 and all K_1 and K_2 such that $K_1 \leq 0 \leq K_2$, the following inequality and integral existence holds:

$$M_1 \leq \int_U \max\{\min\{\alpha, K_2\}, K_1\}\mu \leq M_2.$$

The following is a condensation of some basic remarks about summability. The statements are arranged progressively.

Theorem 4.1. Suppose that μ is in $A(\mathbf{R})(F)^+$ and $\alpha$is in S_μ. Then the following statements are true:

1) If $K_1 \leq 0 \leq K_2$ and V is in F, then

$$\int_V \max\{\min\{\alpha, K_2\}, K_1\}\mu =$$

$$\int_V \max\{\min\{\alpha, K_2\}, 0\}\mu + \int_V \max\{\min\{\alpha, 0\}, K_1\}\mu$$

(the integral existence requirement alone is sufficient for this).

2) There is M_1 and M_2 such that if V is in F and $K_1 \leq 0 \leq K_2$, then

$$M_1 \leq \int_U \max\{\min\{\alpha, 0\}, K_1\}\mu \leq \int_V \max\{\min\{\alpha, 0\}, K_1\}\mu \leq$$

$$0 \leq \int_V \max\{\min\{\alpha, K_2\}, 0\}\mu \leq \int_U \max\{\min\{\alpha, K_2\}, 0\}\mu \leq M_2.$$

3) There is exactly one pair (η_1, η_2), each term in $A_\mu \cap A(\mathbf{R})(F)^+$, such that if $K_1 \leq 0 \leq K_2$, then

$$\int_U |\eta_2 - \int \max\{\min\{\alpha, K_2\}, 0\}\mu| + \int_U |\eta_1 + \int \max\{\min\{\alpha, 0\}, K_1\}\mu| \to 0,$$

$$\min\{-K_1, K_2\} \to \infty.$$

4) There is exactly one element ξ of A_μ such that if $K_1 \leq 0 \leq K_2$, then

$$\int_U |\xi - \int \max\{\min\{\alpha, K_2\}, K_1\}\mu| \to 0, \min\{-K_1, K_2\} \to \infty.$$

Referring to statement 4) of Theorem 4.1, we have the following definition.

Definition 4.2 (see [2,5,11]). If μ is in $A(\mathbf{R})(F)^+$, then σ_μ denotes the function from S_μ into A_μ such that if α is in S_μ. then $\sigma_\mu(\alpha)$ is the element ξ of A_μ such that if $K_1 \leq 0 \leq K_2$, then

$$\int_U |\xi - \int \max\{\min\{\alpha, K_2\}, K_1\}\mu| \to 0, \min\{-K_1, K_2\} \to \infty.$$

We refer the reader to [11] for further properties of σ_μ, such as linearity.

The results below discuss, for α in $r(F)$ and $K_1 \leq x < K_2$, some relations between $\beta(\alpha)(x, \cdot)$ and $\beta(\max\{\min\{\alpha, K_2\}, K_1\})(x, \cdot)$ and some consequences of $\alpha's$ being in S_μ which concern the behavior of integrals involving $\int Q(\beta(\gamma)(\cdot, \cdot)$, where Q is L or G and γ is α or $\max\{\min\{\alpha, K_2\}, K_1\}$.

The argument for the theorem below is quite routine and we leave it to the reader.

Theorem 4.2. If α is in $r(F)$ and $K_1 \leq x < K_2$, then

$$\beta(\alpha)(x,\cdot) = \beta(\max\{\min\{\alpha, K_2\}, K_1\})(x,\cdot).$$

Furthermore,

$$\beta(\max\{\min\{\alpha, K_2\}, K_1\})(K_2, I) = \{1\}$$

for all I in F.

Theorem 4.3. Suppose that α is in S_μ, Q is L or G and V is in F. Then the following three statements are true: 1) If $0 < y$, then

$$\int_0^y x d\int_V Q(\beta(\alpha)(x,\cdot)\mu) \leq \int_V \max\{\min\{\alpha, y\}, 0\}\mu,$$

so that

$$\int_0^\infty x d\int_V Q(\beta(\alpha)(x,\cdot)\mu)$$

exists.

2) If $z < 0$, then

$$\int_V \max\{\min\{\alpha, 0\}, z\}\mu \leq \int_z^0 x d\int_V Q(\beta(\alpha)(x,\cdot)\mu),$$

so that

$$\int_{-\infty}^0 x d\int_V Q(\beta(\alpha)(x,\cdot)\mu)$$

exists.

3)

$$\int_{-\infty}^\infty |x| d\int_V Q(\beta(\alpha)(x,\cdot)\mu)$$

exists.

