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Almost Continuous Functions and Almost Baire Maps

We say that a function $f : X \times Y \to Z$ from the product $X \times Y$ into a space Z is a *Baire–open map* if each set of the form $f(A \times B)$ has non–empty interior provided A and B are non–empty sets of the second category with the Baire property in the spaces X and Y, respectively.

S. Piccard showed in 1939 that addition on the reals is a Baire–open map, which was a duel version of Steinhaus' measure–theoretic result. There have been many other related results and generalizations since then. The aim of this talk is to show an n–dimensional version of Piccard's theorem along with Baire–openess of operations in some topological semi–groups. Most results here are due to William Lindgren and myself.

A subset A of the product $X \times Y$ is said to be *projectively second category* if for each open set $W \subset X \times Y$ intersecting A, both the sets $P_X(A \cap W)$ and $P_Y(A \cap W)$ are of the second category.

A subset A of $X \times Y$ is said to be *projectively solid* if for arbitrary subsets of the first category, E in X and F in Y, both $A \setminus (E \times Y)$ and $A \setminus (X \times F)$ are projectively second category. Clearly, each projectively solid set is projectively second category but not conversely.

Theorem 1 *Let A be a subset of $X \times Y$. Then A is projectively solid if and only if 1. holds for A:*

1. *if U and V are any open subsets of X and Y, respectively, such that $(U \times V) \cap A \neq \emptyset$, then for any sets of the first category $E \subset X$ and $F \subset Y$, $((U \setminus E) \times (V \setminus F)) \cap A \neq \emptyset$.*

Theorem 2 *Let $f : X \times Y \to Z$ be an almost continuous function. Then f is Baire–open if and only if it is quasiopen and satisfies the following condition:*

2. *for any dense subset S of a non–empty open subset of Z, $f^{-1}(S)$ is projectively solid.*

Theorem 3 *Let $f : \mathbb{R}^m \to \mathbb{R}^n$, $m \geq 2n$, be a C^1 function that satisfies the following condition:*

3. *for each point $p = (p_1, p_2, \ldots, p_m) \in \mathbb{R}^m$ for each n-tuple $1 \leq j_1 < j_2, \ldots, < j_n \leq m$ of indices and for each family $\{U_1, U_2, \ldots, U_n\}$ of open subsets of $\mathbb{R}$ such that $p_{j_k} \in U_k$, $k = 1, 2, \ldots, n$, the rank of restriction of f to $G = \{(x_1, x_2, \ldots, x_m) : x_i = p_i$ if $i \neq j_k$ for each $k = 1, 2, \ldots, n$ or $x_i \in U_k$ if $i = j_k\}$ is equal to n.*

If each of the sets B_i, $i = 1, 2, \ldots, m$ is a nonempty subset of $\mathbb{R}$ with the Baire property and is of second category, then $f(B_1 \times B_2 \times \ldots \times B_m)$ has nonempty interior.

Let $(G; +)$ be an algebraic group endowed with a topology under which the operation $+$ is separately continuous and the inverse operation in G is continuous, *i.e.*, the following three functions are continuous: $ax(y) = x + y$, $a^y(x) = x + y$, $i(x) = -x$. Such a structure $(G; +)$ is called a *topological semigroup.*

Theorem 4 *Let $(G; +)$ be a topological semigroup which is a regular first countable Baire space. Then $+$ is Baire–open.*

Theorem 5 (K. Kuratowski and S. Banach) *Let $(G; +)$ be a regular first countable topological semigroup. If H is a non–empty subset of G which is, topologically, of second category and has the Baire property, and, algebraically, is a subgroup of G, then H is open and closed.*