T. Natkaniec, Instytut Matematyki WSP, ul. Chodkiewicza 30, 85-064 Bydgoszcz, Poland.

On Some Problems Concerning Almost Continuity

We consider three classes of real functions defined on $I = [0,1]$. $\mathcal{A}$ denotes the class of almost continuous functions (in the sense of Stallings [7]), $\mathcal{D}$ is the class of all Darboux functions, and $\mathcal{D}^*$ is the class of all functions such that $\overline{f^{-1}(y)} = I$ for each $y \in \Re$. The following inclusions are proper: $\mathcal{A} \subset \mathcal{D}$ and $\mathcal{D}^* \subset \mathcal{D}$.

Obviously $\mathcal{D}$ is closed under composition but $\mathcal{A}$ is not [4]. Since $\mathcal{A} \subset \mathcal{D}$, the class of all compositions of almost continuous functions is included in $\mathcal{D}$. This suggests the following:

Problem 1 *Is every Darboux function the composition of (two) almost continuous functions ? ([4],[6])*

With CH any $f \in \mathcal{D}^*$ is a composition of 2 almost continuous functions [6].

For $\varepsilon > 0$ we define $S_\varepsilon(f) = \{(x,y) : x \in I \;\&\; |y - f(x)| < \varepsilon\}$. By a blocking set we mean a closed subset K of $I \times \Re$ which hits every continuous function on I and such that $K \cap f = \emptyset$ for some function f on I. Note that a function f is almost continuous iff it hits every blocking set [2]. For $f : I \longrightarrow \Re$ we define two conditions:

(α) for sufficiently small $\varepsilon > 0$ and for every blocking set K either $card(dom(K \cap S_\varepsilon(f))) = 2^\omega$ or $(f(x) - \varepsilon, f(x) + \varepsilon) \subset K_x$ for some $x \in I$,

(β) for each $\varepsilon > 0$ and for every blocking set K either $card(dom(K \cap S_\varepsilon(f))) = 2^\omega$ or $int(K \cap S_\varepsilon(f))_x \neq \emptyset$ for some $x \in I$.

Theorem 1 (CH) *For a function f we have: $(\alpha) \longrightarrow f \in \overline{\mathcal{A}} \longrightarrow (\beta)$.*

Note that the inclusion $\mathcal{D}^* \subset \overline{\mathcal{A}}$ follows from the first part of Theorem 1.

Problem 2 *Characterize the class $\overline{\mathcal{A}}$ of all uniform limits of almost continuous functions [3].* (Every function is a pointwise limit of a sequence of almost continuous functions [3].)

Theorem 2 *(1) Each function can be expressed as a sum of two almost continuous functions [3]. (2) (CH) f is a product of two almost continuous functions iff it has a zero in each subinterval in which it changes sign [6]. (3) Every function f can be expressed as $min(max(f_1, f_2), max(f_3, f_4))$, where f_1, f_2, f_3, f_4 are almost continuous [5].*

Let $K^+(f,x)$ denote the right-hand cluster set of f at x and let $K_c^+(f,x)$ be the right-hand c-cluster set of f at x, i.e. $K_c^+(f,x) = \bigcap\{K^+(f|I \setminus B, x) : card(B) < 2^\omega\}$. Similarly we define $K_c^-(f,x)$. It is known that a function f is the maximum of two Darboux functions iff
$(*)$ $f(x) \leq min\{max(K_c^+(f,x)), max(K_c^-(f,x))\}$ for each x [1].

Problem 3 *Is every function satisfying* $(*)$ *the maximum of two almost continuous functions ?*

One can prove that every $f \in \mathcal{D}^*$ is the maximum of two almost continuous functions.

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