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The Improvable Discontinuous Functions

We shall consider bounded real functions f of a real variable. The set $U(f)$ will denote the set of all points of discontinuity of f at which there exists a limit of f which is not equal to the value of f at that point. Of course, the set $U(f)$ is countable for any function f. If we replace the values of f at the points of $U(f)$ with the values of the appropriate limits, then we obtain the new function, which we shall denote by $f_{(1)}$. If $f_{(1)}$ is continuous, then f is said to be *an improvable discontinuous function.*

Professor Świątkowski asked if there can exist a a function f such that f is continuous at no point $\Re$ and yet $f_{(1)}$ is continuous everywhere. The negative answer is given by the following theorem:

Theorem 1 *For any improvable discontinuous function f the set of points of continuity $(C(f))$ is dense. Moreover $C\left(f_{(1)}\right) - C(f)$ is of the first category for any improvable discontinuous function.*

It will be much easier to understand the situation if I give some examples.

Example 1 *Let $D = \left\{\frac{1}{n}; n \in \mathcal{N}\right\}$ and let f be the characteristic function of D. Note that $D = U(f)$ and 0 does not belong to $U(f)$. But after improving we become the continuous function $f_{(1)} = 0$ for every $x \in \Re$. Observe that we can also improve the continuity of f at some points which do not belong to $U(f)$.*

Example 2 *Let D be as above and g be the characteristic function of $D \cup \{0\}$. Note that $D = U(f)$ and 0 does not belong to the set $U(f)$. But $g_{(1)} = \chi_{\{0\}}$. Hence the function g is not improvable discontinuous at 0.*

Example 3 *Let $C \subset [0,1]$. The components of the complement of C are called contiguous intervals. Let C_1 be a set of all central points of contiguous intervals. Let h be the characteristic function of the set C_1. Observe that $C_1 = U(h)$ and each point of the set C belongs to $U(f)$, but $h_{(1)} = 0$ for every $x \in \Re$.*

Now we shall establish necessary and sufficient conditions under which A is a set of all points of continuity of some improvable discontinuous function f. Theorem 1 implies that any such set is dense.

Theorem 2 *If A be is a proper dense $\mathcal{G}_\delta$ subset of $\Re$, then the following conditions are equivalent:*

1. *there is a function $f : \Re \longrightarrow \Re$ such that $C(f) = A$ and $C\left(f_{(1)}\right) = \Re$;*
2. *there exists a $\mathcal{G}_\delta$ set $E \subset A$ such that the set $K = E - A$ is countable and dense in $cl\,(\Re - E)$.*

As corollary we can state:

Corollary 1 *If $f : \Re \longrightarrow \Re$ is a function satisfying (1.), then $U(f) = K$.*

All the problems we discussed before were concerned with functions defined on the whole real line. The results are also valid if we consider functions defined on any dense $\mathcal{G}_\delta$ subset of $\Re$.

Next, we examine properties of improvable discontinuous functions.

Theorem 3 *Any improvable discontinuous function is of the first class of Baire.*

But there are some Baire one functions which are not improvable discontinuous (see Example 2).

Theorem 4 *There is no Darboux function which is improvable discontinuous.*

As our final result we have:

Theorem 5 *The class of all improvable discontinuous functions and continuous functions is (not) closed under uniform (pointwise) convergence.*