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WHAT SETS CAN BE ω-LIMIT SETS IN E^n?

In [ABCP] it was shown that a compact subset of E^1 is an ω-limit set if and only if it is either nowhere dense or a union of finitely many non-degenerate closed intervals. In this paper we address the question of characterizing ω-limit sets in E^n. We are unable to find any characterization as the situation even in E^2 is already very complicated. In contrast to E^1, a nowhere dense, non-empty compact set in E^2 can fail to be an ω-limit set. However, we do obtain a number of results showing a rich variety of compact sets, including the totally disconnected sets and low dimensional continua are ω-limit sets. On the other hand there are many open questions on whether some particular, simple sets, like the union of a line segment and a disk in E^2 can be ω-limit sets.

Notation and Terminology.

Suppose $A \subseteq E^k$ and $f : A \to A$. We define $f^0(x) = x$ and $f^{n+1}(x) = f(f^n(x))$ for each $x \in A$ and natural number n. By the orbit of x under f we mean the set $\gamma(x, f) = \{f^n(x) : n \in \omega_0\}$ where ω_0 is the set of natural numbers. By $\omega(x, f)$, called an ω-limit set, we mean the set of subsequential limit points of the sequence $\{f^n(x)\}_{n=0}^{\infty}$. In this paper we will be dealing with compact ω-limit sets in E^k given by continuous functions.

When $x_k = f^k(x_0)$ we use the notation $\{x_k\}_{k=0}^{\infty}$ to denote the sequence as a function whereas $\gamma(x_0, f)$ is the range of that function.

We say that $\omega(x, f)$ is orbit-enclosing if $\gamma(x, f) \subseteq \omega(x, f)$ or equivalently $x \in \omega(x, f)$. We say that a set B is orbit-enclosing if there exists a continuous function f and x such that $\omega(x, f) = B$ and $\omega(x, f)$ is orbit-enclosing. The notion of orbit-enclosing is essentially equivalent to the notion of topological transitivity (see the discussion preceding Theorem 13). Note that there always exists k such that either $\gamma(x_k, f) \subseteq \omega(x_k, f)$ or $\gamma(x_k, f) \cap \omega(x_k, f) = \phi$ where $x_k = f^k(x_0)$. The symbols

A^0, $\bar{A}$ and A' denote the interior, closure and set of limit points of a set A. To avoid stipulating a set is non-empty we will call a non-empty compact set a compactum, and we use a continuum as a connected compactum.

We will frequently use the following version of the Tietze extension theorem: if A and B are compacta in E^n and E^m, and f maps A continuously into B, then f can be extended continuously to all of E^n.

We begin with a basic tool used in constructing functions realizing given sets as ω-limit sets. This is an extended and simplified version of Theorem 1 of [ABCP].

Theorem 1. *Let M be a compactum in E^n and $\{z_i\}_{i=0}^{\infty}$ be a sequence of distinct points whose set of subsequential limit points is M. Define $s(z_i) = z_{i+1}$ on $\Gamma = \{z_i : i \in \omega_0\}$. Then*

1) *if g is a function from M into M and $s(z_{n_k}) \to g(\lambda)$ whenever $z_{n_k} \to \lambda \in M$, then s is uniformly continuous on Γ.*

2) *if s is uniformly continuous on Γ, then there exists a continuous $f : E^n \to E^n$ such that $M = \omega(z_0, f)$.*

Proof: (1) If s is not uniformly continuous, there exist subsequences $\{z_{n_k}\}_{k=0}^{\infty}$ and $\{z_{m_k}\}_{k=0}^{\infty}$ for which $z_{n_k} - z_{m_k} \to 0$ and $s(z_{n_k}) - s(z_{m_k}) \not\to 0$. By compactness there exist $\lambda \in A$ and $\varepsilon > 0$ and subsequences $\{z'_{n_k}\}_{k=0}^{\infty}$ and $\{z'_{m_k}\}_{k=0}^{\infty}$ such that $z'_{n_k} \to \lambda$, $z'_{m_k} \to \lambda$ and $|s(z'_{n_k}) - s(z'_{m_k})| \geq \varepsilon$. This contradicts the hypothesis.

(2) It is a well-known result that $\bar{s}$ (we identify a function with its graph) is a continuous function from $\bar{\Gamma}$ into $\bar{\Gamma}$ when s is uniformly continuous on Γ and Γ is a bounded subset of E^n. We apply the Tietze extension theorem to extend $\bar{s}$ to a continuous f with domain E^n. Obviously $M = \omega(z_0, f)$.

Next we present several results giving sufficient conditions on homeomorphisms to preserve the property of being an ω-limit set. As an immediate application of Theorem 1 we have

Theorem 2. *Let A and B be compacta in E^n and E^m respectively. Suppose $A = \omega(x_0, f)$ and A and B are homeomorphic. If $\omega(x_0, f)$ is orbit-enclosing, then B is an ω-limit set which is orbit-enclosing.*

Proof: Let g be a homeomorphism from A onto B. Let $z_n = g(x_n)$ where $x_n = f^n(x_0)$. Then the function s defined by $s(z_n) = z_{n+1}$ is gfg^{-1} and is uniformly continuous. By Theorem 1 B is an ω-limit set, which is obviously orbit-enclosing.