Proof: We show statement 1). Suppose that $0 < c'$.

$$\text{Range union of} \max\{\min\{a, y\}, 0\} \subseteq (-c'/(1+\mu(V)); y].$$

Let $c = c'/(1+\mu(V))$. By Theorem 1.A.4,

$$\int_V \max\{\min\{\alpha, y\}, 0\}\mu = \int_{-c}^y x d\int_V Q(\beta(\max\{\min\{\alpha, y\}, 0\})(x,\cdot)\mu) =$$

$$\int_{-c}^{0} xd\int_V Q(\beta(\max\{\min\{\alpha,y\},0\})(x,\cdot)\mu)+$$

$$[\int_0^y xd\int_V Q(\beta(\max\{\min\{\alpha,y\},0\})(x,\cdot)\mu)]_1 \geq -c\mu(V)+[\quad]_1, \qquad (4.3.1)$$

which by Theorem 4.2 is

$$\geq -c\mu(V)+\int_0^y xd\int_V Q(\beta(\alpha)(x,\cdot)\mu) > -c'+\int_0^y xd\int_V Q(\beta(\alpha)(x,\cdot)\mu). \quad (4.3.2)$$

Therefore

$$\int_0^y xd\int_V Q(\beta(\alpha)(x,\cdot)\mu) \leq \int_V \max\{\min\{\alpha,y\},0\}\mu, \qquad (4.3.3)$$

and, clearly, since α is in S_μ,

$$\int_0^\infty xd\int_V Q(\beta(\alpha)(x,\cdot)\mu)$$

exists.

We now show statement 2). Range union of $\max\{\min\{\alpha,0\},z\} \subseteq (z-1;0]$. By Theorem 1.A.4,

$$\int_V \max\{\min\{\alpha,0\},z\}\mu = \int_{z-1}^{0} xd\int_V Q(\beta(\max\{\min\{\alpha,0\},z\})(x,\cdot)\mu) =$$

$$[\int_{z-1}^{z} xd\int_V Q(\beta(\max\{\min\{\alpha,0\},z\})(x,\cdot)\mu)]_2+$$

$$\int_z^0 xd\int_V Q(\beta(\max\{\min\{\alpha,0\},z\})(x,\cdot)\mu), \qquad (4.3.4)$$

which, by Theorem 4.2, is

$$[\quad]_2+\int_z^0 xd\int_V Q(\beta(\alpha)(x,\cdot)\mu) \leq \int_z^0 xd\int_V Q(\beta(\alpha)(x,\cdot)\mu).$$

Therefore

$$\int_V \max\{\min\{\alpha,0\},z\}\mu \leq \int_z^0 xd\int_V Q(\beta(\alpha)(x,\cdot)\mu),$$

and, clearly, since α is in S_μ,

$$\int_{-\infty}^{0} xd\int_V Q(\beta(\alpha)(x,\cdot)\mu)$$

exists.

Statement 3) is an immediate consequence of statements 1) and 2).

Theorem 4.4. Assume the hypothesis of Theorem 4.3. Then the following statements are true:

1) For $0 < K$, $\int_V Q(\beta(\alpha)(K,\cdot)\mu) \to \mu(V)$, $K \to \infty$.

2) For $K < 0$, $\int_V Q(\beta(\alpha)(K,\cdot)\mu) \to 0$, $K \to -\infty$.

Proof: We show statement 1). There is M_2 such that

$$\int_V \max\{\min\{\alpha, K\}, 0\}\mu \leq M_2 \text{ forall} K \geq 0.$$

Suppose that $0 < c$. There is $K^* > 0$ such that $(M_2+1)/K^* < c/2$. Suppose that $K \geq K^*$. From routine considerations concerning common refinements, there is $D << \{V\}$ such that if $E << D$ and a is an α - function on E, then

$$\sum_E \max\{\min\{a(I), K\}, 0\}\mu(I) \leq M_2 + 1$$

and

$$|\int_V Q(\beta(\alpha)(K,\cdot)\mu) - \sum_E Q(\beta(\alpha)(K,\cdot)\mu)(I)| < c/4. \qquad (4.4.1)$$

By Theorem 2.A.1, statement 5), there is $E^* << D$ and a $\beta(\alpha)(K,\cdot)$ - function b on E^* such that

$$\sum_{E^*} |Q(\beta(\alpha)(K,\cdot)\mu)(I) - b(I)\mu(I)| < c/4. \qquad (4.4.2)$$

There is an α - function a on E^* such that for each I in E^*, if $b(I) = 0$, then $a(I) > K$. Now

$$K(\mu(V) - \int_V Q(\beta(\alpha)(K,\cdot)\mu)) =$$

$$K(\mu(V) - \sum_{E^*} b(I)\mu(I) + \sum_{E^*} b(I)\mu(I) - \sum_{E^*} Q(\beta(\alpha)(K,\cdot)\mu)(I) +$$

$$\sum_{E^*} Q(\beta(\alpha)(K,\cdot)\mu)(I) - \int_V Q(\beta(\alpha)(K,\cdot)\mu)) <$$

$$K(\mu(V) - \sum_{E^*} b(I)\mu(I)) + Kc/4 + Kc/4 = \sum_{E^*} K(1 - b(I))\mu(I) + Kc/2 =$$

$$\sum_{E^*} \max\{\min\{a(I), K\}, 0\}(1 - b(I))\mu(I) + Kc/2 \leq$$

$$\sum_{E^*} \max\{\min\{a(I), K\}, 0\}\mu(I) + Kc/2 \leq M_2 + 1 + Kc/2. \tag{4.4.3}$$

