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Differentiability and Integrability in n Dimensions with Respect to α-Regular Intervals

Regularity of an interval $I \subset R^n$ (notation reg I) is the ratio of its shortest and longest edges, hence $0 < \text{reg } I \leq 1$. We denote by $m(J)$ the Lebesgue measure of $J \subset R^n$. An additive function G of interval is said to be *α–regularly differentiable* at $s \in R^n$ to $g \in R$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that $|G(J) - gm(J)| < \varepsilon m(J)$ for every interval $J \subset B(s, \delta) = \{x \in R^n; \max_i |x_i - s_i| \leq \delta\}$ with $s \in J$, reg $J \geq \alpha$. A function $f : I \to R, I \subset R^n$ an interval, is *α–regularly integrable* if there is $c \in R$ such that for every $\varepsilon > 0$ there is a function $\delta : I \to (0, \infty)$ such that $|c - (\Delta) \sum f(t)m(J)| < \varepsilon$ for every finite family Δ of tagged intervals (t, J) such that $t \in J \subset B(t, \delta(t))$, reg $J \geq \alpha$, the intervals J are non–overlapping and their union is I (we then write $c = (\alpha) \int_I f$). Our aim is to show that while the value of regularity is irrelevant for the α–regular differentiability, it is essential for the α-regular integrability.

To prove that α–regular differentiability does not depend on α, we first establish a general property of additive functions of interval. Let an additive function G of interval be defined on an interval $I \subset R^n$, let $t \in \text{Int} I, r > 0$ such that $B(t, r) \subset I$. We denote

$$\Omega = \Omega(t, r, G) = \sup\{|G(J)|; J \subset B(t, r), J \text{interval}\}$$

and, given $\alpha, 0 < \alpha < 1$,

$$\omega = \omega(t, r, G, \alpha) = \sup\{|G(K)|; t \in K = [u_1, v_1] \times \ldots [u_n, v_n], \alpha r \leq v_i - u_i \leq r\}.$$

(Note that $K \subset B(t, r)$; reg $K \geq \alpha$.)

Proposition. *There is a constant $k = k(n, \alpha)$ such that*

$$\omega \leq \Omega \leq k\omega$$

for every additive interval function G on I and any $B(t, r) \subset I$.

Putting $G(J) = F(J) - fm(J)$ in Proposition, we obtain as a corollary.

Theorem 1. *Let $0 < \beta < \alpha < 1$, let an additive function F be α–differentiable to f at t. Then F is β–differentiable to f at t, as well.*

(An analogous result holds if we replace "differentiable" by "lipschitzian", defining α–lipschitzianity in the obvious way.)

The other result has the character of a counterexample.

Theorem 2. *Given $\alpha, 0 < \alpha < 1$, there exists a function $f = f_\alpha : R^n \to R$ which is α_1-regularly integrable on $I = [-1,2]^n$ for every $\alpha_1, \alpha < \alpha_1 < 1$, and is not α_2-regularly integrable for every $\alpha_2, 0 < \alpha_2 < \alpha$.*

Let us mention that the function f can be constructed in such a way that the set of points at which the primitive F is not α–lipschitzian is closed and has an arbitrarily small Hausdorff measure.

In the discussion at the Conference, a question was raised by W. F. Pfeffer whether f is α–integrable. Since then, it was proved that for each $\alpha, 0 < \alpha < 1$, there exist functions g, h such that g is β–integrable for $\beta > \alpha$ but not for $\beta \leq \alpha$ while h is β–integrable for $\beta \geq \alpha$ but not for $\beta < \alpha$.

Theorems 1 and 2 have an interesting consequence relative to the property of "α–variational normality of F" (also called "good behavior on sets of zero measure"). Recall that given $0 < \alpha < 1, A \subset I$, then an additive function of interval F defined on I is said to be *α–variationally normal on A* if for every set $N \subset A$ with measure zero and every $\varepsilon > 0$ there is a function $\delta : I \to (0, \infty)$ such that $(\Delta) \sum |F(J)| \leq \varepsilon$ for every finite family of tagged intervals (t, J) such that $t \in J \subset B(t, \delta(t))$, reg $J \geq \alpha$, the intervals J are non–overlapping and $t \in N$ for every $(t, J) \in \Delta$.

The following theorem was proved (in a more general form) by the authors in [1] (Theorem 4.2):

Theorem 3. *A function $f : I \to R$ is α–regularly integrable with a primitive F iff*

(i) F is additive;

(ii) F is α–regularly differentiable to $f(t)$ at almost every $t \in I$;

(iii) F is α–variationally normal on I.

Consequently, the property (iii) is not independent of the value of regularity α.

The detailed account of the results will appear in Resultate der Mathematik, special volume in honour of the 65th birthday of Prof. H.-W. Knobloch.

Reference

[1] Kurzweil J. and Jarník J.: Equiintegrability and controlled convergence of Perron–type integrable functions. Real Analysis Exchange (1991), in print.