As one application of Theorem 2, in the sequel we will show that the circle cross [0,1] is an orbit-enclosing ω-limit set. Since this set is homeomorphic to an annulus, it too is an ω-limit set. In general, however, when an ω-limit set is nowhere dense, it is rarely orbit-enclosing. The next two results partially remedy this deficiency.

Theorem 3. *Suppose A and B are homeomorphic compacta of E^n. Then, if A is an ω-limit set, so is B.*

Proof: Let h be a homeomorphism from A onto B. Assume $A = \omega(x_0, f)$. Without loss of generality we may assume that diam $(A \cup \gamma(x_0, f)) < 2^{-1}$. We may also assume that A is infinite, because B is obviously a finite ω-limit set when A is finite. Choose N to be a dense subset of A and enumerate it as $\{a_n\}_{n=1}^{\infty}$. Clearly either $\gamma(f^k(x_0), f) \subseteq A$ for some k or $\gamma(x_0, f) \cap A = \phi$. In the former case we are through by Theorem 2. So we may assume that $\gamma(x_0, f) \cap A = \phi$. Then clearly $A^0 = \phi$ and since $B \subseteq E^n$ we also have $B^0 = \phi$.

First we find a double sequence $\{c_{nk}\}_{n,k=1}^{\infty}$ such that $\gamma(x_0, h) = \{c_{nk} : n \geq 1,\ k \geq 1\}$; $c_{nk} \neq c_{mj}$ whenever $(n,k) \neq (m,j)$; for each n, $c_{nk} \to a_n$ as $k \to \infty$; and for each n, $|c_{nk} - a_n| < 2^{-n}$ for all k. To show this we proceed as follows:

Let $\{\alpha_n\}_{n=1}^{\infty}$ be a 1 - 1 enumeration of $N \times \omega_0$. If $\alpha_1 = (a_\xi, m)$ pick $y_1 \in \gamma(x_0, f)$ such that $|a_\xi - y_1| < 2^{-m-\xi}$. Having picked $y_1, \ldots, y_k$, let $\alpha_{k+1} = (a_\xi, m)$ and choose $y_{k+1} \in \gamma(x_0, f) - \{y_1, \ldots, y_k\}$ such that $|a_\xi - y_{k+1}| < 2^{-m-\xi}$. Let $\{d_{nk}\}_{k=1}^{\infty}$ be an enumeration of $\{y_j : 1^{st}$ coord. $\alpha_j = a_n\}$. Then $d_{nk} \to a_n$ and $|d_{nk} - a_n| < 2^{-n}$ for all k. Moreover $d_{nk} \neq d_{mj}$ whenever $(n,k) \neq (m,j)$.

Let $M = \gamma(x_0, f) - \{d_{nk} : n \geq 1,\ k \geq 1\}$. For each $n \geq 0$ the set $M_n = \{x \in M : 2^{-(n+1)} \leq \operatorname{dist}(x, A) < 2^{-n}\}$ is finite and $M = \cup_{n=0}^{\infty} M_n$. If $M_n \neq \phi$ and $M_n = \{u_1, \ldots, u_m\}$, define $c_{nk} = u_k$, if $1 \leq k \leq m$, and $c_{nk} = d_{n,k-m}$, if $k > m$. If $M_n = \phi$, then put $c_{nk} = d_{nk}$ for each k. Then clearly $\{c_{nk}\}_{n,k=1}^{\infty}$ is the desired sequence.

Since $h(N)$ is dense in B and $B^0 = \phi$, we may use an easy induction argument to define a double sequence $\{e_{nk}\}_{n,k=1}^{\infty}$ such that $e_{nk} \neq e_{mj}$; whenever $(n,k) \neq (m,j)$; for each $n, e_{nk} \to h(a_n)$ as $k \to \infty$; for each n, $|e_{nk} - h(a_n)| < 2^{-n}$ for all k; and when $E = \{e_{nk} : n \geq 1,\ k \geq 1\}$, $E \cap B = \phi$ and $E' = B$.

Now extend h as follows: If $x \in \gamma(x_0, f)$ and $x = c_i$, put $h(x) = e_{ij}$. Then h is 1 - 1 from $\gamma(x_0, f) \cup A$ onto $E \cup B$. To show h is a homeomorphism it suffices to show that $d_{n_k t_k} \to \lambda \in A$ implies $e_{n_k t_k} \to h(\lambda) \in B$.

Suppose $c_{n_k t_k} \to \lambda$. If $\{n_k\}_{k=1}^{\infty}$ is eventually equal to some m, then $d_{n_k t_k} \to a_m$ and $e_{n_k t_k} \to h(a_m)$. So we can assume $n_k \to \infty$. Then we have

$$|a_{n_k} - \lambda| \leq |a_{n_k} - c_{n_k t_k}| + |c_{n_k t_k} - \lambda| \leq 2^{-n_k} + |c_{n_k t_k} - \lambda|.$$

Therefore, $a_{n_k} \to \lambda$ and $h(a_{n_k}) \to h(\lambda)$. Then

$$\begin{aligned} |e_{n_k t_k} - h(\lambda)| &\leq |e_{n_k t_k} - h(a_{n_k})| + |h(a_{n_k}) - h(\lambda)| \\ &\leq 2^{-n_k} + |h(a_{n_k}) - h(\lambda)|. \end{aligned}$$

Therefore, $e_{n_k t_k} \to h(\lambda)$.