Therefore

$$\mu(V) - \int_V Q(\beta(\alpha)(K,\cdot)\mu) < (M_2+1)/K + c/2 < c/2 + c/2 = c. \tag{4.4.4}$$

Therefore

$$\int_V Q(\beta(\alpha)(K,\cdot)\mu) \to \mu(V), K \to \infty.$$

The proof of statement 2) follows in a similar fashion and we leave it to the reader.

5. Finite sequences of summable set functions and joint distribution functions.

In this section we prove the Main Theorem, as stated in the introduction.

Definition 5.1. Suppose that each of γ and ω is in $r(F)$. Let $\delta^*(\gamma,\omega)$ denote the function with domain F such that if I is in F, then $\delta^*(\gamma,\omega)(I) = 0$ if $\gamma(I) \subseteq \omega(I)$ and $\delta^*(\gamma,\omega)(I) = 1$ otherwise.

Theorem 5.1. If g is a function from $\mathbf{R}^N$ into $\mathbf{R}$ and $\{\alpha_k\}_{k=1}^N$ is a sequence of elements of S_μ, then as $\min\{-H_1, ..., -H_N, K_1, ..., K_N\} \to \infty$, the following tends to zero

$$\int_U L(\delta^*(g(\alpha_1,...,\alpha_N), g(\max\{\min\{\alpha_1,K_1\},H_1\},...,\max\{\min\{\alpha_N,K_N\},H_N\}))\mu)$$

Proof: If I is in F, then

$$\delta^*(g(\alpha_1,...,\alpha_N), g(\max\{\min\{\alpha_1,K_1\},H_1\},...,\max\{\min\{\alpha_N,K_N\},H_N\}))(I)$$

is in

$$\{0\} \cup \max\{\max\{\beta(\alpha_i)(H_i,I), 1-\beta(\alpha_i)(K_i,I)\} | i = 1,...,N\}. \tag{5.1.1}$$

Therefore

$$\int_U L(\delta^*(g(\alpha_1,...,\alpha_N), g(\max\{\min\{\alpha_1,K_1\},H_1\},...,\max\{\min\{\alpha_N,K_N\},H_N\}))\mu)$$

$$\leq [\sum_{i=1}^N \int_U L(\beta(\alpha_i)(H_i,\cdot)\mu)]_1 + \sum_{i=1}^N \int_U L((1-\beta(\alpha_i)(K_i,\cdot)\mu) =$$

$$[\quad]_1 + \sum_{i=1}^{N}(\mu(U) - \int_U G(\beta(\alpha_i)(K_i, \cdot)\mu)),$$

which by Theorem 4.4,

$$\to 0, \min\{-H_1, ..., -H_N, K_1, ..., K_N\} \to \infty. \tag{5.1.2}$$

Theorem 5.2. Suppose that $0 < M$, g is a function from $\mathbf{R}^N$ into $[-M; M]$, $\{\alpha_k\}_{k=1}^{N}$ is a sequence of elements of S_μ, and for each $H_1, ..., H_N, K_1, ..., K_N$ as in Theorem 5.1, $\int_U g(\max\{\min\{\alpha_1, K_1\}, H_1\}, ..., \max\{\min\{\alpha_N, K_N\}, H_N\})\mu$ exists. Then:

1) $\int_U g(\alpha_1, ..., \alpha_N)\mu$ exists, and

2) As $\min\{-H_1, ..., -H_N, K_1, ..., K_N\} \to \infty$ the following tends to 0

$$\int_U |g(\alpha_1, ..., \alpha_N) - g(\max\{\min\{\alpha_1, K_1\}, H_1\}, ..., \max\{\min\{\alpha_N, K_N\}, H_N\})|\mu$$

Proof: Let

$$A = \alpha_1, ..., \alpha_N,$$

$$A^* = \max\{\min\{\alpha_1, K_1\}, H_1\}, ..., \max\{\min\{\alpha_N, K_N\}, H_N\}$$

and

$$\delta^* = \delta^*(g(A), g(A^*)). \tag{5.2.1}$$

For each I in F,

$$(g(A)(1 - \delta^*))(I) \subseteq (g(A^*)(1 - \delta^*))(I). \tag{5.2.2}$$

We first show 1).

