Real Analysis Exchange Vol 15 (1989-90)

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Some Applications of an L^1 Version of the Gauss Integral Theorem

This paper presents some applications of a Gauss integral theorem for functions with partial derivatives defined in an integral sense. The applications are to a Green's representation formula, interchanging the order of partial differentiations and a form of Weyl's lemma, all for functions with L^1 partial derivatives. This notion of differentiability is now recalled.

__Definition:__ A function $u \in L^p(\mathrm{R}^n)$, $1 \leq p \leq \infty$ is said to have an L^p partial derivative with respect to x_k if there is an L^p function u_{x_k} such that if $h = (h_1, \ldots, h_r)$;

$$\lim_{h_k \to 0} \left(\int_{\mathrm{R}^n} \left| \frac{u(x+h) - u(x)}{h_k} - u_{k_k}(x) \right|^p dx \right)^{1/p} = 0. \tag{1}$$

The function u_{x_k} is called the L^p first partial derivative of u with respect to x_k.
Let Ω be an open set in R^n satisfying the following conditions:

(i) $\partial\Omega$ is a simple closed $n-1$ dimensional surface.

(ii) $0 \in \mathrm{int}(\Omega)$

(iii) Surface measure on $\partial\Omega$ is given by $ds = |n(\varsigma, \eta)| d\varsigma d\eta$ where $n(\varsigma, \eta)$ is the unit outer normal to the point on $\partial\Omega$ given by parameters ς, η in a C_1 parametrization.

The following version of the Gauss integral theorem was proved in Arnold [1].

THEOREM 1. Let Ω satisfy conditions (i), (ii) and (iii) and let $u \in L^1(\mathrm{R}^n)$ have L^1 first partials $u_{y_i} \in L^1(\mathrm{R}^n)$. Then for almost all $x \in \mathrm{R}^n$:

$$\int_{x+\Omega} u_{y_i} dt = \int_{\partial(x+\Omega)} u n_i ds \tag{2}$$

This theorem will now be put into a form more useful in applications. First it is shown that (2) actually holds for all $x \in \mathrm{R}^n$. This should have been done in Arnold [1]. It is enough to prove that both integrals in (2) are continuous functions of x. With χ denoting characteristic function, we have the following estimate:

$$\begin{aligned}\left|\int_{x+\Omega} u_{y_i} dt - \int_{z+\Omega} u_{y_i} dt\right| &= \left|\int_{\mathbb{R}^n} u_{y_i}(\chi_{x+\Omega} - \chi_{z+\Omega}) dt\right| \\ &\leq \int_{\mathbb{R}^n} |u_{y_i}|\chi_{S_{x,z}} dt \\ &= \int_{S_{x,z}} |u_{y_i}| dt \\ S_{x,z} &= (x+\Omega)\Delta(z+\Omega)\end{aligned}$$

Since $u_{y_i} \in L^1(\mathbb{R}^n)$, the absolute continuity of the Lebesgue integral will give the result if it can be shown that the symmetric difference $(x+\Omega)\Delta(z+\Omega)$ has small Lebesgue measure when $|x-z|$ is small.

By the absolute continuity of the integrals, Ω may be assumed to have finite Lebesgue measure. Then finite disjoint unions of cubes form a Vitali cover and it suffices to prove the theorem for a cube. The result then reduces to showing that if Ω is a cube in $\mathbb{R}^n$ containing 0, that the measure of $S_{x,z}$ is small with $|x-z|$. This follows from an elementary geometric argument.

The same argument holds in the $n-1$ dimensional manifold $\partial\Omega$, transforming the $n-1$ dimensional Lebesgue measure by coordinate charts, giving the continuity in x of the surface integral in (2). Since (2) now holds for all x we may drop the translation notation in (2): any region Ω is the translation of one containing 0.

Next, the domain of u may be taken to be an open set Ω instead of all of $\mathbb{R}^n$ by setting $u \equiv 0$ off Ω. For this, the limit (1) is required to hold with $\mathbb{R}^n$ replaced by Ω. This is enough to prove that the extension by zero of u has L^1 partials on $\mathbb{R}^n$. In this case u is said to have L^1 partials on Ω. These improvements are summarized in the following restatement of Theorem 1.