Now define $x_k = f^k(x_0)$ and $z_k = h(x_k)$ so that $z_{k+1} = hfh^{-1}(z_k)$. Since hfh^{-1} is uniformly continuous on $\{z_k : k \in \omega_0\}$, B is an ω-limit set by Theorem 1.

Theorem 3 does not work if the compact, homeomorphic sets lie in different dimensional Euclidean spaces. For example, let $B = [0,1] \cup \{-2^{-n} : n \in \omega_0\}$ which is not an ω-limit set in E^1 by [ABCP]. Yet $B \times \{0\}$ is an ω-limit set in E^2 by Theorem 13 below and B and $B \times \{0\}$ are homeomorphic. The construction in the proof of Theorem 3 breaks down because we can't pick the double sequence $\{e_{nk}\}_{n,k=1}^{\infty}$ outside the above set B. This example illustrates the anomaly that a set can be an ω-limit set yet when embedded in a lower dimensional space it is not an ω-limit set.

Theorem 4. Let A and B be homeomorphic compacta in E^n and E^m respectively. Then if A is an ω-limit set and B has empty interior, then B is an ω-limit set.

Proof: The proof is the same as that of Theorem 3 but with a different reason for $B^0 = \phi$, namely the assumption.

The following result is well-known and follows from [K] p. 441.

Theorem 5. Each totally disconnected compactum in E^n is homeomorphic to a compact, nowhere-dense subset of [0,1].

Theorem 6. Each totally disconnected compactum in E^n is an ω-limit set.

Proof: Apply Theorems 4 and 5.

Since each 0-dimensional compactum in E^n is an ω-limit set, a natural starting point for analyzing higher dimensional compacta would be in terms of their components. First of all, suppose a compactum A in E_n has only one component, i.e., it is a continuum. The next result shows that if dim $A < n$, then the continuum A is an ω-limit set.

Theorem 7. If M is a continuum with empty interior in E^n, then M is an ω-limit set $\omega(x_0, f)$ where $f(\lambda) = \lambda$ for all $\lambda \in M$.

Proof: First we show the following.

1) If $\mathbb{G}$ is any finite cover of M by open balls and $G \in \mathbb{G}$, then $\mathbb{G}$ can be enumerated by $G_1, G_2, \ldots, G_m$ where $G_1 = G_m = G$ and $G_i \cap G_{i+1} \neq \varphi$ for all $i < m$.

   To show this we proceed as follows: For $T \in \mathbb{G}$ let T^* consist of all $S \in \mathbb{G}$ such that there exist $S_1, \ldots, S_k \in \mathbb{G}$ such that $S_1 = T$, $S = S_k$ and $S_i \cap S_{i+1} \neq \varphi$ for each $i < k$. Clearly for $T, W \in \mathbb{G} \cup T^* = \cup W^*$ or $\cup T^* \cap \cup W^* = \varphi$. However, in the latter case M would be disconnected by $\cup T^*$ and $\cup S : S \notin T^*\}$. Therefore for each $T \in \mathbb{G}$, $T^* = \mathbb{G}$.

   Next enumerate $\mathbb{G}$ by $\{H_1, \ldots, H_m\}$ and apply the above property to get a chain joining each consecutive pair in the list $H_1, \ldots, H_m$, $H_{m-1}, H_{m-2}, \ldots, H_2, H_1$ and concatenating these chains we obtain the desired result (1).

Let $\mathbb{B}_1$ be a finite covering of M by open balls of radius 2^{-1}, each of which hits M. Then by the above property

$$\mathbb{B}_1 = \{B_1^1, B_2^1, \ldots, B_{n_1}^1\}$$

where $B_1^1 = B_{n_1}^1$ and $B_i^1 \cap B_{i+1}^1 \neq \phi$ for all i.

Next let $\mathbb{B}_2$ be a refinement of $\mathbb{B}_1$ which is a finite covering of M by open balls of radius $< 2^{-2}$, each of which hits M. Then

$$\mathbb{B}_2 = \{B_1^2, \ldots, B_{n_2}^2\}$$

where

$$B_1^2 \subseteq B_1^1 \text{ and } B_1^2 = B_{n_2}^2 \subseteq B_{n_1}^2 \text{ and } B_i^2 \cap B_{i+1}^2 \neq \phi \text{ for all } i.$$

We continue in this manner obtaining the following array:

$$\begin{array}{ccccc} B_1^1 & B_2^1 & \cdot & \cdot & B_{n_1}^1 \\ B_1^2 & B_2^2 & \cdot & \cdot & B_{n_2}^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ B_1^k & B_2^k & \cdot & \cdot & B_{n_k}^k \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

Using the lexicographic order re-label the entries above as $\{C_n\}_{n=0}^{\infty}$.