$$\int_U L(g(A)\mu) - \int_U G(g(A)\mu) \leq$$

$$\int_U L(g(A)(1 - \delta^*)\mu + g(A)\delta^*\mu) - \int_U G(g(A)(1 - \delta^*)\mu + g(A)\delta^*\mu) \leq$$

$$\int_U L(g(A)(1-\delta^*)\mu) - \int_U G(g(A)(1-\delta^*)\mu) + \int_U L(g(A)\delta^*\mu) - \int_U G(g(A)\delta^*\mu) \leq$$

$$\int_U L(g(A^*)(1 - \delta^*)\mu) - \int_U G(g(A^*)(1 - \delta^*)\mu) +$$

$$[\int_U L(g(A^*)\delta^*\mu) - \int_U G(g(A^*)\delta^*\mu)]_1, \tag{5.2.3}$$

which, by Theorem 2.A.2, is

$$\int_U L(-g(A^*)\delta^*\mu) + \int_U g(A^*)\mu - [\int_U G(-g(A^*)\delta^*\mu) + \int_U g(A^*)\mu] + [\quad]_1 =$$

$$\int_U L(-g(A^*)\delta^*\mu) + \int_U L(g(A^*)\delta^*\mu) + [\quad]_1 \leq 2M \int_U L(\delta^*\mu) + 2M \int_U L(\delta^*\mu) \to 0,$$

$$\min\{-H_1, ..., -H_N, K_1, ..., K_N\} \to \infty. \tag{5.2.4}$$

Therefore

$$\int_U L(g(A)\mu) - \int_U G(g(A)\mu) = 0, \tag{5.2.5}$$

so that $\int_U g(A)\mu$ exists. Therefore 1) is true.

We now show 2). Suppose that I is in F.

If $\delta^*(I) = 1$, then for each x in $g(A)(I)$ and y in $g(A^*)(I)$,

$$|x - y|(1 - \delta^*(I))\mu(I) = 0 \leq L(g(A^*)\mu)(I) - G(g(A^*)\mu)(I). \tag{5.2.6}$$

Suppose that $\delta^*(I) = 0$. Then

$$g(A)(I) \subseteq g(A^*)(I), \tag{5.2.7}$$

so that if x is in $g(A)(I)$ and y is in $g(A^*)(I)$, then x is in $g(A^*)(I)$, so that

$$|x - y|(1 - \delta^*(I))\mu(I) = \max\{x, y\}\mu(I) - \min\{x, y\}\mu(I) \leq$$

$$L(g(A^*)\mu)(I) - G(g(A^*)\mu)(I). \tag{5.2.8}$$

Therefore

$$\int_U |g(A) - g(A^*)|\mu \leq \int_U L(|g(A) - g(A^*)|(1 - \delta^*)\mu) + \int_U L(|g(A) - g(A^*)|\delta^*\mu) \leq$$

$$\int_U [L(g(A^*)\mu) - G(g(A^*)\mu)] + 2M \int_U L(\delta^*\mu) =$$

$$0 + 2M \int_U L(\delta^*\mu) \to 0, \min\{-H_1, ..., -H_N, K_1, ..., K_N\} \to \infty. \tag{5.2.9}$$

Therefore 2) is true.

Theorem 5.3. Suppose that $\{\alpha_k\}_{k=1}^N$ is a sequence of elements of S_μ and $B^{(N)}$ is a function defined on $\Xi^N \times F$ as in the Main Theorem, stated in the introduction. Then for all R in Ξ^N,

$$\int \cdots \int_R [\max\{|x_1|, ..., |x_N|\} \int_U B^{(N)}(\cdot, \cdot)] \leq$$

$$\sum_{k=1}^{N}\int_{-\infty}^{\infty}|x_k|d\int_U Q_k(\beta(\alpha_k)(x_k,\cdot)\mu).$$

Proof: Suppose that $R=[a_1;b_1]\times\cdots\times[a_N;b_N]$, $v=1,...,N$, I is in F and for each subinterval $W=[p_1;q_1]\times\cdots\times[p_N;q_N]$ of R,

$$B'(W)=\prod_{k=1}^{N}(\int_I Q_k(\beta(\alpha_k)(q_k,\cdot)\mu)-\int_I Q_k(\beta(\alpha_k)(p_k,\cdot)\mu)). \tag{5.3.1}$$

Then, by Theorem 2.I.2,

$$\int\cdots\int_R|x_v|B'=(\int_{a_v}^{b_v}|x_k|d\int_I Q_v(\beta(\alpha_v)(x_v,\cdot)\mu))\cdot$$

$$\prod_{k\neq v}^{N}(\int_I Q_k(\beta(\alpha_k)(b_k,\cdot)\mu)-\int_I Q_k(\beta(\alpha_k)(a_k,\cdot)\mu)), \tag{5.3.2}$$

so that
$$\int\cdots\int_R|x_v|B^{(N)}(\cdot,I)=\int\cdots\int_R[|x_v|B'/(\mu(I)^{N-1})]=$$

$$(\int_{a_v}^{b_v}|x_v|d\int_I Q_v(\beta(\alpha_v)(x_v,\cdot)\mu))\cdot$$

$$[(\prod_{k\neq v}^{N}(\int_I Q_k(\beta(\alpha_k)(b_k,\cdot)\mu)-\int_I Q_k(\beta(\alpha_k)(a_k,\cdot)\mu)))/(\mu(I)^{N-1})]\leq$$

$$(\int_{a_v}^{b_v}|x_v|d\int_I Q_v(\beta(\alpha_v)(x_v,\cdot)\mu))\cdot 1^{N-1}. \tag{5.3.3}$$

