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About net convergences of measurable functions

The importance of Condition (B) (for sequences) was stressed by N.N. Luzin in his dissertation, cf. 6Luz53Y. The idea to consider nets instead sequences naturally arises in measure and integration theory dealing with $L(\mathbf{X},\mathbf{Y})$-valued measures, where both $\mathbf{X},\mathbf{Y}$ are locally convex topological vector spaces, cf. 6Hal90Y, 6Hal92Y. Condition (B) for nets (and the measure) in the classical setting was introduced and investigated by B. F. Goguadze, cf. 6Gog79Y.

I.

Let $\mathcal{R}$ denote the real line and let $\mathcal{N}$ denote the set of all natural numbers. Let Δ be an algebra of subsets of $\mathcal{R}$. Recall that a set function $m : \Delta \to [0,\infty]$ is said to be a charge on Δ if $m(\emptyset) = 0$ and $m(E \cup F) = m(E \setminus F) + m(F)$ for every $E, F \in \Delta$. A charge space is a triple $(\mathcal{R}, \Delta, m)$.

Definition (B). *A charge m is said to satisfy Condition* (B) *if for every $E \in \Delta$, $m(E) < \infty$, and every net of sets $E_i \in \Delta, i \in I$, $E_i \subset E$, $\limsup_{i\in I} E_i \neq \emptyset$ whenever there exists a real number $\delta > 0$ such that $m(E_i) \geq \delta$ for every $i \in I$.*

A charge m is said to be a purely atomic measure if (i) it is a measure, (ii) every $E \in \Delta, m(E) < \infty$, can be written as a countable union of atoms ($A \in \Delta$ is an atom if $m(A) > 0$ and $B \subset A$ implies either $B \notin \Delta$ or $B = \emptyset$). Every purely atomic measure satisfies Condition (B), cf. 6Gog79Y.

Let $(\mathcal{R}, \Delta, m)$ be a charge space. A real valued function f on $\mathcal{R}$ is said to be measurable if for every $\varepsilon > 0$, there exists a partition $\{F_0, F_1, F_2, \dots, F_n\}$ of $\mathcal{R}$ in Δ such that $m(F_0) < \varepsilon$ and $|f(t) - f(t')| < \varepsilon$ for every $t, t' \in F_i$ for every $i = 1, 2, \dots, n$, cf.6Bha83Y, Definition 4.4.6. We then write $f \in \mathcal{M}$. We say that a net $f_i \in \mathcal{M}, i \in I$, converges to a function $f \in \mathcal{M}$ on $E \in \Delta$ in charge if for every $\varepsilon > 0, \delta > 0$, there exists $i_0 \in I$ such that for every $i \geq i_0, i \in I$, we have $m(\{t \in E; |f_i(t) - f(t)| \geq \delta\} < \varepsilon$, where $m(A) = \inf_{A\subset B, B\in\Delta} m(B), A \subset \mathcal{R}$.

Theorem 1. *The followig statements are equivalent:*

(i) *The charge m satisfies Condition* (B).

(ii) *If a net of measurable functions converges everywhere to a measurable function f on $E \in \Delta, m(E) < \infty$, then it converges on E to f in charge.*

Theorem 2. *There exists a non-zero non-atomic charge which satisfies Condition* (B).

II.

Let χ_E denote the indicator function of a set $E \subset \mathcal{R}$. Let $\mathcal{F} \subset \mathcal{M}$ be a vector lattice containing $\{\chi_E; E \in \Delta, m(E) < \infty\}$ and $J\colon \mathcal{F} \times \Delta \to \mathcal{R}$ a map such that

(a) $f, g \in \mathcal{F},\ \alpha, \beta \in \mathcal{R},\ E \in \Delta \Rightarrow J(\alpha f + \beta g, E) = \alpha J(f, E) + \beta J(g, E)$,

(b) $E, F \in \Delta,\ f \in \mathcal{F} \Rightarrow J(f, E \cup F) = J(f, E \setminus F) + J(f, F)$,

(c) $m(E_i) \searrow 0,\ E_i \in \Delta, i \in I, f \in \mathcal{F} \Rightarrow J(f, E_i) \searrow 0$,

(d) $0 \le g \le f, E \in \Delta, g, f \in \mathcal{F} \Rightarrow 0 \le J(g, E) \le J(f, E)$.

Theorem 3. *Let a charge $m\colon \Delta \to [0, \infty]$ satisfy Condition* (B). *Let $E \in \Delta, m(E) < \infty$. Let a net $f_i \in \mathcal{F}, i \in I$, converge everywhere to a function $f \in \mathcal{M}$. If there exists a function $g \in \mathcal{F}$, such that $|f_i| \le g$ for every $i \in I$, then $\lim_{i \in I} J(f_i, E) = J_1(f, E)$ exists.*

Remark. Let $\mathcal{F}_1 (\subset \mathcal{M})$ denote the closure of $\mathcal{F}$ given by Theorem 2. It can be proved that $J_1(f, E)$ in Theorem 2 does not depend on the choice of a net $f_i, i \in I$, and Conditions (a) – (d) remain valid if $\mathcal{F}$ is replaced by $\mathcal{F}_1$ and J is replaced by $J_1\colon \mathcal{F}_1 \times \Delta \to \mathcal{R}$.

III.

Let $(\mathcal{R}, \Lambda, \lambda)$ denote the Lebesgue measure space. The measure λ satisfies Condition (B) for sequences, but does not satisfy Condition (B) for arbitrary directed sets I. So, there is a question how to restrict the class of all directed sets $\{I\}$ in Condition (B) to a smaller, more suitable class $\{I\} \subset \{I\}$, that the Lebesgue measure satisfies Condition (B) with respect to the class $\{I\}$.

Problem. Find a directed set I essentially different from $\mathcal{N}$, such that for every $E \in \Lambda, \lambda(E) < \infty$, and every net of sets $E_i \in \Lambda, i \in I,\ E_i \subset E$, we have $\limsup_{i \in I} E_i \ne \emptyset$ whenever there exists a real number $\delta > 0$ such that $\lambda(E_i) \ge \delta$ for every $i \in I$.

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