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Remarks on Nonabsolutely Convergent Integrals in the Real Line

Let [a,b] be a compact interval of the real line $\mathbf{R}$. Unless stated otherwise, a function is a real valued function defined on [a,b].

Definition *A class of continuous functions* $\mathbf{V}$ *is c alled an ordinary (descriptive) integral on [a,b] if i)* $\mathbf{V}$ *is a vector space, ii) each element of* $\mathbf{V}$ *is a.e. derivable, iii)* $F \in \mathbf{V}$ *and* $F' = 0$ *a.e. imply that* F *is a constant.*
A function f *is* $\mathbf{V}$*-integrable on [a,b] if there is* $F \in \mathbf{V}$ *so that* $F' = f$ *a.e. in [a,b]. In this case the value* $F(b) - F(a)$ *is the (definite)* $\mathbf{V}$*-integral of* f *on [a,b], denoted by* $(V)\int_a^b f$.

Given an integral $\mathbf{V}$ the class of all $\mathbf{V}$-integrable functions is denoted by $\mathcal{I}_V$.

Theorem 1 *Let* $\mathcal{M}$ *be the class of all functions that are (Lebesgue) measurable and almost everywhere finite on [a,b]. The "axiom of choice" implies that there exist infinitely many ordinary integrals* $\mathbf{V}$ *such that*

1) $ACG_* \subset \mathbf{V}$

2) $\mathcal{I}_V = \mathcal{M}$

3) if $F \in \mathbf{V}$, $F' \geq 0$ *a.e. and* $F' \notin L^1[a,b]$ *then* F *does not satisfy Lusin's condition (N) (abbreviated as* $F \notin (N)$*)*

4) if $F \in \mathbf{V}$, F' *is Denjoy-Perron integrable on each* $[\alpha + \varepsilon', \beta - \varepsilon''] \subset [a,b]$ *and if* $\lim_{\varepsilon',\varepsilon'' \to 0} (\mathcal{D}) \int_{\alpha+\varepsilon'}^{\beta-\varepsilon''} f$ *does not exists then* $F \notin (N)$.

Theorem 2 *To each ordinary integral* $\mathbf{V}$ *with* $ACG_* \subsetneqq \mathbf{V} \subset (N)$ *there is an ordinary integral* $\tilde{\mathbf{V}}$ *so that* $\mathbf{V} \cap \tilde{\mathbf{V}} = ACG_*$ *and* $\mathcal{I}_V = \mathcal{I}_{\tilde{V}}$.

Theorem 3 *Denote by* Δ *the class of all differentiable functions and set* $\mathbf{V} = \langle AC, \Delta \rangle$. *To each* $f \in \mathcal{I}_V$ *and to each positive* ε *and* M *there is a positive gage* δ *such that*

$$|\sum_{i=1}^{p} f(x_i)(\beta_i - \alpha_i) - (V)\int_a^b f| < \varepsilon,$$

for every δ-fine partition $\{([\alpha_i, \beta_i], x_i) : i = 1, 2, \ldots, p\}$ that satisfies the condition

$$\sum_{i=1}^{p} d([\alpha_i, \beta_i]; x_i) < M;$$

here $d([\alpha_i, \beta_i]; x_i)$ denotes the distance between $[\alpha_i, \beta_i]$ and x_i.

Remark The regularity condition of Theorem 3 is related to that employed in [4].

Theorem 4 *Denote by $\mathcal{R}_b^*$ the class of all functions f that satisfy the condition of Theorem 3. Then $\mathcal{R}_b^*$ is a generalized Riemann integral that is properly contained in the integral $\mathcal{R}_s^*$ considered in [1].*

Problem *Is the inclusion $\mathcal{I}_{\langle AC,\Delta\rangle} \subset \mathcal{R}_b^*$ proper ? If so, characterize the convergence of generalized Riemann sums of $\langle AC, \Delta\rangle$-integrable functions.*

References

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