Therefore

$$\int\cdots\int_R[\max\{|x_1|,...,|x_N|\}\int_U B^{(N)}(\cdot,\cdot)]\leq\int\cdots\int_R[\sum_{k=1}^{N}|x_k|\int_U B^{(N)}(\cdot,\cdot)]=$$

$$\sum_{k=1}^{N}\int\cdots\int_R[|x_k|\int_U B^{(N)}(\cdot,\cdot)], \tag{5.3.4}$$

which, by Corollary 2.4 is

$$\sum_{k=1}^{N}\int_U[\int\cdots\int_R|x_k|B^{(N)}(\cdot,\cdot)]\leq$$

$$\sum_{k=1}^{N}\int_U[\int_{a_k}^{b_k}|x_k|d\int_I Q_k(\beta(\alpha_k)(x_k,\cdot)\mu)] =$$

$$\sum_{k=1}^{N}\int_{a_k}^{b_k}[|x_k|d\int_U Q_k(\beta(\alpha_k)(x_k,\cdot)\mu)] \leq$$

$$\sum_{k=1}^{N}\int_{-\infty}^{\infty}[|x_k|d\int_U Q_k(\beta(\alpha_k)(x_k,\cdot)\mu)]. \tag{5.3.5}$$

We next prove a theorem which uses and is an extension of Theorem 4.4. We begin with a lemma and a preliminary theorem.

Lemma 5.4. Suppose that μ is in $A(\mathbf{R})(F)^+$, for $i = 1,2$, $(S_i, \leq_i^*)$ is a partially ordered system and η_i is a function with domain S_i such that if x is in S_i, then $\eta_i(x)$ is in $AB(\mathbf{R})(F)$, $\mu - \int|\eta_i(x)|$ is in $A(\mathbf{R})(F)^+$ and $\int_U|\mu - \eta_i(x)| \to 0$, $\leq_i^*$. Then for all (x_1,x_2) in $S_1 \times S_2$, $\int(\eta_1(x_1)\eta_2(x_2)/\mu)$ is in $AB(\mathbf{R})(F)$, $\mu - \int|\int(\eta_1(x_1)\eta_2(x_2)/\mu)|$ is in $A(\mathbf{R})(F)^+$ and

$$\int_U|\mu - \int(\eta_1(x_1)\eta_2(x_2)/\mu)| \to 0, \ \leq_i^*, \ i = 1,2.$$

Indication of proof: We refer to Corollary 2.3 . For $i = 1,2$, let $\eta^{(i)} = \eta_i(x_i)$. If V is in F, then

$$\int_V|\int(\eta^{(1)}\eta^{(2)}/\mu)| = \int_V|\int[(\eta^{(1)}/\mu)(\eta^{(2)}/\mu)]\mu| = \int_V|[(\eta^{(1)}/\mu)(\eta^{(2)}/\mu)]\mu| =$$

$$\int_V(|\eta^{(1)}/\mu||\eta^{(2)}/\mu|\mu) \leq \mu(V).$$

$$\int_U|(\int(\eta^{(1)}\eta^{(2)}/\mu)) - \mu| = \int_U|(\eta^{(1)}\eta^{(2)}/\mu) - \mu| = \int_U|(\eta^{(1)}\eta^{(2)}/\mu) - \mu\mu/\mu| =$$

$$\int_U|[(\eta^{(1)} - \mu)\eta^{(2)}/\mu] + (\eta^{(2)} - \mu)\mu/\mu| \leq \int_U|\eta^{(1)} - \mu|1 + \int_U|\eta^{(2)} - \mu|1.$$

Theorem 5.4. Suppose that N is a positive integer ≥ 2, μ is in $A(\mathbf{R})(F)^+$, for each $k = 1,...,N$, $(S_k, \leq_k^*)$ is a partially ordered system and $\{\rho_k\}_{k=1}^N$ is a sequence such that if $k = 1,...,N$, then ρ_k is a function with domain S_k such that if x is in S_k, then $\rho_k(x)$ is in $AB(\mathbf{R})(F)$, $\mu - \int|\rho_k(x)|$ is in $A(\mathbf{R})(F)^+$ and $\int_U|\mu - \rho_k(x)| \to 0$, $\leq_k^*$. Then $\mu - \int|(\prod_{k=1}^N \rho_k(x_k))/\mu^{N-1}|$ is in $A(\mathbf{R})(F)^+$ for all $(x_1,...,x_N)$ in $S_1 \times \cdots \times S_N$ and

$$\int_U|\int[(\prod_{k=1}^{N}\rho_k(x_k))/\mu^{N-1}] - \mu| \to 0, \leq_k^*, k = 1,...,N.$$