THEOREM 2. Let Ω be an open set in $\mathbb{R}^n$ and let u have L^1 first partials in Ω. If Ω satisfies (i) and (iii) then

$$\int_\Omega u_{y_i} dt = \int_{\partial\Omega} u n_i ds$$

Theorem 2 can be used to prove the divergence theorem with L^1 partials in place of the usual partial derivatives. This theorem will be called the L^1 version of the divergence theorem. Green's theorem in the plane for L^1 partials is then a corollary. These theorems will be used several times below. Their proofs and those of Lemmas 1 and 3 below follow exactly as in John [2] Ch. 4 by substituting

Theorem 1 and the L^1 version just mentioned of the divergence theorem for their classical counterparts.

If the L^1 first partials of u, themselves have L^1 first partials, written $u_{y_i y_j}$, u will be said to have L^1 second partials. The set of functions on Ω whose L^1 first partials have L^1 first partials is denoted $L^{1,2}(\Omega)$. The Laplacian formed with these derivatives is:

$$\Delta u = \sum_{i=1}^{n} u_{y_i y_i}$$

LEMMA 1.

$$\int_{\Omega} \Delta u dt = \int_{\partial\Omega} u_n ds$$

where

$$u_n = \sum_{i=1}^{n} u_{y_i} n_i$$

The next fact is the analog of Lebesgue's differentiation theorem for spheres rather than balls. The dimension restriction is rather surprising.

LEMMA 2. Let $n \geq 3$, $f \in L^p_{loc}$, $P > \frac{n}{n-1}$

$$S(0,1) = \{x : |x| = 1\}$$

then

$$\lim_{\rho \to 0} \int_{S(0,1)} f(x - \rho y) ds(y) = f(x)$$

for almost all x.

Proof: Stein and Wainger [3].

Note: This result is false for any n if $p \leq \frac{n}{n-1}$. It is an open problem whether it is true when $n = 2$, $p > 2$. See Stein and Wainger [3] for a discussion.

The final preliminary result we require is the L^1 analog of a standard application of the divergence theorem. Again, the proof is just as in John [2], p. 96 with the L^1 version of the divergence theorem replacing the usual one¿

LEMMA 3. Let $u, v \in L^{1,2}(\Omega)$, $B(\xi, \rho) = \{y : |y - \xi| < \rho\}$, $\rho \geq 0$, $S(\xi, \rho) = \partial B(\xi, \rho)$, $\Omega_\rho = \Omega - B(\xi, \rho)$, $\Delta v = 0$ on Ω_p. Then

$$\int_{\Omega_\rho} v \Delta u dt = \int_{\partial\Omega} (v u_n - u v_n) ds + \int_{S(\xi,\rho)} (v u_n - u v_n) ds$$

A Green's representation theorem for a function with L^1 second partials can now be given along classical lines with a reliance on the absolute continuity of the Lebesgue integral and Lemma 2 replacing continuity assumptions.

Let $E(r)$ be the fundamental solution to Laplace's equation:

$$E(r) = \frac{r^{2-n}}{(2-n)\omega_n}, \quad n > 2$$

$$\frac{\log r}{2\pi}, \quad n = 2$$

Here ω_n = measure of the unit sphere. Choosing $v = E(r) = E(|t-\xi|)$, Lemma 3 produces

$$\int_{\Omega_\rho} v\Delta u dt = \int_{\partial\Omega} (vu_n - uv_n) ds - E(\rho) \int_{S(\xi,\rho)} u_n ds - \rho^{1-n} \int_{S(\xi,\rho)} u ds$$

Now let $\rho \to 0$. By absolute continuity of integrals and the fact that $\Delta u \in L^1(\mathsf{R}^n)$ we get

$$\int_{B(\xi,\rho)} \Delta u dt = \int_{S(\xi,\rho)} u_n ds \to 0$$

In addition, Lemma 2 gives for almost all ξ:

$$\frac{\rho^{1-n}}{\omega_n} \int_{S(\xi,\rho)} u ds \to u(\xi)$$

Hence the following representation theorem has been established.

