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LIPSCHITZIAN HOMEOMORPHISMS WITH LARGE SETS OF DIRECTIONAL DERIVATIVES

1. Introduction. In the calculus for vector-valued functions of several variables the first step is to define the notion of directional derivative. Let $f\colon D \to \mathbb{R}^n$ (D is an open set in $\mathbb{R}^m$) be a function, $e \in \mathbb{R}^m$ and $x \in D$. Then the derivative of f at x in the direction e is

$$\lim_{t \to 0} \frac{f(x+te) - f(x)}{t}$$

if the limit exists. This limit need not exist even if f is lipschitzian. But then we can define $Df(x;e)$ (the contingent one-side derivative) to be the set of all limits

$$\lim_{n \to \infty} \frac{f(x+t_n e) - f(x)}{t_n}$$

where $t_n \to 0^+$ as $n \to \infty$. $Df(x;e)$ is nonempty and compact and the natural question arises as to what it can look like. We shall give three examples which show that it can have unexpected shape even if both f and its inverse f^{-1} are lipschitzian.

2. Examples. Let a function $f\colon \mathbb{R}^k \to \mathbb{R}^k$ satisfy the following condition

(1) $$K\|x-y\| \leq \|f(x)-f(y)\| \leq L\|x-y\|$$

for all $x,y \in \mathbb{R}^k$, where $0 < K \leq L$ (f is bilipschitzian). It follows from the well-known Rademacher's theorem (see [3]) that f has a Fréchet derivative $f'(x)$ (being a linear mapping from $\mathbb{R}^k$ into itself) for almost all x in $\mathbb{R}^k$. For such an x and for all $e_1, e_2 \in \mathbb{R}^k$ we have

(2) $$K\|e_1-e_2\| \leq \|f'(x)e_1 - f'(x)e_2\| \leq L\|e_1-e_2\|$$

which is easy to prove. The above condition says that $f'(x)(\cdot)$ is lipschitzian and invertible in the same way as f is.

If f is not differentiable at the point x, then one can formulate for $Df(x;e)$ an equivalent of (2). Namely, for any $e_1, e_2 \in \mathbb{R}^k$ and for any $p_1 \in Df(x;e_1)$ there exists $p_2 \in Df(x;e_2)$ such that

(3) $$K\|e_1-e_2\| \leq \|p_1-p_2\| \leq L\|e_1-e_2\| \quad .$$

Indeed, let

$$p_1 = \lim_{n\to\infty} \frac{f(x+t_n e_1)-f(x)}{t_n} \quad .$$

Since the sequence

$$\left\{ \frac{f(x+t_n e_2)-f(x)}{t_n} \right\}$$

is bounded, we may assume (choosing a subsequence if necessary) that it converges to a point $p_2 \in Df(x;e_2)$. Now, observing that

$$\|p_2-p_1\| = \lim_{n\to\infty} \frac{\|f(x+t_n e_2)-f(x+t_n e_1)\|}{t_n}$$

and using (1) we arrive at the desired inequalities (3).

It follows from (3) that

$$H(Df(x;e_1),Df(x;e_2)) \leqslant L\|e_1-e_2\|$$

(where H denotes the Hausdorff metric, see [2]). Therefore the function $Df(x;\bullet)$ is lipschitzian with the same constant as f. The natural question is: Could one obtain a similar estimation from below for the Hausdorff distance between $Df(x;e_1)$ and $Df(x;e_2)$? Condition (3) enables us to state only that

$$(4)\qquad K\|e_1-e_2\| \leqslant H(Df(x;e_1),Df(x;e_2))$$

whenever one of the sets $Df(x;e_1)$, $Df(x;e_2)$ is a singleton. In general (4) is not true as the following example shows.

<u>Example 1.</u> Let $k = 2$ and

$$f(x,y) = \begin{cases} \big(x\cos(\ln r)-y\sin(\ln r)\ ,\ x\sin(\ln r)+y\cos(\ln r) \big) & \text{if } (x,y) \neq (0,0) \\ (0,0) & \text{if } (x,y) = (0,0) \end{cases}$$

where $r = (x^2+y^2)^{1/2}$. (The point $(x,y) \neq (0,0)$ is rotated about the origin through the angle $\ln r$).

Then, for any $e \in \mathbb{R}^2$, with $\|e\| = 1$,

$$Df((0,0);e) = \{p \in \mathbb{R}^2 : \|p\| = 1\} \ .$$

(The function $Df((0,0);\bullet)$ is constant on the unit circle).

First, we should prove (1). To do this we note that all components of the Jacobi matrices of f and f^{-1} are absolutely bounded by 3 at each point $(x,y) \neq (0,0)$ and $\|f(x,y)\| = \|f^{-1}(x,y)\| = (x^2+y^2)^{1/2}$ for every $(x,y) \in \mathbb{R}^2$. Now it suffices to see that, for any $e \in \mathbb{R}^2$, with $\|e\| = 1$, the quotient $f(te)/t$ turns along the unit circle infinitely many times as $t \to 0^+$.

