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Strong Fubini Theorems from Measure Extension Axioms

Fubini's theorem asserts that if $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are σ-finite measure spaces and $f : X \times Y \to \mathbb{R}$ is a measurable function, then the iterated integrals $\int [\int f(x,y) d\mu(x)]\, d\nu(y)$ and $\int [\int f(x,y) d\nu(y)]\, d\mu(x)$ exist and are equal.

Strong Fubini theorems are statements about the existence and equality of iterated integrals of functions which are not necessarily measurable. The simplest one asserts that the iterated integrals of a nonnegative function are equal, if only they can be defined.

It is easy to give in ZFC an example showing that the above quoted strong Fubini theorem (SFT) is in general false.

On the other hand, in the case where $X = Y = \mathbb{R}$ and $\mu = \nu$ is Lebesgue measure, SFT is false under CH (Sierpinski), but it is consistent with ZFC (Friedman [2]). Laczkovich proved that it follows from the assumption that if χ is the least cardinality of a nonmeasurable subset of $\mathbb{R}$, then the union of χ-many null sets does not cover $\mathbb{R}$.

Connections between various strengthenings of SFT for $\mathbb{R} \times \mathbb{R}$ and other cardinal conditions were investigated by Shipman [3].

Our aim is to present an alternative approach to strong Fubini theorems via measure extension axioms.

PMEA(λ, χ, ρ) where λ, χ, and ρ are infinite cardinals and $\rho \leq 2^\omega$, is the version of the Product Measure Extension Axiom asserting that for any family $\mathcal{S}$ of χ-many subsets of 2^λ, the usual product measure m_λ on 2^λ can be extended to a ρ-additive measure $\overline{m}_\lambda$ with $\mathcal{S} \subset dom(\overline{m}_\lambda)$.

Kunen proved Con(ZFC + $\forall\lambda$ PMEA$(\lambda, 2^\lambda, 2^\omega)$) starting from Con(ZFC + $\exists$ strongly compact cardinal).

Carlson [1] proved Con(ZFC + $\forall\chi < 2^\omega$ PMEA$(2^\omega, \chi, \chi^+)$) without large cardinal assumptions.

Theorem 1 *Assume* PMEA(λ, χ, χ^+) *and suppose that* $D \subset 2^\lambda \times 2^\chi$. *If for* m_λ*-a.a.* $x \in 2^\lambda$ *and for* m_χ*-a.a.* $y \in 2^\chi$ *the sets* $D_x = \{y \in 2^\chi : (x,y) \in D\}$

and $D^y = \{x \in 2^\chi : (x,y) \in D\}$ *are measurable, and* $\mathrm{m}_\lambda(\{x \in 2^\lambda : \nu(D_x) > 0\}) = 0$*, then* $\mathrm{m}_\chi(\{y \in 2^\chi : \mu(D^y) > 0\}) = 0$.

Corollary 1 PMEA(λ, χ, χ^+) *implies* SFT *for* $X \times Y$*, whenever* $(X, \mathcal{A}, \mu)$ *and* $(Y, \mathcal{B}, \nu)$ *are* σ*-finite Radon measure spaces of Maharam types* $\leq \lambda$ *and* $\leq \chi$*, respectively.*

References

[1] T. Carlson, Extending measure by infinitely many sets, *Pac. J. Math.* 115 (1984), 33–45.

[2] H. Friedman, A consistent Fubini–Tonelli theorem for nonmeasurable functions, *Illinois J. Math.* 24 (1980), 390–395.

[3] J. Shipman, Cardinal conditions for strong Fubini theorems, *Trans. Amer. Math. Soc.* 321 (1990), 465–481.