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INTERVALS OF FINITELY ADDITIVE SET FUNCTIONS

Abstract. Suppose that U is a set, F is a field of subsets of U, $A(\mathbb{R})(F)$ is the set of all real – valued finitely additive functions defined on F, $A(\mathbb{R})(F)^+$ is the set of all nonnegative – valued elements of $A(\mathbb{R})(F)$, each of ξ_1 and ξ_2 is in $A(\mathbb{R})(F)$, $\xi_2 - \xi_1$ is in $A(\mathbb{R})(F)^+$, α is a function with domain F and range a collection of subsets of $\mathbb{R}$ with bounded union, and for i = 1,2, the integral $\int_U \alpha\xi_i$, as a refinement – wise limit of sums, exists. Let T denote the transformation with domain $\{\rho : \rho \text{ in } A(\mathbb{R})(F), \text{ each of } \xi_2 - \rho \text{ and } \rho - \xi_1 \text{ in } A(\mathbb{R})(F)^+\}$ and range $\subseteq A(\mathbb{R})(F)$ given by $T(\rho)(V) = \int_V \alpha\rho$. Continuity, closure, maximum value, minimum value and convergence properties of T are studied.

1. Introduction. Suppose that U is a set, F is a field of subsets of F, r(F) is the set of all functions from F into $\exp(\mathbb{R})$, rB(F) is the set of all elements of r(F) with bounded range union, $A(\mathbb{R})(F)$ is the set of all functions from F into $\mathbb{R}$ that are finitely additive, $AB(\mathbb{R})(F)$ is the set of all bounded elements of $A(\mathbb{R})(F)$, and $A(\mathbb{R})(F)^+$ is the set of all nonnegative – valued elements of $A(\mathbb{R})(F)$; $A(\mathbb{R})(F)^+ \subseteq AB(\mathbb{R})(F)$ and shall be denoted by $AB(\mathbb{R})(F)^+$. For each of ζ_1 and ζ_2, each in $A(\mathbb{R})(F)$ with $\zeta_2 - \zeta_1$ in $AB(\mathbb{R})(F)^+$, we shall let $[\zeta_1;\zeta_2]$ denote

$$\{\rho : \rho \text{ in } A(\mathbb{R})(F), \text{ each of } \zeta_2 - \rho \text{ and } \rho - \zeta_1 \text{ in } AB(\mathbb{R})(F)^+\}.$$

We shall now suppose for the remainder of this paper that each of ξ_1 and ξ_2 is in $A(\mathbb{R})(F)$, $\xi_2 - \xi_1$ is in $AB(\mathbb{R})(F)^+$, α is in rB(F), $M = \sup\{|x| : x \text{ in range union of } \alpha\}$, and for i = 1,2, the integral (section 2)

$$\int_U \alpha(I)\xi_i(I)$$

exists.

Theorem 3.1 (section 3). If ρ is in $[\xi_1;\xi_2]$, then $\int_U \alpha(I)\rho(I)$ exists.

Now let T denote

$$\{(\rho,\int\alpha\rho)\colon \rho \text{ in } [\xi_1;\xi_2]\}$$

(see section 2 for the notion of integral function). In this paper we investigate continuity, representation, closure and convergence properties of T. The following theorem (section 3) implies that T is continuous with respect to variation norm.

Theorem 3.2. If each of ρ_1 and ρ_2 is in $[\xi_1;\xi_2]$, then each of $\rho_1-\rho_2$ and $T(\rho_1)-T(\rho_2)$ is in $AB(\mathbb{R})(F)$ and $M\int|\rho_1-\rho_2| - \int|T(\rho_1)-T(\rho_2)|$ is in $AB(\mathbb{R})(F)^+$.

Now, note that certain of the properties of T given in the theorems below (see section 4) are analogous to "standard" properties of real – valued functions continuous on a number interval.

Theorem 4.1. If each of η_1 and η_2 is in $[\xi_1;\xi_2]$, then there is ζ_1 and ζ_2, each in $[\xi_1;\xi_2]$ such that

$$\int\max\{T(\eta_1),T(\eta_2)\} = \int\alpha\zeta_1 \text{ and } \int\min\{T(\eta_1),T(\eta_2)\} = \int\alpha\zeta_2.$$

Theorem 4.2. If ρ is in $[\xi_1;\xi_2]$, then each of

$$\int\max\{T(\xi_1),T(\xi_2)\} - T(\rho) \text{ and } T(\rho) - \int\min\{T(\xi_1),T(\xi_2)\}$$

is in $AB(\mathbb{R})(F)^+$.

