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Some Remarks on Sup–Measurability

It is well–known that the superposition of two (Lebesgue) measurable functions need not be measurable. In particular, consider superpositions of the form $F(x, f(x))$ which play an important role in differential equations. We say that $F : \mathbb{R}^2 \to \mathbb{R}$ is sup-measurable if $x \mapsto F(x, f(x))$ is measurable for any measurable (or, equivalently, Borel) function $f : \mathbb{R} \to \mathbb{R}$. One can easily find a measurable F which is not sup–measurable; it suffices to take a nonmeasurable set $H \subset \mathbb{R}$ and, for a fixed measurable $f : \mathbb{R} \to \mathbb{R}$, define F as the characteristic function of the graph $Gr(f|H)$. Grande and Lipiński [1] and, independently, Harazišvili (Georgia, USSR) observed that, assuming the Continuum Hypothesis (CH), one can construct a nonmeasurable set $E \subset \mathbb{R}^2$ which meets the graph of any Borel function $f : \mathbb{R} \to \mathbb{R}$ on a countable set. Thus the characteristic function of E is nonmeasurable and sup–measurable. The analogous result holds for category. We extend that construction to more general cases and replace CH by a weaker assumption.

We consider σ–ideals of subsets of $\mathbb{R}$. For any σ–ideal $\mathcal{I}$ we always assume that $\mathbb{R} \notin \mathcal{I}$ and either $\mathcal{I} = \{\emptyset\}$ or $\mathcal{I}$ contains all singletons. Let $S(\mathcal{I})$ be the σ–field generated by sets from $\mathcal{I}$ and all Borel sets. For two σ–ideals $\mathcal{I}$ and $\mathcal{J}$, we denote

$$\mathcal{I} \otimes \mathcal{J} = \left\{E \subset \mathbb{R}^2 : \exists \text{ a Borel } B \supset E \text{ such that } \{x \in \mathbb{R} : B_x \notin \mathcal{J}\} \in \mathcal{C}\right\},$$

where $B_x = \{y \in \mathbb{R} :< x, y >\in B\}$. Then $\mathcal{I} \otimes \mathcal{J}$ forms a σ–ideal called the Fubini product of $\mathcal{I}$ and $\mathcal{J}$.

We shall let c denote the cardinality of the continuum and shall let

$$\operatorname{non}(\mathcal{I}) = \min\{|E| : E \subset \mathbb{R} \text{ and } E \notin \mathcal{I}\}.$$

We say that an $F : \mathbb{R}^2 \to \mathbb{R}$ is $S(\mathcal{I})$ –sup–measurable if $x \mapsto F(x, f(x))$ is $S(\mathcal{I})$–measurable (i.e. the preimage of any open set belongs to $S(\mathcal{I})$) for any Borel $f : \mathbb{R} \to \mathbb{R}$.

Our generalization of the theorem of Grande and Lipiński is the following

Proposition 1 *There exists an $F : \mathbb{R}^2 \to \{0,1\}$ such that*

- *F is $S(\mathcal{I})$-sup-measurable for each $\mathcal{I}$ with $non(\mathcal{I}) = c$.*
- *F is not $S(\mathcal{I} \otimes \mathcal{J})$-measurable for any $\mathcal{I} \neq \{\emptyset\}$ and $\mathcal{J} \neq \{\emptyset\}$.*

Letting $\mathbf{L}$ denote the σ–ideal of all Lebesgue null sets, this yields

Corollary 1 *If $non(\mathbf{L}) = c$, there exists an $F : \mathbb{R}^2 \to \{0,1\}$ which is sup–measurable and not (Lebesgue) measurable.*

Recently, that result has been improved by Marcin Penconek from Warsaw University, who showed the above extence assuming that $non(\mathbf{L}) = cf(c)$. (We always have $cf(c) \leq c$.)

Problem 1 *Can existence of a sup–measurable and nonmeasureable function be proved within ZFC?*

An easier version of that problem is

Problem 2 *Can existense of a quasi–sup–measurable and nonmeasurable function be proved within ZFC?*

A function $F : \mathbb{R}^2 \to \mathbb{R}$ is called quasi–sup–measurable if

$$\{y \in \mathbb{R} : x \mapsto F(x, f(x) + y) \text{ is nonmeasurable}\}$$

is countable for any Borel $f : \mathbb{R} \to \mathbb{R}$. It is obvious that sup–measurability implies quasi–sup–measurability. Observe that the converse does not hold: it suffices to consider the characteristic function of $H \times \{0\}$ where $H \subset \mathbb{R}$ is nonmeasurable.

All of these remarks and problems can be formulated quite analogously for the category case.

References

[1] Z. Grande and J. Lipiński, Un example d'une fonction sup–mesurable qui n'est pas mesurable, *Colloq. Math.* 39(1978), 77-79.