Remark 1. Suppose that the bilipschitzian function $f\colon \mathbb{R}^k \to \mathbb{R}^k$,

$$K\|x-y\| \leqslant \|f(x)-f(y)\| \leqslant L\|x-y\| \tag{5}$$

has a one-side directional derivative

$$df(x;e_0) = \lim_{t\to 0^+} \frac{f(x+te_0)-f(x)}{t} = p_0$$

at the point x in the direction $e_0 \neq 0$. Then for every $e \neq 0$ we have

$$\operatorname{dist}(Df(x;e),1^+_{p_0}) \geqslant K\operatorname{dist}(e,1^+_{e_0}) \quad ,$$

where $1^+_{p_0} = \{sp_0\colon s>0\}$ and $1^+_{e_0} = \{se_0\colon s>0\}$.

Indeed, if

$$Df(x;e) \ni p_1 = \lim_{n\to\infty} \frac{f(x+t_n e)-f(x)}{t_n} \quad ,$$

then for $s>0$ we get

$$\|p_1-sp_0\| = \lim_{n\to\infty} \left\| \frac{f(x+t_n e)-f(x)}{t_n} - sp_0 \right\| =$$

$$= \lim_{n\to\infty} \left\| \frac{f(x+t_n e)}{t_n} - \frac{f(x+st_n e_0)}{t_n} + s\left(\frac{f(x+st_n e_0)-f(x)}{st_n} - p_0\right) \right\| =$$

$$= \lim_{n\to\infty} \frac{\|f(x+t_n e)-f(x+st_n e_0)\|}{t_n} \geqslant K\|e-se_0\| \geqslant K\operatorname{dist}(e,1^+_{e_0})$$

and therefore

$$\operatorname{dist}(p_1,1^+_{p_0}) \geqslant K\operatorname{dist}(e,1^+_{e_0}) \quad .$$

As a corollary we get that if there exists $e \in \mathbb{R}^k \setminus \{0\}$ such that

$$\left\{ q \in \mathbb{R}^k\colon \text{ there exists } p \in Df(x;e) \text{ such that } q = \frac{p}{\|p\|} \right\}$$

is the entire unit sphere in $\mathbb{R}^k$, then the function f has no one-side directional derivative $df(x;e_0)$ at the point x in any direction $e_0 \neq 0$.

Remark 2. Example 1 can suggest that if for a bilipschitzian function $f: \mathbb{R}^2 \to \mathbb{R}^2$ there exists $e_0 \in \mathbb{R}^2$ such that

$$\left\{q \in \mathbb{R}^2: \text{ there exists } p \in Df(x;e_0) \text{ such that } q = \frac{p}{\|p\|}\right\}$$

is the unit circle, then the same holds for all directions $e \neq 0$. The above conjecture is not true as Example 2 shows.

Example 2. We define $g_i: [\frac{1}{4},1] \to \mathbb{R}$ $(i=1,2,3)$ and $h: [\frac{1}{4},1] \times [-\pi,\pi] \to [\frac{1}{4},1] \times \mathbb{R}$ in the following way:

$$g_1(r) = \begin{cases} 2\pi r - 3\pi , & \frac{3}{4} \leq r \leq 1 \\ -2\pi r , & \frac{1}{2} \leq r \leq \frac{3}{4} \\ -4\pi r + \pi , & \frac{3}{8} \leq r \leq \frac{1}{2} \\ 4\pi r - 2\pi , & \frac{1}{4} \leq r \leq \frac{3}{8} \end{cases} ;$$

$$g_2(r) = \begin{cases} 4\pi r - 4\pi , & \frac{3}{4} \leq r \leq 1 \\ -4\pi r + 2\pi , & \frac{1}{2} \leq r \leq \frac{3}{4} \\ -8\pi r + 4\pi , & \frac{3}{8} \leq r \leq \frac{1}{2} \\ 8\pi r - 2\pi , & \frac{1}{4} \leq r \leq \frac{3}{8} \end{cases} ;$$

$$g_3(r) = g_1(r) + 2\pi \text{ for } r \in [\tfrac{1}{4},1] ;$$

$$h(r,\varphi) = \begin{cases} \left(r , -\frac{\varphi}{\pi} g_1(r) + (\frac{\varphi}{\pi} +1)g_2(r)\right) & \text{for } r \in [\frac{1}{4},1] , \varphi \in [-\pi,0] \\ \left(r , (1-\frac{\varphi}{\pi})g_2(r) + \frac{\varphi}{\pi} g_3(r)\right) & \text{for } r \in [\frac{1}{4},1] , \varphi \in [0,\pi] . \end{cases}$$