By induction pick a sequence $\{z_n\}_{n=0}^{\infty}$ such that $z_0 \in C_0 - M$ and $z_{n+1} \in C_{n+1} - M - \{z_0, z_1, \ldots, z_n\}$. It is clear that the cluster set of $\{z_n\}_{n=0}^{\infty}$ is M and that $|z_n - z_{n+1}| \to 0$. Since $z_n \notin M$ for all n, the hypothesis of Theorem 1 is obviously satisfied so that $M = \omega(x_0, f)$ for some x_0 and continuous $f : E^n \to E^n$. Moreover since $|z_n - z_{n+1}| \to 0$ it follows that $f(x) = x$ for all $x \in M$.

A special case of Theorem 7 was previously known, namely: If M is a continuum in E^n, then $M \times \{0\}$ is an ω-limit set in E^{n+1} (see [B]).

We will discover in the sequel that the closed disk and an annulus in E^2 are ω-limit sets. Moreover, it is unknown whether a closed disk with two holes can be an ω-limit set. However, there are 2-dimensional continua in E^2 which are not ω-limit sets as shown by the next example.

Example 1. There exists a 2-dimensional continuum in E^2 which is not an ω-limit set.

Proof: Let D be a closed disk and A be a hereditarily indecomposable continuum such that $A - D \notin \phi$ and $D \cap A = \{p\}$.

By definition A contains no subcontinuum which is the union of two proper subcontinua (see [HY], p. 142-3). Suppose $D \cup A = \omega(x_0, f)$ with $x_k = f^k(x_0)$. Since $D^0 \neq \phi$ it follows that $\{x_k\}_{k=0}^{\infty}$ is eventually in $D \cup A$. Suppose A is not a subset of $f(D)$. Then pick $z \in A - f(D)$ and an open disc W containing z with $W \cap D = \phi$. Then $W \cap \gamma(x_0, f)$ is infinite and we can find $x_m \in D^0$ and an i such that $x_{m+i} \in W \cap A$. Then there exists a closed disc S in D^0 containing x_m for which $f^i(S) \subseteq W$. Since $f^i(S)$ contains infinitely many points of $\gamma(x_0, f)$, $f^i(S)$ is a nondegenerate continuum which is locally connected. Hence, it is the union of two proper subcontinua [HY, p. 139], a contradiction. Hence $A \subseteq f(D)$.

Now define h on D as follows: $h(x) = p$ if $f(x) \notin A$; $h(x) = f(x)$ if $f(x) \in A$. Then f maps D continuously onto A. This is a contradiction by the above argument since D is a locally connected continuum.

Now suppose a compactum in E^n has a finite number of components. It is easily seen that if such a set is an ω-limit set then each component is mapped by the function onto another component in a cyclical manner. Therefore, for example an arc union either a point or an indecomposable continuum can't be an ω-limit set. Hence, in contrast to the situation in E^1 a nowhere dense compactum in E^2 can fail to be an ω-limit set.

The next result shows that the union of finitely many mutually disjoint copies of the same ω-limit set is an ω-limit set.

Theorem 8. If $A_1, \ldots, A_m$ are mutually disjoint and mutually homeomorphic subsets of E^n, and A_1 is an ω-limit set, then so is $\bigcup_{i=1}^m A_i$.

Proof: Let $A_1 = \omega(x_0, g_1)$ and h_i be a homeomorphism from A_1 onto A_i with

h_1 the identity map. By the proof of Theorem 3 there exist y_i and g_i such that $A_i = \omega(y_i, g_i)$ for each $i > 1$ and each h_i can be extended to a homeomorphism H_i mapping $\gamma(x_0, g_1) \cup A_1$ onto $\gamma(y_i, g_i) \cup A_i$ with H_1 the identity map. Moreover, we may assume that the sets dom H_i, $\quad i = 1, \ldots, m$ are mutually disjoint. (It also turns out from the proof of Theorem 3 that each $g_i = H_i g_1 H_i^{-1}$ and $y_i = H_i(x_0)$.)

For any natural number j there exist unique non-negative integers i and k for which $j = im + k$ where $k < m$. Put $z_j = H_k(x_i)$ where $x_i = g_i(x_0)$. Define $s(z_j) = z_{j+1}$.

By Theorem 1 it suffices to show s is uniformly continuous on $\{z_n : n \in \omega_0\} = Z$. But for each i, the function $s \rceil (\gamma(y_i, g_i) \cap Z)$ is $H_{i+1}H_i^{-1}$ if $i < m$ and $g_1 H_m^{-1}$ if $i = m$. Since these are uniformly continuous it follows that s is too.

Finding a necessary and sufficient condition for a union of two continua to be an ω-limit set remains an open question. In fact, it is unknown whether a disk D union a line segment A can be an ω-limit set. If $D \cup A = \omega(x_0, f)$, then it follows that $A = \omega(y_0, f^2)$ where f^2 has the property that all its level sets except possibly two are uncountable. Such continuous functions from A to A exist, yet whether one can realize A as an ω-limit set is another problem.

