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Measurable Fields of Metric Spaces

The concept of measurability can be generalized to the situation where the value space varies with the argument of a function.

Let $(T, \mathcal{T})$ be a measureable space, $(A_t, d_t)_{t\in T}$ a family of metric spaces, and $X \subset \prod\{A_t : t \in T\}$. Elements of X will play the role of measurable functions. Then $((T, \mathcal{T}), (A_t, d_t)_{t\in T}, X)$ is called a *measurable field of metric spaces* if it satifies the following axioms:

1. X is closed under pointwise limits of sequences.
2. If $x, y \in X$ and $S \in \mathcal{T}$, then the function z defined by $z(t) = x(t)$ for $t \in S$ and $z(t) = y(t)$ for $t \in T \setminus S$ also belongs to X.
3. For each $x, y \in X$ the real-valued function $t \mapsto d_t(x(t), y(t))$ is measurable.
4. For each $t \in T$ the set $\{x(t) : x \in X\}$ is dense in A_t.

The notion of a measurable field of metric spaces was introduced by Delode, Arino, and Peno [1]. Independently, Evstigneev and Kuznetsov [2] defined a similar concept of "skew products" of measurable spaces.

A set-valued function ϕ defined on T and such that $\phi(t) \subset A_t$, $t \in T$, is said to be *measurable* if for each $x \in X$ the real-valued mapping $t \mapsto dist_t(x(t), \phi(t))$ is measurable (*cf.* [2]).

The family $f_t : A_t \to \mathbb{R}$, $t \in T$, is called a *measurable field of functionals* if for each $x \in X$ the function $t \mapsto f_t(x(t))$ is measurable (*cf.* [3]). We say that $(f_t)_{t\in T}$ is a *normal field of functionals* if each f_t is upper semicontinuous, and the set-valued map

$$E(t) = \{(a, r) \in A_t \times \mathbb{R} : r \leq f_t(a)\}, t \in T$$

is measurable with respect to the product measurable field of metric spaces $((T, \mathcal{T}), (B_t, \rho_t)_{t\in T}, Y)$, where $B_t = A_t \times \mathbb{R}$, ρ_t is the product metric, and Y

is the set of all pairs (x, h), where $x \in X$ and h is a measurable real–valued function on T (*cf.* [2]).

We shall consistently assume that the measurable field of metric spaces $((T, \mathcal{T}), (A_t, d_t)_{t\in T}, X)$ is *separable*, *i.e.*, there is a countable subset $X_0 \subset X$ such taht $\{x(t) : x \in X_0\}$ is dense in A_t, $t \in T$.

The following theorems are generalizations of some results for normal integrands. They are motivated by applications to optimization and mathematical economy.

Theorem 1 *If all (A_t, d_t) are complete and $(f_t)_{t\in T}$ is a normal field of functionals, then there exists a sequence of measurable fields of functionals $(f_t^n)_{t\in T}$, $n \in N$, such that each f_t^n is continuous, and $f_t^n \downarrow f_t$. If all A_t are compact, then the converse statement holds.*

Theorem 2 *If $(f_t)_{t\in T}$ is a normal field of functionals such that for each $t \in T$ and each $r \in \mathbb{R}$ the set $\{a \in A_t : r \leq f_t(a)\}$ is relatively compact in A_t, then the function*

$$v(t) = \sup\{f_t(a) : a \in A_t\},\ \ t \in T$$

is measurable, and there exists $x^* \in X$ such that $v(t) = f_t(x^*(t))$ for all $t \in T$.

References

[1] D. Delode, O. Arino, and J. P. Penot, Champs mesurables et multisections, *Ann. Inst. H. Poincaré*, Sect. B 12 (1976), 11–42.

[2] I. V. Evstigneev and S. E. Kuznetsov, Skew products of measurable spaces, *Selected Problems of Probability and Mathematical Economics*, CEMI Akad Nauk SSSR, Moscow 1977, pp. 28–37 (in Russian).

[3] T. Jdanok, Opérateurs et fonctionelles aléatoires dans les champs mesurables, *Travaux Sém. Anal. Convexe Montpellier* (1983), 1–36