We have the following immediate corollary to Theorems 4.1 and 4.2; the reader can easily supply its proof:

Corollary 4.1. There is ζ_1 and ζ_2, each in $[\xi_1;\xi_2]$, such that if ρ is in $[\xi_1;\xi_2]$, then each of

$T(\zeta_1) - T(\rho)$ and $T(\rho) - T(\zeta_2)$ is in $AB(\mathbb{R})(F)^+$.

The next theorem is a result that deals with closest and farthest approximations.

Theorem 4.3. Suppose that λ is in $A(\mathbb{R})(F)$ and for some, and hence all (see section 3) elements ρ of $[\xi_1;\xi_2]$, $\lambda - T(\rho)$ is in $AB(\mathbb{R})(F)$. Then there is μ_1 and μ_2, each in $[\xi_1;\xi_2]$, such that if ρ is in $[\xi_1;\xi_2]$, then each of $\int|\lambda - T(\rho)| - \int|\lambda - T(\mu_1)|$ and $\int|\lambda - T(\mu_2)| - \int|\lambda - T(\rho)|$ is in $AB(\mathbb{R})(F)^+$.

Theorem 4.4 (intermediate value theorem). Suppose that λ is in $A(\mathbb{R})(F)$, each of η_1 and η_2 is in $[\xi_1;\xi_2]$ and each of $\lambda - T(\eta_1)$ and $T(\eta_2) - \lambda$ is in $AB(\mathbb{R})(F)^+$. Then there is μ in $[\xi_1;\xi_2]$ such that μ is in $[\int\min\{\eta_1,\eta_2\};\int\max\{\eta_1,\eta_2\}]$ and $\lambda = T(\mu)$.

Corollary 4.4. (stronger intermediate value theorem). Suppose that λ is in $A(\mathbb{R})(F)$ and for each V in F there is η_1 and η_2, each in $[\xi_1;\xi_2]$, such that

$$T(\eta_1)(V) \le \lambda(V) \le T(\eta_2)(V).$$

Then there is ζ in $[\xi_1;\xi_2]$ such that $\lambda = T(\zeta)$.

Theorem 4.5. Suppose that each of η_1 and η_2 is in $[\xi_1;\xi_2]$ and $T(\eta_2) - T(\eta_1)$ is in $AB(\mathbb{R})(F)^+$. Then there is μ_1 and μ_2, each in $[\xi_1;\xi_2]$, such that $\mu_2 - \mu_1$ is in $AB(\mathbb{R})(F)^+$ and

$$T([\mu_1;\mu_2]) = [T(\eta_1);T(\eta_2)].$$

We end the paper with the following Helly – type integral convergence theorem:

Theorem 4.6. Suppose that $\{\eta_i\}_{i=1}^{\infty}$ is a sequence of elements of $[\xi_1;\xi_2]$ and ζ is a function from F into $\mathbb{R}$ such that for each V in F,

$$\eta_n(V) \to \zeta(V) \text{ as } n \to \infty.$$

Then ζ is in $[\xi_1;\xi_2]$ and

$$T(\eta_n)(U) \to T(\zeta)(U) \text{ as } n \to \infty.$$

2. Preliminary theorems and definitions.

We adopt the convention that if δ is a function from F into $\mathbb{R}$, then δ shall be regarded as "equivalent" to the following element of r(F);

$$\{(V,\{\delta(V)\}) : V \text{ in } F\}.$$

If V is in F, then the statement that D is a subdivision of V means that D is a finite collection of mutually exclusive sets of F with union V. The statement that H is a refinement of E, denoted by H << E, means that for some W in F, each of H and E is a subdivison of W and each element of H is a subset of some element of E.

We shall, unless otherwise specified, use the following notational device: Suppose that each of P and Q is a collection of sets such that each set of P is a subset of some set of Q. Then, if I is in Q and includes some element of P, we shall let

$$P(I) = \{ J : J \text{ in } P, J \subseteq I\}.$$

If S is a set, γ is a function with domain S and range a collection of sets and $T \subseteq S$, then the statement that b is a γ – function on T means that b is a function with domain T such that if x is in T, then b(x) is in $\gamma(x)$.

Suppose that γ is in r(F) and V is in F. The statement that K is an integral of γ on V means that K is in $\mathbb{R}$ and if $0 < c$, then there is D << {V} such that if E << D and b is a

$\gamma-$function on E, then

$$|K-\Sigma_E b(I)| < c. \tag{2.1}$$

There is no more than one K' such that K' is an integral of γ on V; if, then, K is an integral of γ on V, then K is unique and shall be denoted, variously, by

$$\int_V \gamma(I),\ \int_V \gamma(J),\ \int_V \gamma, \text{ etc.}, \tag{2.2}$$

depending upon circumstances. We shall use the phrase "$\int_V \gamma(I)$ exists" to mean that there is K such that K is an integral of γ on V. Now, if $\int_U \gamma$ exists, then for each W in F, $\int_W \gamma$ exists and

$$\{(W,\int_W \gamma) : W \text{ in } F\}, \tag{2.3}$$

which we shall denote by $\int\gamma$, is in $A(\mathbb{R})(F)$.