Now let us consider the case when a compactum has infinitely many components. First we have the following. The proof in E^1 is found in [S].

Theorem 9. If $\omega(x_0, f)$ has infinitely many components, then each component contains at most one orbit point and $\omega(x_0, f)$ has empty interior.

Proof: Let $x_k = f^k(x_0)$ for each k. Suppose C is a component of $\omega(x_0, f)$ which contains some x_i and x_{i+m}. Then the orbit is eventually in $\omega(x_0, f)$. Also $f^m(C) \cap C \neq \phi$ so that $f^m(C) \subseteq C$. For $0 \leq j \leq m$ let B_j be that component containing $f^{i+j}(C)$. Then for all $k \geq m$, $x_k \in \bigcup_{j=0}^m f^{i+j}(C) \subseteq \bigcup_{j=0}^m B_j$. Hence, there are at most k orbit points outside $\bigcup_{j=0}^m B_j$. Let $\{C_n\}_{n=1}^{k+1}$ be $k+1$ distinct components outside $\{B_0, B_1, \ldots, B_m\}$. We may separate these by $k+1$ mutually disjoint open sets each missing $\bigcup_{j=0}^m B_j$. But each of these open sets must contain an orbit point so that there are $k+1$ orbit points outside $\bigcup_{j=0}^m B_j$, a contradiction.

Theorem 9 cannot be improved to conclude that $\omega(x_0, f)$ is not orbit-enclosing. The product of $[0,1]$ with the Cantor set is an orbit-enclosing ω-limit set by Theorem 11.

In view of Theorem 8 and the example following Theorem 3 there is, in contrast, no restriction on the number of lower-dimensional components of an ω-limit set (except when infinite it must have cardinality $\aleph_0$ or $2^{\aleph_0}$).

Now we turn our attention to constructing ω-limit sets via limits, products and unions to produce new ω-limit sets. First we look at limits.

Theorem 10. Let $p \in E^n$ and $\{P_k\}_{k=1}^{\infty}$ be a sequence of mutually disjoint nowhere dense compacta in E^n such that diam $(P_k \cup \{p\}) \to 0$, and such that each P_k is a continuous image of P_{k+1}. Let $M = \{p\} \cup (\bigcup_{k=1}^{\infty} P_k)$. Then there exists $x_0 \in E^n$ and a continuous $h : E^n \to E^n$ such that $\omega(x_0, h) = M$, $h(P_1) = \{p\}$ and $h(P_{k+1}) = P_k$ for all k. Moreover M is not orbit-enclosing.

Proof: Choose a sequence $\{c_k\}_{k=1}^{\infty}$ in $E^n - M$ converging to p. Choose $\{I_k\}_{k=1}^{\infty}$ to be a sequence of mutually disjoint open sets such that $P_k \subseteq I_k$ for each k and $c_m \notin I_k$ for all m and k. Let $\boldsymbol{W}$ be a countable base of open sets for the topological space $M - \{p\}$ with $\cup \boldsymbol{W} \subseteq \bigcup_{k=1}^{\infty} I_k$. Suppose $\boldsymbol{W}$ is well-ordered by ω_0. By the hypothesis we may find a continuous $g : M \to M$ such that $g(P_1) = \{p\}$, $g(p) = p$ and $g(P_{k+1}) = P_k$.

By induction we will define a particular enumeration $\{W_k\}_{k=1}^{\infty}$ of $\boldsymbol{W}$ and a sequence $\{z_k\}_{k=1}^{\infty}$ in E^n. Put $n_k = \sum_{i=1}^{k} i = k(k+1)2^{-1}$ and note that $n_{k+1} - n_k = k + 1$. For each k we will define W_k and z_s for each s such that $n_k \leq s < n_{k+1}$ as follows:

To begin with, let W_1 be the first member of $\boldsymbol{W}$ hitting P_1 and pick $z_1 = c_1$ and $z_2 \in W_1 - M$. Now suppose $k > 1$ and we have picked W_i for all $i < k$ and z_m for all $m < n_k$.

Then choose W_k to be the first member of $\boldsymbol{W} - \{W_1, \ldots, W_{k-1}\}$ to hit $\bigcup_{i=1}^{k-1} P_i$. Suppose $W_k \in I_j$. Then $j < k$ and we may pick $x_1, x_2, \ldots, x_k \in M$ with $x_i \in P_i$ and $x_j \in W_k$ for which $g(x_{i+1}) = x_i$ for all $i < k$.

Put $z_{n_k} = c_k$. For $m \in \{1, \ldots, k\}$ choose $z_{n_k+m} \in I_{k-m+1} - M - \{z_\xi : \xi < n_k\}$ such that $|z_{n_k+m} - x_{k-m+1}| < 2^{-k}$. In addition we may insist that $x_{n_k+j} \in W_k$.

Clearly $\{z_k\}_{k=1}^{\infty}$ is a sequence outside M whose cluster set is M. We need only to verify by Theorem 1 that $z_{m_k} \to \lambda$ implies that $z_{m_k+1} \to g(\lambda)$. Then we have 2 cases.

