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# Measurability of Multifunctions of Two Variables

**1. Definition** (see [1]). A measurable space with negligibles is a triple $(X, \mathfrak{M}(X), \mathfrak{J}(X))$ where $X$ is a set, $\mathfrak{M}(X)$ is a $\sigma$-algebra of subsets of $X$ and $\mathfrak{J}(X) \subset \mathfrak{P}(X)$ is a $\sigma$-ideal of the Boolean algebra $\mathfrak{P}(X)$ generated by $\mathfrak{J}(X) \cap \mathfrak{M}(X)$.

Denote by $\hat{\mathfrak{M}}$ the completion of $\mathfrak{M}(X)$ with respect to $\mathfrak{J}(X)$.

**2. Definition.** Let $(X, \mathfrak{M}(X))$ be a measurable space, $(Y, \mathcal{T}(Y))$ a topological space and $F : X \to Y$ a multifunction. We say that $F$ is lower (resp. upper) $\mathfrak{M}(X)$-measurable if $F^{-}(G) = \{x \in X : F(x) \cap G \neq \emptyset\} \in \mathfrak{M}(X)$ (resp. $F^{+}(G) = \{x \in X : F(x) \subset G\} \in \mathfrak{M}(X)$) for each $G \in \mathcal{T}(Y)$.

Let $(X, \mathfrak{M}(X), \mathfrak{J}(X))$ and $(Y, \mathfrak{M}(Y), \mathfrak{J}(Y))$ be two measurable spaces with negligibles. Let $\mathfrak{M}(X) \otimes \mathfrak{M}(Y)$ be the $\sigma$-algebra generated by $\mathfrak{M}(X) \times \mathfrak{M}(Y)$ and let $\mathfrak{J}(X) \otimes \mathfrak{J}(Y)$ denote the $\sigma$-ideal generated by all sets of the form $X \times J$ and $I \times Y$ where $I \in \mathfrak{J}(X)$ and $J \in \mathfrak{J}(Y)$. In many important cases the product $X \times Y$ is, in a natural way, endowed with the $\sigma$-ideal $\mathfrak{J}(X) \otimes \mathfrak{J}(Y) \supset \mathfrak{J}(X) \otimes \mathfrak{J}(Y)$ with the following "Fubini property": for any set $A \in \hat{\mathfrak{M}}(X \otimes Y)$ the following implications hold

$$\{x \in X : A_x \in \mathfrak{J}(Y)\} \in \mathfrak{J}(X) \ \Rightarrow \ A \in \mathfrak{J}(X \otimes Y) \quad \text{and} \tag{1}$$

$$\{y \in Y : A^y \in \mathfrak{J}(X)\} \in \mathfrak{J}(Y) \ \Rightarrow \ A \in \mathfrak{J}(X \otimes Y), \tag{2}$$

where $A_x = \{y \in Y : (x, y) \in A\}$ and $A^y = \{x \in X : (x, y) \in A\}$ denote the sections of $A$ and $\hat{\mathfrak{M}}(X \otimes Y)$ denotes $\mathfrak{J}(X \otimes Y)$-completion of $\mathfrak{M}(X) \otimes \mathfrak{M}(Y)$.

**3. Definition.** Let$(X, \mathcal{T}(X))$ and $(Y, \mathcal{T}(Y))$ be two topological spaces and let $F : X \to Y$ be a multifunction. $F$ is called lower (resp. upper) semicontinuous at a point $x_0 \in X$ if the implication $F(x_0) \cap U \neq \emptyset \Rightarrow \bigvee_{G \in \mathcal{T}(X)} x_0 \in G \wedge \bigwedge_{x \in G} F(x) \cap U \neq \emptyset$ (resp. $F(x_0) \subset U \Rightarrow \bigvee_{G \in \mathcal{T}(X)} x_0 \in G \wedge \bigwedge_{x \in G} F(x) \subset F(x) \subset U$) hold for each $U \in \mathcal{T}(Y)$. $F$ is continuous at $x_0$ if it is lower and upper semicontinuous at $x_0$.

Let $\mathcal{U}(x_0)$ be a filterbase of open neighbourhoods of $x_0 \in X$ and let $\nabla(x_0) = \{A(x_0) \subset X : \bigwedge_{U \in \mathcal{U}(x_0)} A(x_0) \cap U \neq \emptyset\}$. Following Jacobs [2] define

$p-\limsup_{x\to x_0} F(x) = \bigcap_{U\in\mathcal{U}(x_0)} \mathrm{Cl}(\bigcup_{x\in U} F(x)) \subset Y$ and $p-\liminf_{x\to x_0} F(x) = \bigcap_{A\in\mathcal{U}(x_0)} \mathrm{Cl}(\bigcup_{x\in A} F(x)) \subset Y$. If $\mathcal{B}$ is a basis for $X$ and we replace $\nabla\mathcal{U}(x_0)$ in the latter definition by $\nabla\mathcal{U}(x_0)\cap\mathcal{B} = \{U\in\mathcal{B} : x_0 \in \mathrm{Cl}(U)\}$ we denote the resulting operation by $q$-lim inf.

**4. Theorem.** *Let $(X, \mathfrak{M}(X), \mathfrak{J}(X))$ be a complete mesurable space with negligibles, $(Y, \mathcal{T}(Y))$ a second countable topological space and $\mathfrak{J}(Y)\subset\mathcal{B}(Y)$ a Borel $\sigma$-ideal in $Y$ such that there exists a $\sigma$-ideal $\mathfrak{J}(X\otimes Y)\supset\mathfrak{J}(X)\otimes\mathfrak{J}(Y)$ such that (1) and (3) hold, where*

$$A\in\mathfrak{M}(X)\otimes\mathcal{B}(Y) \longrightarrow \pi_x(A) = \{x\in X : \bigvee_{y\in Y}(x,y))\in A\}\in\hat{\mathfrak{M}}(X). \tag{3}$$

*Let $(Z, \mathcal{T}(Z))$ be a second countable perfectly normal topological space. Assume that $F : X\times Y\to Z$ is a closed-valued multifunction with the following three properties:*

*(i) All the section $F^y$, $y\in Y$, are lower $\hat{\mathfrak{M}}(X)$-measurable.*

*(ii) For all $x\in X$ the set $D(F_x)$ of discontinuity points of the section $F_x$ is $\mathfrak{J}(Y)$-negligible.*

*(iii) For all $(x,y)\in X\times Y$ the inequality*

$$G_*(x,y)\subset F(x,y)\subset G^*(x,y) \tag{4}$$

*holds, where for some fixed countable dense subset $S$ of $Y$ (the existence of which we assume)*

$$G_*(x,y) = q-\liminf_{t\to Y}(F_x|_S)(t)\subset Z, \quad \text{and} \tag{5}$$

$$G^*(x,y) = p-\limsup_{t\to Y}(F_x|_S)(t)\subset Z. \tag{6}$$

*Then $F$ is lower $\hat{\mathfrak{M}}(X\otimes Y)$-measurable.*

We provide examples and applications in the setting of category bases.

## REFERENCES


[1] D. H. Fremlin, Measure-additive coverings and measurable selectors, *Dissertationes Mathematicae* CCLX (1987), 1–120.

[2] M. Q. Jacobs, Measurable multivalued mappings and Lusin theorem, *Trans. Amer. Math. Soc.* 146(1968), 471–481.