Again, suppose that γ is in r(F). If V is in F, then the statement that γ is $\Sigma-$bounded on V with respect to D means that $D << \{U\}$ and

$$\{\Sigma_E b(J) : E << \{V\},\ b \text{ a } \gamma-\text{function on } E,\ E \subseteq H \text{ for some } H << D\} \tag{2.4}$$

is bounded. We have the following results:

Theorem 2.A.1 (see [4]). If γ is in r(F) and is $\Sigma-$bounded on U with respect to D, then the following statements are true:

1) If V is in F, then γ is $\Sigma-$bounded on V with respect to D.

2) Suppose that $L_D(\gamma)$ and $G_D(\gamma)$ denote the functions with domain F given, for each I in F as, respectively, the sup and inf of

$$\{\Sigma_E b(J) : E << \{I\},\ b \text{ a } \gamma-\text{function on } E,\ E \subseteq H \text{ for some } H << D\}.$$

Then, if V is in F, $H_1 << \{V\}$, $H_2 << \{V\}$ and for $i = 1,2$, $M << H_i$, then

$$\Sigma_{H_1} G_D(\gamma)(I) \le \Sigma_M G_D(\gamma)(J') \le \Sigma_M L_D(\gamma)(J') \le \Sigma_{H_2} L_D(\gamma)(I). \tag{2.A.1.1}$$

3) If V is in F, then the following existence and inequality holds:

$$\int_V G_D(\gamma) \leq \int_V L_D(\gamma). \tag{2.A.1.2}$$

4) If V is in F, then $\int_V \gamma$ exists iff

$$\int_V G_D(\gamma) = \int_V L_D(\gamma), \tag{2.A.1.3}$$

in which case

$$\int_V G_D(\gamma) = \int_V \gamma = \int_V L_D(\gamma). \tag{2.A.1.4}$$

5) If V is in F, Q is L_D or G_D, $E << \{V\}$ and $0 < c$, then there is $H << E$ and a $\gamma-$function a on H such that

$$\Sigma_H |Q(\gamma)(J) - a(J)| < c. \tag{2.A.1.5}$$

We now state Kolmogoroff's differential equivalence theorem.

<u>Theorem 2.K.1</u> (see [4,5]). If γ is in r(F) and $\int_U \gamma$ exists, then for each I in F, $\int_I \gamma$ exists and the following existence and equality holds:

$$\int_U |\gamma(I) - \int_I \gamma| = 0, \tag{2.K.1.1}$$

i. e., if $0 < c$, then there is $D << \{U\}$ such that if $E << D$ and a is a $\gamma-$function on E, then

$$\Sigma_E |a(I) - \int_I \gamma| < c, \tag{2.K.1.2}$$

so that if $H \subseteq E$ and b is a $\gamma-$function on H, then

$$\Sigma_H |b(I) - \int_I \gamma| < c. \tag{2.K.1.3}$$

We refer the reader, again to [4], for certain of the more immediate consequences of Theorem 2.K.1; these consequences treat conditions under which, given an element γ of r(F) such that $\int_U \gamma$ exists, $\int_I \gamma$ and $\gamma(I)$ can be interchanged. Throughout this paper there will be portions of arguments in which assertions of integral existence or integral equivalence follow from Theorem 2.K.1 and these consequences. In such cases we shall feel free to simply make these assertions and leave the details to the reader.

Now, before we continue with some more specialized matters, we remark that we shall assert and use, without preamble, certain simple inequality and linearity existence and equivalence properties of set function integrals.

Theorem 2.A.2 (see [4]). Suppose that each of ξ and η is in $A(\mathbb{R})(F)$. Then the following statements are true:

1) If V is in F, $D << \{V\}$ and for $i = 1,2$, $E_i << D$, then

$$\Sigma_{E_1} \min\{\xi(J),\eta(J)\} \leq \Sigma_D \min\{\xi(I),\eta(I)\} \leq \Sigma_D \max\{\xi(I),\eta(I)\} \leq \Sigma_{E_2} \max\{\xi(J),\eta(J)\} \tag{2.A.2.1}$$

2) If V is in F, then the following two statements are equivalent:

a) $\{\Sigma_H \min\{\xi(J),\eta(J)\} : H << \{V\}\}$ is bounded below.

b) $\int_V \min\{\xi,\eta\}$ exists.