Case 1. $\lambda \neq p$.

Then $\{z_{m_n}\}_{n=1}^{\infty}$ is eventually (a) in some I_{j+1} or (b) in I_1. In subcase (b) z_{m_n+1} by construction is in $\{c_k : k \in \omega_0\}$. Hence $z_{m_n+1} \to p = g(\lambda)$. In subcase (a) z_{m_n} is within 2^{-t_n} of some point x_{m_n} in P_j and z_{m_n+1} is within 2^{-t_n} of $g(x_{m_n})$ where $t_n \to \infty$. Hence $|z_{m_n} - x_{m_n}| \to 0$ and $|z_{m_n+1} - g(x_{m_n})| \to 0$. Hence $x_{m_n} \to \lambda$ and $g(x_{m_n}) \to g(\lambda)$ and $x_{m_n+1} \to g(\lambda)$.

Case 2. $\lambda = p$.

Then z_{m_n} is some c_{s_n} or belongs to some I_{s_n} where $s_n \to \infty$. If $z_{m_n} \in I_{s_n}$, then for sufficiently large n, $z_{m_n+1} \in I_{s_n-1}$. If $z_{m_n} = c_{s_n}$, then $z_{m_n+1} \in I_{t_n}$ where $t_n \to \infty$. Hence, $z_{m_n+1} \in I_{s_n-1} \cup I_{t_n}$ where $s_n \to \infty$ and $t_n \to \infty$. Hence, $z_{m_n+1} \to p = g(p)$.

The hypothesis of Theorem 1 (1) is satisfied so there exists a continuous $h : E^n \to E^n$ such that $\omega(z_1, h) = M$. By construction it follows that $h(p) = p$ and $h(P_{i+1}) = P_i$ for each i. By Theorem 9 it follows that M is not orbit-enclosing.

Theorem 10 is known in E^1 where the proof is simpler (see [BS]).

Theorem 10 doesn't work if the P_k sets have interior because of Theorem 9. Note that the P_k sets do not have to be ω-limit sets themselves. For example, if each P_k is a copy of $[0,1] \cup \{2\}$, then the resulting ω-limit set consists of a sequence of points and a sequence of arcs converging to the same point, but the individual copies of $[0,1] \cup \{2\}$ are not ω-limit sets as we remarked following example 1.

If $f : A \to A$ and for any non-empty subsets U and V of A, both relatively open in A, there exists m such that $f^m(U) \cap V \neq \phi$, then we say that f is <u>topologically transitive on</u> A (see [D] p. 49). Then the following result is well-known and due to Šarkovskii [S], whose one dimensional proof extends immediately to E^n.

<u>Theorem 11.</u> Let A be a compactum in E^n. Then there exists a continuous $f : A \to A$ such that f is topologically transitive on A if and only if A is orbit-enclosing.

The outline of the non-trivial direction is: Let $\{O_k\}_{k=1}^{\infty}$ be a base of open sets in A. Put $T_k = \{x \in A : \gamma(x, f) \cap O_k = \phi\}$. It is easily verified that each T_k is closed and nowhere dense in A. Therefore by the Baire Category Theorem relative to the compact metric space A, $A - \bigcup_{k=1}^{\infty} T_k \neq \phi$. Picking $x_0 \in A - \bigcup_{k=1}^{\infty} T_k$ we have $A = \omega(x_0, f)$. Note that if A is perfect then the set of $x \in A$ for which $\omega(x, f) = A$ is a residual G_δ in A.

We say that $\omega(x_0, f)$ is <u>topologically mixing</u> if for non-void relative open subsets U and V of $\omega(x_0, f)$ there exists m such that $f^n(\overline{U}) \cap V \neq \phi$ for all $n \geq m$. It is well known that there exists $x_0 \in (0,1)$ such that $\omega(x_0, h)$ is topologically mixing where h is the hat function (e.g. see [BCR] Th. 12). Hence the unit interval I and also the unit circle C are "topologically mixing."

The next result gives a sufficient condition in terms of this notion for a product of two ω-limit sets to be an ω-limit set.

<u>Theorem 12.</u> Suppose A and B are compact and $A = \omega(x_0, f) \subseteq E^i$ and $B = \omega(y_0, g) \subseteq E^j$. If A or B is topologically mixing, then there exists $z_0 \in E^{i+j}$ such that $A \times B = \omega(z_0, F)$ where $F(x, y) = (f(x), g(y))$ and $z_0 = (x_0, y_0)$.

Moreover, if both A and B are orbit-enclosing, then $\omega(z_0, F)$ is orbit-enclosing too.

Proof: The function F is continuous and maps $A \times B$ into $A \times B$. It is easily checked that F is topologically transitive so $A \times B$ is an ω-limit set by Theorem 11. Moreover it is clear that $A \times B = \omega(z_0, F)$ is orbit-enclosing whenever A and B are orbit-enclosing.

As a consequence of Theorem 12 the following sets are ω-limit sets: the solid square, the annulus, the surface of a torus and the solid torus. These sets are homeomorphic to $I \times I$, $I \times C$, $C \times C$ and $I \times C \times I$ respectively.

