Washek F. Pfeffer, Department of Mathematics, University of California, Davis, CA 95616

A Voltera Type Derivative of the Legesgue Integral

Let $m \geq 1$ be an integer and let $\Omega \subset \mathbf{R}^m$ be an open set. The norms in $L^1(\Omega)$ and $L^\infty(\Omega)$ are denoted by $|\cdot|_1$ and $|\cdot|_\infty$, respectively. We consider only real-valued functions. Given a nonnegative function θ defined on Ω, we set

$$S_\theta = \{x \in \Omega : \theta(x) > 0\}$$

and denote by d_θ the diameter of S_θ. We say that $\theta \in L^1(\Omega)$ is *normalized* whenever $|\theta|_1$ equals the measure of S_θ.

A function $\theta \in L^1(\Omega)$ is of *bounded variation* if its distributional gradient $D\theta$ is a vector-valued Borel measure in Ω whose variation $|D\theta|$ is finite. By BV_+ we denote the family of all nonnegative functions $\theta \in L^\infty(\Omega)$ such that θ is of bounded variation, vanishes outside a compact subset of Ω, and $|\theta|_1 > 0$. The *regularity* of $\theta \in BV_+$ is the number

$$r(\theta) = \frac{|\theta|_1}{d(\theta)|D\theta|}.$$

Fix a function $f \in L^1_{\mathrm{loc}}(\Omega)$. A point $x \in \Omega$ is called *regular* if given $\varepsilon > 0$, we can find a $\delta > 0$ so that

$$\left| f(x)|\theta|_1 - \int_\Omega f\theta \right| < \varepsilon|\theta|_1$$

for each $\theta \in BV_+$ for which x is contained in the closure of S_θ, $d_\theta < \delta$, and $r(\theta) > \varepsilon$. If the inequality holds only when θ is, in addition, normalized and $|\theta|_\infty < 1/\varepsilon$, then x is called *weakly regular*.

Each regular point is weakly regular, however, the converse is false when $m > 1$; if $m = 1$ the two concepts coincide.

Proposition. *If $x \in \Omega$ is a weakly regular point and f is essentially bounded in a neighborhood of x, then x is a regular point.*

Theorem. *Almost all points of Ω are weakly regular. In particular, almost all points of Ω are regular whenever $f \in L^\infty_{\mathrm{loc}}(\Omega)$.*

The theorem is used to show that a conditionally convergent integral in $\mathbf{R}^m$ defined by BV_+ partitions of unity is invariant with respect to lipeomorphisms.