Indication of proof: We give the essential parts of an induction argument. Suppose that m is a positive integer ≥ 2 for which the statement of the theorem holds. Now suppose that the hypothesis of the theorem is satisfied for $N = m+1$. We shall take some notational liberties, omit some variables and leave the details to the reader. Let

i) $S_1' = S_1 \times \cdots \times S_m$,
ii) $S_2' = S_{m+1}$,
iii) $\leq^* {}'_1 = \{((x_1, ..., x_m), (y_1, ..., y_m)) | (x_k, y_k) \text{ in } \leq *_k,\ k = 1, ..., m\}$,
iv) $\leq^* {}'_2 = \leq^*_{m+1}$,
v) $\eta_1 = \int [(\prod_{k=1}^N \rho_k)/\mu^{N-1}]$
and
vi) $\eta_2 = \rho_{m+1}$.

We see that the hypothesis of Lemma 5.4 is satisfied, so that

$$\int_U | \int [(\prod_{k=1}^{m+1} \rho_k)/\mu^m] - \mu| = \int_U | \int (([(\prod_{k=1}^{m} \rho_k)/\mu^{m-1}][\rho_{m+1}/\mu]) - \mu| =$$

$$\int_U |(\int (\eta_1[\eta_2/\mu])) - \mu| = \int_U | \int (\eta_1\eta_2/\mu) - \mu| \to 0, \leq *_k, k = 1, ..., m+1. \quad (5.4.1)$$

Theorem 5.5. For $B^{(N)}$ defined as in the Main Theorem, for each R in Ξ^N, $\mu - \int B^N(R, \cdot)$ is in $A(\mathbf{R})(F)^+$ and

$$\int_U | \int B^{(N)}(R, \cdot) - \mu| \to 0, R = [H_1; K_1] \times \cdots \times [H_N; K_N],$$

$$\min\{-H_1, ..., -H_N, K_1, ..., K_N\} \to \infty.$$

Indication of proof: Suppose that $i = 1, ..., N$. Let

$$S_i = \{(H, K) | H \leq 0 \leq K\}$$

and

$$\leq_i^* = \{((H', K'), (H'', K'')) | H'' \leq H' \leq 0 \leq K' \leq K''\}.$$

Clearly, $(S_i, \leq_i^*)$ is a partially ordered system. Suppose that $i = 1, ..., N$. For each (H, K) in S_i let

$$\rho_i(H, K) = \int Q_i(\beta(\alpha_i)(K, \cdot)\mu) - \int Q_i(\beta(\alpha_i)(H, \cdot)\mu).$$

Clearly

$$\mu - \rho_i(H, K) = \mu - \int |\rho_i(H, K)|,$$

which is in $A(\mathbf{R})(F)^+$, and, by Theorem 4.4,

$$\int_U |\mu - \rho_i(H,K)| \to 0, \leq_i^* .$$

If $R = [H_1; K_1] \times \cdots \times [H_N; K_N]$, then

$$B^{(N)}(R,\cdot) = (\prod_{i=1}^{N} \rho_i(H_i, K_i))/\mu^{N-1}.$$

Therefore by Theorem 5.4,

$$\mu - \int B^{(N)}(R,\cdot) = \mu - \int | \int B^{(N)}(R,\cdot)|,$$

which is in $A(\mathbf{R})(F)^+$ and

$$\int_U |\mu - \int B^{(N)}(R,\cdot)| \to 0, \min\{-H_1, ..., -H_N, K_1, ..., K_N\} \to \infty.$$

We state another well - known multiple integral theorem that we shall use in proving Theorems 5.6 and the Main Theorem.

Theorem 5.I.1. Suppose that each of R' and R'' is in Ξ^N, $R' \subseteq R''$ and A is a real - nonnegative - valued function, defined and finitely additive on the subintervals of R''. Suppose that each of g and h is a function defined on R'' such that $h - |g|$ is nonnegative - valued on R'' and each of the integrals $\int \cdots \int_{R''} gA$ and $\int \cdots \int_{R''} hA$ exists. Then the following existence and inequality holds:

$$|\int \cdots \int_{R''} gA - \int \cdots \int_{R'} gA| \leq \int \cdots \int_{R''} hA - \int \cdots \int_{R'} hA.$$

Theorem 5.6. Suppose that $\{\alpha_k\}_{k=1}^N$ is a sequence of elements of S_μ and g is a continuous function from $\mathbf{R}^N$ into $\mathbf{R}$ with bounded range. Then the following existence and limiting assertion holds:

$$\int \cdots \int_R [g \int_U B^{(N)}(\cdot,\cdot)] \to \int_U g(\alpha_1, ..., \alpha_N)\mu,$$

$$R = [H_1; K_1] \times \cdots \times [H_N; K_N], \min\{-H_1, ..., -H_N, K_1, ..., K_N\} \to \infty.$$