It is easy to observe that h and h^{-1} are lipschitzian and they generate $\tilde{f}$ and $\tilde{f}^{-1}$ from the annulus $P = \left\{x \in \mathbb{R}^2: \frac{1}{4} \leq \|x\| \leq 1\right\}$ onto P , which are also lipschitzian mappings. Now the following extension f

of $\tilde{f}$

$$f(x) = \begin{cases} 2^{-2k}\tilde{f}(2^{2k}x) & \text{for } 2^{-2(k+1)} \leqslant \|x\| \leqslant 2^{-2k}, \ k=2,3,\ldots \\ 0 & \text{for } x = 0 \\ x & \text{for } \|x\| \geqslant 1 \end{cases}$$

is bilipschitzian and for $e_1 = (1,0)$ we have

$$\left\{q \in \mathbb{R}^2: \text{ there exists } p \in Df(0;e_1) \text{ such that } q = \frac{p}{\|p\|}\right\} =$$
$$= \left\{q \in \mathbb{R}^2: \|q\| = 1\right\} \neq$$
$$\neq \left\{q \in \mathbb{R}^2: \text{ there exists } p \in Df(0;-e_1) \text{ such that } q = \frac{p}{\|p\|}\right\} =$$
$$= \left\{q = (q_1,q_2) \in \mathbb{R}^2: \|q\| = 1, \ q_1 \leqslant 0\right\}.$$

Now we show what we can obtain after a modification of Example 1 to $\mathbb{R}^3$.

Example 3. Let $f(0,0,0) = (0,0,0)$ and

$$f(x,y,z) =$$
$$= [\, x\cos(\ln r)-y\sin(\ln r), \ (x\sin(\ln r)+y\cos(\ln r))\cos(\sqrt{2}\ln r) - z\sin(\sqrt{2}\ln r),$$
$$(x\sin(\ln r)+y\cos(\ln r))\sin(\sqrt{2}\ln r) + z\cos(\sqrt{2}\ln r)\,]$$

if $(x,y,z) \neq (0,0,0)$, where $r = (x^2+y^2+z^2)^{1/2}$.

Proceeding as in Example 1 it follows that both f and f^{-1} are lipschitzian which yields condition (1). Now, let us take $e_1 = (1,0,0)$ and $e_2 = (0,0,1)$. Then, for $t > 0$, we have

$$\frac{f(te_1)}{t} = [\, \cos(\ln t), \ \cos(\sqrt{2}\ln t)\sin(\ln t), \ \sin(\sqrt{2}\ln t)\sin(\ln t)\,],$$
$$\frac{f(te_2)}{t} = [\, 0, \ -\sin(\sqrt{2}\ln t), \ \cos(\sqrt{2}\ln t)\,].$$

Employing arguments analogous to those used in Example 1 one can prove that

$$Df((0,0,0);e_2) = \left\{(x,y,z) \in \mathbb{R}^3: \ x = 0, \ y^2+z^2 = 1\right\}.$$

We claim that $Df((0,0,0);e_1)$ is the whole unit sphere in $\mathbb{R}^3$. Indeed, let us fix any $(x,y,z)\in\mathbb{R}^3$ with $x^2+y^2+z^2 = 1$ and take $t_0>0$ to satisfy $x = \cos(\ln t_0)$. Let

$$t_k = t_0 e^{-2k\pi}, \; k = 1,2,\dots .$$

Then $t_k \to 0$ as $k\to\infty$ and $\cos(\ln t_k) = x$, $\sin(\ln t_k) = \sin(\ln t_0)$ for $k = 1,2,\dots$.

In the case of $x^2 = 1$ we thus obtain the claim since

$$\frac{f(t_k e_1)}{t_k} = (x,y,z) \quad \text{for} \quad k = 1,2,\dots .$$

Now, let $x^2<1$. Then $y^2+z^2>0$ and $\sin(\ln t_0)\neq 0$. Therefore it suffices to note that the sequence $\{(\cos(\sqrt{2}\ln t_k),\sin(\sqrt{2}\ln t_k))\}$, $k=1,2,\dots$, is dense in the unit circle on the plane (see for example [4]).

<u>Remark 3.</u> Let us observe that the mapping given in Example 3 is a superposition of two relevant lipschitzian transformations, which allows us to generalize this example to the case of $\mathbb{R}^k$ with $k>3$. We must only apply the following theorem, due to Kronecker (see [1]):

If $y_1,\dots,y_{n-1}\in\mathbb{R}\setminus\mathbb{Q}$ and $y_i(y_j)^{-1}\in\mathbb{R}\setminus\mathbb{Q}$ for $i\neq j$, then for every $\mu>0$ and $x_1,\dots,x_{n-1}\in\mathbb{R}$ there exist $t\in\mathbb{N}$ and $p_1,\dots,p_{n-1}\in\mathbb{Z}$ such that $|ty_i+p_i-x_i|<\mu$, for $i = 1,2,\dots,n-1$.

<u>Open problem.</u>

Can one construct a bilipschitzian function $f\colon \mathbb{R}^3\to\mathbb{R}^3$ such that for some $x\in\mathbb{R}^3$ we have

$$\left\{q\in\mathbb{R}^3\colon \text{there exists } p\in Df(x;e) \text{ such that } q = \frac{p}{\|p\|}\right\} = \left\{q\in\mathbb{R}^3\colon \|q\|=1\right\}$$

for every $e\in\mathbb{R}^3\setminus\{0\}$?

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