Milan Kuchta, Mathematical Institute, Slovak Academy of Sciences, Stefánikova 49, 814 73 Bratislava, Czechoslovakia

SHADOWING PROPERTY OF MAPS WITH ZERO TOPOLOGICAL ENTROPY

Let $C^0(I,I)$ denote the class of continuous maps $I \to I$, where I is a compact real interval. The orbit of $x \in I$ with respect to f is the sequence $\text{orb}(x) = \{f^n(x)\}_{n=0}^{\infty}$ where f^n denotes the nth iterate of f. Interval $J \subset I$ is called a periodic interval with period $\text{per}(J) = k \in \mathbb{N}$ if $f^k(J) = J$ and $f^i(J) \cap f^j(J) = \emptyset$ for $0 \le i \ne j < k$. If J is degenerate to a point then it may be called a periodic point. Denote the set of all periodic points of f by $\text{Per}(f)$ and the topological entropy of f by $E(f)$. We will denote a closed interval with $x \le y$ by $[x,y]$ and a closed interval where no information about order of x, y is provided by $[x,y]^*$.

Definition 1. *If $f \in C^0(I,I)$ and $\delta > 0$ is given, a sequence $\mathbf{X}_\delta = \{\mathbf{x}_i\}_{i=0}^{\infty}$ of points in I is called a δ-chain of f (or δ-pseudo orbit of f) provided that*

$$|f(\mathbf{x}_i) - \mathbf{x}_{i+1}| \le \delta \qquad \textit{for every} \quad i \ge 0$$

Given $\varepsilon > 0$, a δ-chain $\mathbf{X}_\delta$ is said to be ε-shadowed by $y \in I$, if

$$|f^i(y) - \mathbf{x}_i| \le \varepsilon \qquad \textit{for every} \quad i \ge 0$$

f is said to have the shadowing property if for any $\varepsilon > 0$ there is $\delta > 0$ such that every δ-chain of f can be ε-shadowed by a point in I.

Definition 2. *Let $f \in C^0(I,I)$. We will call f a shrink function if and only if for every sequence $\{J_k\}_{k=0}^{\infty}$ of periodic intervals such that $J_{k+1} \subset J_k$ and $\text{per}(J_{k+1}) > \text{per}(J_k)$ we have that $\lim_{k\to\infty} |J_k| = 0$.*

Definition 3. *We will call an one-side neighborhood $[p,q]^*$ of the periodic point p an m-f-non-trapping neighborhood of p if $f^m(p) = p$ and for every $x \in [p,q]^*$; $x \in f^m([p,x]^*)$.*

Definition 4. *We will call $f \in C^0(I,I)$ a non-degenerate function if the following condition holds*

> *If $x \in I$, $p \in \text{Per}(f)$, $[p,q]^*$ is an m-f-non-trapping neighborhood of p and $\lim_{n\to\infty} f^{mn}(x) = p$, then for every neighborhood O_x of x and for all $z_1, z_2 \in (p,q)^*$ there is an $n_0 \in \mathbb{N}$ such that $[z_1,z_2]^* \subset f^{mn_0}(O_x)$.*

Main Theorem. *Let $f \in C^0(I,I)$ and $E(f) = 0$. Then f has the shadowing property if and only if f is a non-degenerate shrink function.*

Remark 5. Our condition is necessary for any continuous function ($E(f) \geq 0$) to have the shadowing property and it is quite easy to prove. Moreover, if we use the results from [2], we can easily obtain similar results for continuous maps of the circle.

The proof of the sufficiency is based on the following lemma.

Lemma 6. *Let $f \in C^0(I,I)$, $E(f) = 0$ and f be a non-degenerate shrink function. Then for all $\varepsilon > 0$ there is $\varepsilon^* > 0$ and a non-decreasing function $h \in C^0(I,I)$ such that for all $x, y \in I$ we have*

$$|h(x) - h(y)| \leq |x - y|\,,$$
$$\text{if} \qquad |h(x) - h(y)| < \varepsilon^* \qquad \text{then} \qquad |x - y| < \varepsilon\,,$$
$$h \circ f = g \circ h$$

where g is a non-degenerate continuous function of the type 2^n (it means that if $p \in \operatorname{Per}(g)$ then $g^{2^n}(p) = p$).

Now using the following result we are easily done.

Theorem 7. (T. Gedeon, M. Kuchta [1]) *Let $f \in C^0(I,I)$ be of the type 2^n. Then f has the shadowing property if and only if f is a non-degenerate function.*

References

[1] T. Gedeon, M. Kuchta, *Shadowing property of continuous maps*, Proc. Amer. Math. Soc. (to appear).

[2] M. Kuchta, *Characterization of chaos for continuous maps of the circle*, Comment. Math. Univ. Carolin. **31** (1990), 383–390.