Proof: There is $M > 0$ such that range of $g \subseteq [-M; M]$. Suppose that $0 < c$. By Theorem 5.5 and 5.2, there is $[H'_1; K'_1] \times \cdots \times [H'_N; K'_N]$, with $H'_i \leq 0 \leq K'_i$, $i = 1, ..., N$, such that if R is in Ξ^N and

$$[H'_1; K'_1] \times \cdots \times [H'_N; K'_N] \subseteq R = [H_1; K_1] \times \cdots \times [H_N; K_N],$$

then

$$|\int_U g(\alpha_1, ..., \alpha_N)\mu - \int_U g(\max\{\min\{\alpha_1, K_1\}, H_1\}, ..., \max\{\min\{\alpha_N, K_N\}, H_N\})\mu|$$

$$< c/2 \tag{5.6.1}$$

and

$$0 \leq \mu(U) - \int_U B^{(N)}(R, \cdot) < c/2M. \tag{5.6.2}$$

Now suppose that $[H'_1; K'_1] \times \cdots \times [H'_N; K'_N] \subseteq R = [H_1; K_1] \times \cdots \times [H_N; K_N]$. For each $W = [p_1; q_1] \times \cdots \times [p_N; q_N]$ in Ξ^N and V in F, let

$$B^{*(N)}(W, V) =$$

$$[\prod_{i=1}^{N}(\int_V Q_i(\beta(\max\{\min\{\alpha_i, K_i + 1\}, H_i\})(q_i, \cdot)\mu) -$$

$$\int_V Q_i(\beta(\max\{\min\{\alpha_i, K_i + 1\}, H_i\})(p_i, \cdot)\mu))]/[\mu(V)^{N-1}]. \tag{5.6.3}$$

Now,

$$|\int_U g(\alpha_1, ..., \alpha_N)\mu - \int \cdots \int_R [g(x_1, ..., x_N) \int_U B^{(N)}(\cdot, \cdot)]| =$$

$$[|\int_U g(\alpha_1, ..., \alpha_N)\mu - \int \cdots \int_R [g(x_1, ..., x_N) \int_U B^{*(N)}(\cdot, \cdot)]|]_1 \tag{5.6.4}$$

by Theorem 4.2. Let $R'' = [H_1 - 1; K_1 + 1] \times \cdots \times [H_N - 1; K_N + 1]$. We see that

$$[\quad]_1 \leq |\int_U g(\alpha_1, ..., \alpha_N)\mu - \int \cdots \int_{R''} [g(x_1, ..., x_N) \int_U B^{*(N)}(\cdot, \cdot)]| +$$

$$[|\int \cdots \int_{R''} [g(x_1, ..., x_N) \int_U B^{*(N)}(\cdot, \cdot)] -$$

$$\int \cdots \int_R [g(x_1, ..., x_N) \int_U B^{*(N)}(\cdot, \cdot)]|]_3 =$$

$$[|\int_U g(\alpha_1,...,\alpha_N)\mu - \int_U g(\max\{\min\{\alpha_1, K_1+1\}, H_1\}, ...,$$
$$\max\{\min\{\alpha_N, K_N+1\}, H_N\})\mu|]_2 + [\quad]_3 \tag{5.6.5}$$

by Theorem 3.1. Continuing, we see that

$$[\quad]_2 + [\quad]_3 < c/2 + [\quad]_3 \le c/2 + M(\int_U B^{*(N)}(R'',\cdot) - \int_U B^{*(N)}(R,\cdot)) =$$
$$c/2 + M(\mu(U) - \int_U B^{*(N)}(R,\cdot)) = c/2 + M(\mu(U) - \int_U B^{(N)}(R,\cdot)) <$$
$$c/2 + Mc/2M = c. \tag{5.6.6}$$

Therefore the limiting assertion holds. We are now almost ready to prove the Main Theorem, as stated in the introducton. First, though, we state a characterization theorem which clearly implies that the continuity and boundedness conditions on the function g of the hypothesis of the Main Theorem are not excessive.

Theorem 5.A.1 [5]. Suppose that g is a function from $\mathbf{R}^N$ into $\mathbf{R}$. Then the following two statements are equivalent:

1) If F is a field of subsets of U, μ is in $A(\mathbf{R})(F)^+ and \{\alpha_k\}_{k=1}^N$ is a sequence of elements of S_μ, then $g(\alpha_1,...,\alpha_N)$ is in S_μ.

2) g is continuous and $\frac{|g(\alpha_1,...,\alpha_N)|}{\max\{|x_1|,...,|x_N|}$: $1 \le \max\{|x_1|,...,|x_N|\}\}$ is bounded.

We now prove the Main Theorem.

Proof of the Main Theorem: There is $M > 0$ such that if $(x_1,...,x_N)$ is in $\mathbf{R}^N$ and $1 \le \max\{|x_1|,...,|x_N|\}$, then $|g(\alpha_1,...,\alpha_N)| \le M\max\{|x_1|,...,|x_N|\}$.

