Jacek Hejduk, Institute of Mathematics, Łodź University, ul. S. Banacha 22, 90-238 Łodź, Poland

Non–Baire Sets in Category Bases

Around 1975 John C. Morgan II introduced a theory of category bases. Its main feature is to present, in a common framework, measure and category and some other properties of the point set classification. I would like to give some conditions to place on category bases such that each set which is not meager will contain a non–Baire set.

A *category base* on a set X is a pair $(X, \mathcal{S})$ such that X is a non–empty set and $\mathcal{S}$ is a family of non–empty subsets of X, called regions, satisfying the following axioms:

1. $\bigcup \mathcal{S} = X$.
2. Let A be a region and $\mathcal{D}$ a non–empty family of disjoint regions of cardinality less than the cardinality of $\mathcal{S}$. Then
   (a) if $A \cap (\bigcup \mathcal{D})$ contains a region, then there is a region $B \in \mathcal{D}$ such that $A \cap B$ contains a region,
   (b) if $A \cap (\bigcup \mathcal{D})$ contains no region, then there is a region $B \subset A$ which is disjoint from $\bigcup \mathcal{D}$.

Standard examples of category bases include topologies without the empty set or sets of positive measure with respect to a σ–finite measure.

We say a set $C \subset X$ is *singular* if, for every region A, there exists a region $B \subset A$ such that $B \cap C = \emptyset$. A set $M \subset X$ is *meager* if M is a countable union of singular sets. The class of meager sets in a base $(X, \mathcal{S})$ will be denoted by $\mathcal{M}(\mathcal{S})$. A set $G \subset X$ is *Baire* if, for every region A, there exists a region $B \subset A$ such that $B \cap G$ is meager or $B \cap (X \setminus G)$ is meager. By *a base of any family of sets* $\mathcal{P}$ we shall understand a subfamily $\mathcal{P}'$ such that each member of $\mathcal{P}$ is contained in some member of $\mathcal{P}'$.

Theorem 1 *Let $(X, \mathcal{S})$ be a category base such that the following conditions are satisfied:*

1. $\mathcal{M}_0 \subset \mathcal{M}(\mathcal{S})$ *where* $\mathcal{M}_0 = \{A \subset X : \ card\, A < card\, X\}$.
2. *there exists a base of a σ–ideal of $\mathcal{M}(\mathcal{S})$ of cardinality not greater than card X.*

Then a set C is meager if and only if each subset of C is a Baire set.

In the case of the category base generated by the family of sets of positive Lebesgue measure over the real line, we can conclude by Theorem 1 the existence of a nonmeasurable set. Similarly, in the case of the category base generated by the natural topology we can conlude the existence of a set without the Baire property.

As a simple corollary of Theorem 1 we can establish the following

Theorem 2 *Let $(X, \mathcal{S})$ be a point meager base (i.e. each singleton is meager) such that there exists a base of the family of meager sets of cardinality not greater than card X. Then each set A of cardinality $\aleph_1$ is meager if and only if each subset of A is Baire.*

This theorem can be compared with a theorem of Morgan [1].

Theorem 3 *Let $(X, \mathcal{S})$ be category point meager base fulfilling c.c.c. (i.e. each family of pairwise disjoint regions has cardinality not greater than $\aleph_0$). Then each set A of cardinality $\aleph_1$ is meager if and only if each subset of A is Baire.*

There are examples of category bases $(X, \mathcal{S})$ with c.c.c., but for which $\mathcal{M}(\mathcal{S})$ does not possess any base of cardinality not greater than *card X*. Conversely, under the assumption that $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$ there exists a category base $(X, \mathcal{S})$ possessing a base of $\mathcal{M}(\mathcal{S})$ having cardinality not greater than *card X*, but the c.c.c. is not satisfied.

References

[1] John C. Morgan II, Point set theory, *Pure and Applied Mathematics*, Marcel Dekker, New York–Basel, 1990.