It is an open question whether the product of any two ω-limit sets is again an ω-limit set. However, if $A \times B \subseteq E^2$ the answer is yes because in case A and B are nowhere dense, $A \times B$ is an ω-limit set by Theorem 6 and in case one set is not nowhere dense it is a union of finitely many closed intervals and Theorems 8 and 12 make $A \times B$ an ω-limit set.

The next results give conditions under which a union of two ω-limit sets is an ω-limit set.

Theorem 13. Suppose A and B are nowhere dense compacta in E^k with $A = \omega(x_0, f)$ and $B = \omega(y_0, g)$ and $f(x) = g(x)$ whenever $x \in A \cap B$. If there exists $\lambda \in A \cap B$ for which $f(\lambda) = g(\lambda) = \lambda$, then $A \cup B$ is an ω-limit set $\omega(x_0, h)$ where h coincides with f on A and with g on B.

Proof: If A or B is orbit-enclosing we will adjust it so that they are not as follows: Let $x_n = f^n(x_0)$ and $y_n = g^n(y_0)$. Since $A \cup B$ is nowhere dense we may choose sequences $\{x'_n\}_{n=0}^{\infty}$ and $\{y'_n\}_{n=0}^{\infty}$ both missing $A \cup B$ such that $x'_n \neq x'_m$ and $y'_n \neq y'_m$ whenever $n \neq m$ and $x'_n \neq y'_m$ for all n and m and such that $|x_n - x'_n| < 2^{-n}$ and $|y_n - y'_n| < 2^{-n}$ for each n. Put $s(x'_n) = x'_{n+1}$ and let $x'_{n_k} \to \mu$. Since $x'_{n_k} \notin A$ for all k we must have $n_k \to \infty$ and $x_{n_k} \to \mu$ and $x_{n_k+1} \to f(\mu)$. It follows that $x'_{n_k+1} = s(x'_{n_k}) \to f(\mu)$ too. According to Theorem 1 there exists an f' such that $A = \omega(x'_0, f')$. It is clear that $f = f'$ on A. Likewise we can find g' such that $\omega(y'_0, g') = B$. Therefore dropping the primes we may assume that $\gamma(x_0, f) \cap (A \cup B) = \phi$, $\gamma(y_0, g) \cap (A \cup B) = \phi$ and $\gamma(x_0, f) \cap \gamma(y_0, g) = \phi$.

Let $\{z_n\}_{n=0}^{\infty}$ be given by concatenating the following inductively defined strings of points. ($u \to v$ means $v = z_{n+1}$ if $u = z_n$.)

The first stage is

$$x_0 \to x_1 \to \cdots \to x_a \to y_0 \to y_1 \to \cdots \to y_b \to$$

where x_a is the first x_j such that $|\lambda - x_j| < 2^{-1}$ and y_b is the first y_j such that $|\lambda - y_j| < 2^{-1}$. Now suppose that n^{th} stage has been described where x_α and y_β are the last orbit points to appear in the n^{th} stage. Then the $n + 1^{st}$ stage is

$$\to x'_\alpha \to x_{\alpha+1} \to \cdots \to x_c \to y'_\beta \to y_{\beta+1} \to \cdots \to y_d \to$$

where c is the first $j > \alpha$ such that $|x_j - \lambda| < 2^{-n-1}$ and d is the first $j > \beta$ such that $|y_j - \lambda| < 2^{-n-1}$ and where x'_α and y'_β are chosen to miss $A \cup B \cup \lambda(x, f) \cup \lambda(y, g)$, and to miss all previously chosen x'_ξ and y'_ξ points and to satisfy $|x'_\alpha - x_\alpha| < 2^{-n}$ and $|y'_\beta - y_\beta| < 2^{-n}$.

Clearly the set of subsequential limit points of $\{z_n\}_{n=0}^\infty$ is $A \cup B$. We need only to verify the hypothesis of Theorem 1. It suffices to show that if $z_{n_k} \to \mu$, then $z_{n_k+1} \to h(\mu)$.

The pair (z_{n_k}, z_{n_k+1}) can take one of the following forms:

$$(x'_{\alpha_k}, x_{\alpha_k+1}),\ (x_{m_k}, x_{m_k+1}),\ (x_{c_k}, y'_{\beta_k}),\ (y_{m_k}, y_{m_k+1}),\ (y'_{\beta_k}, y_{\beta_k+1}) \text{ or } (y_{d_k}, x'_{\alpha_k}).$$

Without loss of generality we may assume that each of these forms occurs infinitely often giving rise to six subsequences of $\{n_k\}_{k=0}^\infty$. Without loss of generality we may assume that each subsequence is the whole sequence. Then

<u>Case 1.</u> $(z_{n_k}, z_{n_k+1}) = (x_{\alpha_k}{}', x_{\alpha_k+1})$ for all k.

Since $|x'_{\alpha_k} - x_{\alpha_k}| \to 0$ and $x'_{\alpha_k} \to \lambda$ we have $x_{\alpha_k} \to \lambda$. So $\mu = \lambda$ and $x_{\alpha_k+1} = f(x_{\alpha_k}) \to f(\lambda) = \lambda = h(\mu)$.