Suppose that $0 < c$. From the immediately preceding condition on g and from Theorem 5.3 it follows that there is $R' = [H_1'; K_1'] \times \cdots \times [H_N'; K_N']$ such that:

i) $H_i' \le 0 \le K_i'$, $i = 1,...,N$,,

ii) if $(x_1,...,x_N)$ is in $\mathbf{R}^N$ and for some $v = 1,...,N$, $|x_v| \ge \min\{-H_v', K_v'\}$, then $|g(\alpha_1,...,\alpha_N)| \le M\max\{|x_1|,...,|x_N|\}$, and

iii) if $R' \subseteq R_1 \subseteq R_2$, each in Ξ^N, then

$$0 \le \int\cdots\int_{R_2}[\max\{|x_1|,...,|x_N|\}\int_U B^{(N)}(\cdot,\cdot)] - \int\cdots\int_{R_1}[\max\{|x_1|,...,|x_N|\}\cdot$$
$$\int_U B^{(N)}(\cdot,\cdot)] < c/4M, \tag{5.7.1}$$

so that

$$0 \le \int\cdots\int_{R_2}[|g(x_1,...,x_N)|\int_U B^{(N)}(\cdot,\cdot)] -$$

$$\int \cdots \int_{R_1} [|g(x_1, ..., x_N)| \int_U B^{(N)}(\cdot, \cdot)] < Mc/4M = c/4. \qquad (5.7.2)$$

Now, since from Theorem 5.A.1 $g(\alpha_1, ..., \alpha_N)$ is in S_μ, there is P and P' such that $P \leq 0 \leq P'$ and if $H \leq P \leq 0 \leq P' \leq K$, then

$$|\sigma_M(g(\alpha_1, ..., \alpha_N))(U) - \int_U \max\{\min\{g(\alpha_1, ..., \alpha_N), K\}, H\}\mu| < c/4.$$

Furthermore, if $R' \subseteq R_1 \subseteq R_2$, each in Ξ^N, and $H \leq 0 \leq K$, then

$$|\int \cdots \int_{R_2} [\max\{\min\{g(x_1, ..., x_N), K\}, H\} \int_U B^{(N)}(\cdot, \cdot)] -$$

$$\int \cdots \int_{R_1} [\max\{\min\{g(x_1, ..., x_N), K\}, H\} \int_U B^{(N)}(\cdot, \cdot)]| \leq$$

$$\int \cdots \int_{R_2} [|\max\{\min\{g(x_1, ..., x_N), K\}, H\}| \int_U B^{(N)}(\cdot, \cdot)] -$$

$$\int \cdots \int_{R_1} [|\max\{\min\{g(x_1, ..., x_N), K\}, H\}| \int_U B^{(N)}(\cdot, \cdot)] \leq$$

$$\int \cdots \int_{R_2} [|g(x_1, ..., x_N)| \int_U B^{(N)}(\cdot, \cdot)] -$$

$$\int \cdots \int_{R_1} [|g(x_1, ..., x_N)| \int_U B^{(N)}(\cdot, \cdot)] < c/4. \qquad (5.7.3)$$

This and Theorem 5.6 clearly imply that if $R' \subseteq R^*$ in Ξ^N, then

$$c/2 > |\int \cdots \int_{R^*} [\max\{\min\{g(x_1, ..., x_N), K\}, H\} \int_U B^{(N)}(\cdot, \cdot)] -$$

$$\int_U \max\{\min\{g(\alpha_1, ..., \alpha_N), K\}, H\}\mu|. \qquad (5.7.4)$$

Now, suppose that $R' \subseteq R$ in Ξ^N. There is H^* and K^* such that $H^* \leq P \leq P' \leq K^*$ and if $(x_1, ..., x_N)$ is in R, then

$$g(x_1, ..., x_N) = \max\{\min\{g(x_1, ..., x_N), K^*\}, H^*\}.$$

Therefore

$$|\sigma_\mu(g(\alpha_1, ..., \alpha_N))(U) - \int \cdots \int_R [g(x_1, ..., x_N) \int_U B^{(N)}(\cdot, \cdot)]| =$$

$$|\sigma_\mu(g(\alpha_1, ..., \alpha_N))(U) - \int \cdots \int_R [\max\{\min\{g(x_1, ..., x_N), K^*\}, H^*\} \int_U B^{(N)}(\cdot, \cdot)]|$$

$$\leq |\sigma_\mu(g(\alpha_1, ..., \alpha_N))(U) - \int_U \max\{\min\{g(\alpha_1, ..., \alpha_N), K^*\}, H^*\}\mu| +$$

$$|\int_U \max\{\min\{g(\alpha_1, ..., \alpha_N), K^*\}, H^*\}\mu -$$

$$\int \cdots \int_R [\max\{\min\{g(x_1, ..., x_N), K^*\}, H^*\} \int_U B^{(N)}(\cdot, \cdot)]| <$$

$$c/4 + c/2 = 3c/4 < c, \tag{5.7.5}$$

and the theorem is established.

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