THEOREM 3. Let $p > n/n-1$, $n \geq 3$. Let $u \in L^p(\Omega) \cap L^{1,2}(\Omega)$, Ω satisfying (i) and (iii). Then for almost all $\xi \in \Omega$

$$u(\xi) = \int_\Omega E(|t-\xi|) \Delta u(t) dt - \int_{\partial\Omega} [E(|s-\xi|) u_n(s) - u(s) E_n(|s-\xi|) ds$$

This theorem can be used to give a natural and quick proof of a version of Weyl's Lemma for the L^1 derivatives considered here.

THEOREM 4. Let $u \in L^p(\Omega) \cap L^{1,2}(\Omega)$, $\rho > n/n-1$, $n \geq 3$ be continuous in a region Ω satisfying (i) and (iii); and let u have L^1 second partials in Ω. Then $\Delta u = 0$ implies that u is harmonic in Ω.

Proof: Let Ω in Theorem 3 be a ball $B(\xi,\rho)$ and replace E by $G(\xi,t) = E(|t-\xi|) - E(\rho)$ there. From Lemma 1, it follows that the formula in Theorem

3 remains valid. Note that $G = 0$ and $G_n = \frac{1}{\omega_n}\rho^{1-n}$ on $\partial\Omega$. Since $\Delta u = 0$ in Ω we get the mean value property

$$u(\xi) = \frac{1}{\omega_n \rho^{n-1}} \int_{S(\xi,\rho)} u(s)ds.$$

This is sufficient to imply, in view of the continuity of u, that u is harmonic. (c.f. Stein and Weiss [1], p. 41).

Remark: The methods used here are natural extensions of potential theoretic arguments. However, the restrictions on p necessary for the use of Lemma 2 are not necessary for the proof of Weyl's lemma by other methods.

The next application is a theorem on interchanging the order of partial differentiation. In this two dimensional situation partial derivatives are written as subscripted x and y. For smooth functions it follows easily from Green's theorem in the plane that $u_{xy} = y_{yx}$. This proof can be used as a model to obtain the result for functions u with L^1 second partials.

THEOREM 5. Let $u(x, y)$ be defined in a plane open set Ω. Suppose $u \in L^{1,2}(\Omega)$ with L^1 second partials u_{xy} and u_{yx} and that u is absolutely continuous in each variable separately. Then $u_{xy} - u_{yx}$ a.e. in Ω.

Remark: There is a harmless redundancy in the hypotheses since if $u(x, y)$ is absolutely continuous with respect to x for fixed y then $u_x(x, y)$ exists for almost every x for this fixed y. A similar remark holds with x and y interchanged.

Proof: Let R be a rectangle contained in Ω which is parameterized linearly with x and y as parameters on the horizontal and vertical sides respectively. The hypotheses imply that u restricted to the boundary ∂R is absolutely continuous with respect to each parameter. Hence, the fundamental theorem of calculus for Lebesgue integrals provides:

$$\int_{\partial R} u_x dx + u_y dy = \int_{\partial R} \frac{du}{ds} ds = 0.$$

Now Green's theorem in the plane for L^1 partials gives:

$$\int_R (u_{xy} - u_{yx}) dt = 0.$$

Now let C be the class of all plane sets $S \subset \Omega$ for which

$$\int_S (u_{xy} - u_{yx}) dt = 0.$$

C is a monotone class containing the σ-field of all countable disjoint unions of finite rectangles. By the monotone class theorem C contains all the Borel sets in Ω. Since $u_{xy} - u_{yx} \in L^1(\mathsf{R}^2)$ we conclude $u_{xy} = u_{yx}$ a.e. in Ω.

References:

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3. E.M. Stein and S. Wainger, Problems in harmonic analysis related to curvature, Bull. AMS. (84), 1978, pp. 1239-1295.

4. E.M. Stein and G. Weiss, Introduction to Fourier Analysis on Euclidean spaces, Princeton University Press, 1971.

Received March 2, 1989