<u>Case 2.</u> $(z_{n_k}, z_{n_k+1}) = (y_{\beta_k}{}', y_{\beta_k+1})$ for all k.

This is similar to case 1 and yields $y_{\beta_k+1} \to h(\mu)$.

<u>Case 3.</u> $(z_{n_k}, z_{n_k+1}) = (x_{m_k}, x_{m_k+1})$ for all k.

Then $x_{m_k} \to \mu$ and $x_{m_k+1} = f(x_{m_k}) \to f(\mu) = h(\mu)$.

<u>Case 4.</u> $(z_{n_k}, z_{n_k+1}) = (y_{m_k}, y_{m_k+1})$ for all k.

This is similar to case 3 and yields $y_{m_k+1} \to h(\mu)$.

<u>Case 5.</u> $(z_{n_k}, z_{n_k+1}) = (x_{c_k}, y_{\beta_k}{}')$ for all k.

Then $x_{c_k} \to \lambda = \mu$, $y_{\beta_k} \to \lambda$ and $|y_{\beta_k} - y_{\beta_k}{}'| \to 0$. Hence $y_{\beta_k}{}' \to \lambda = h(\mu)$.

<u>Case 6.</u> $(z_{n_k}, z_{n_k+1}) = (y_{c_k}, x_{\alpha_k}{}')$ for all k.

This is similar to case 5 and yields $x'_{\alpha_k} \to h(\mu)$.

Hence, from the proof of Theorem 1, part 2, there exists $t : E^k \to E^k$ such that $A \cup B = \omega(z_0, t)$ and $t = \overline{s}$ on $A \cup B \cup \{z_n : n \in \omega_0\}$. From the above $x_{n_k} \to x$ implies $s(z_{n_k}) \to h(x)$. Therefore $t = h$ on $A \cup B$.

Theorem 13 need not be valid when one of the sets has non-void interior. For example, Theorem 9 shows that the union of a disk D with a sequence of points S outside it but converging to a boundary point of D is not an ω-limit set in E^2. However, D and S do satisfy the remaining hypotheses of Theorem 13. The set S is an ω-limit set with the limit point a fixed point (by Theorem 10). And D is an ω-limit set with some boundary point fixed. (See discussion following Theorem 14.)

One application of Theorem 13, as mentioned previously, is that the set $B \times \{0\}$ where $B = [0,1] \times \{2^{-n} : n \in \omega\}$ is an ω-limit set in E^2. By Theorem 7, $[0,1]$ is an ω-limit set with all points fixed and the sequence together with 0 is an ω-limit set with 0 fixed. Hence Theorem 13 applies. The next result is a variant of Theorem 13 applicable to a set in E^2 having non-void interior.

Theorem 14. Suppose $A = \omega(z_0, f)$ is a compactum in E^2 lying in the upper-half plane. If $f(z) = z$ for all $z \in A$ with $\mathrm{Im}\ z = 0$, then (1) $A^* = \{z : \overline{z} \in A\}$ is an ω-limit set and (2) $A \cup A^*$ is an ω-limit set.

Proof: If $h(z) = \overline{z}$, then h is a homeomorphism from A onto A^* so by Theorem 3, A^* is an ω-limit set. If $A^* \cap A = \phi$, then Theorem 8 gives $A \cup A^*$ is an ω-limit set. In case $A^* \cap A \neq \phi$ we have $f(z) = z$ for all $z \in A^* \cap A$. If A is nowhere dense, then Theorem 13 can be applied to give $A \cup A^*$ an ω-limit set. If A has non-empty interior we can assume $\gamma(z_0, f) \subseteq A$. Then $F(x) = \overline{x}$ if $x \in A$ and $F(x) = \overline{f(x)}$ if $x \in A^*$. Then it is easily checked that F is topologically transitive on $A \cup A^*$ so that $A \cup A^*$ is an ω-limit set by Theorem 11.

Since the hat function h has 0 and 2/3 as fixed points, it follows by Theorem 12 that $I^2 = [0,1]^2 = \omega(x_0, F)$ where $F(x,y) = h(x), h(y))$ and (0,0), (0,2/3), and (2/3,0) are fixed points of F lying on the boundary of I^2. Hence, using this and Theorem 14 we can get the union of two tangent disks as an ω-limit set and a "necklace" of an even number of disks each tangent to two others.

If $I^2 = \omega(x_0, f)$ and each point on the bottom side were a fixed point, then we would be able to obtain a disk with two holes as an ω-limit set as follows: map I^2 homeomorphically onto an "E" whose 3 ends lie along the x-axis. The 3 segments lying on the x-axis consist of fixed points so applying Theorem 14 we obtain a rectangle with two holes as an ω-limit set. However, the existence of such a function f remains an open question.

The results of this paper generate a rich variety of ω-limit sets in the plane. However, some simple compacta are unknown to be ω-limit sets. For example, 1) a disk union a line segment; 2) a disk with finitely many segments protruding from it; and 3) a disk with two or more holes in it.

In a subsequent paper we will address these problems.

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