3) If V is in F, then the following two statements are equivalent:

a) $\{\Sigma_H \max\{\xi(J),\eta(J)\} : H << \{V\}\}$ is bounded above.

b) $\int_V \max\{\xi,\eta\}$ exists.

Theorem 2.1. Suppose that each of ξ and η is in $A(\mathbb{R})(F)$. Then the following statements are equivalent:

1) There is λ in $A(\mathbb{R})(F)$ such that each of $\xi - \lambda$ and $\eta - \lambda$ is in $AB(\mathbb{R})(F)$.

2) $\int_U \max\{\xi,\eta\}$ exists.

3) $\int_U \min\{\xi,\eta\}$ exists.

4) $\xi - \eta$ is in $AB(\mathbb{R})(F)$.

Proof: We first show that 1) implies 2). Suppose that 1) is true. Suppose that D $<< \{U\}$. Then

$$\sum_D \max\{\xi(I),\eta(I)\} = \sum_D \max\{\xi(I) - \lambda(I) + \lambda(I),\eta(I) - \lambda(I) + \lambda(I)\} \le$$
$$\sum_D \max\{\int_I |\xi - \lambda| + \lambda(I),\int_I |\eta - \lambda| + \lambda(I)\} = \sum_D [\max\{\int_I |\xi - \lambda|,\int_I |\eta - \lambda|\} + \lambda(I)] =$$
$$[\sum_D \max\{\int_I |\xi - \lambda|,\int_I |\eta - \lambda|\}] + \lambda(U) \le [\sum_D \int_I |\xi - \lambda| + \int_I |\eta - \lambda|] + \lambda(U) =$$
$$\int_U |\xi - \lambda| + \int_U |\eta - \lambda| + \lambda(U). \tag{2.1.1}$$

It therefore follows from Theorem 2.A.2 that $\int_U \max\{\xi,\eta\}$ exists. Therefore 1) implies 2).

We can use the fact that 1) implies 2) to show that 1) implies 3). Suppose that 1) is true. If ρ is ξ or η, then $\int|\rho - \lambda| = \int|-\rho - -\lambda|$, so that by the fact that 1) implies 2) and some elementary observations, we have the following existence and equality:

$$\int_U \max\{-\xi,-\eta\} = \int_U -\min\{\xi,\eta\} \tag{2.1.2}$$

so that by linearity, $\int_U \min\{\xi,\eta\}$ exists. Therefore 1) implies 3).

Now, suppose that 2) is true. Then each of $\int\max\{\xi,\eta\} - \xi$ and $\int\max\{\xi,\eta\} - \eta$ is in $AB(\mathbb{R})(F)^+ \subseteq AB(\mathbb{R})(F)$. Therefore 2) implies 1).

If 3) is true, then each of $\xi - \int\min\{\xi,\eta\}$ and $\eta - \int\min\{\xi,\eta\}$ is in $AB(\mathbb{R})(F)^+ \subseteq AB(\mathbb{R})(F)$. Therefore 3) implies 1).

Therefore, so far, 1), 2) and 3) are equivalent.

The fact that 4) implies 1) is immediate, since $\eta - \eta = 0$. If 1) is true, then $\xi - \eta = (\xi - \lambda) + (\lambda - \eta)$, which is in $AB(\mathbb{R})(F)$. Therefore 1) implies 4).

Therefore, 1), 2), 3) and 4) are equivalent.

The next theorem, which we use in section 4, partly involves existence assertions that follow from Theorem 2.1.

Theorem 2.2. Suppose that $X \subseteq A(\mathbb{R})(F)$, λ is in $A(\mathbb{R})(F)$ and for each ρ in X, $\lambda - \rho$ is in $AB(\mathbb{R})(F)^+$ ($\rho - \lambda$ is in $AB(\mathbb{R})(F)^+$), so that by Theorem 2.1, for each ρ_1 and ρ_2 in X, each of $\int_U \max\{\rho_1,\rho_2\}$ and $\int_U \min\{\rho_1,\rho_2\}$ exists. Suppose that if each of ρ_1 and ρ_2 is in X, then

$\int\max\{\rho_1,\rho_2\}$ is in X ($\int\min\{\rho_1,\rho_2\}$ is in X). Suppose that μ is the function with domain F given by

$$\mu(I) = \sup\{\rho(I) : \rho \text{ in } X\} \ (\mu(I) = \inf\{\rho(I) : \rho \text{ in } X\}) \tag{2.2.1}$$

Then μ is in A($\mathbb{R}$)(F), $\lambda - \mu$ is in AB($\mathbb{R}$)(F)$^+$ ($\mu - \lambda$ is in AB($\mathbb{R}$)(F)$^+$), $\mu - \rho$ is in AB($\mathbb{R}$)(F) for all ρ in X, and, if $0 < c$, then there is ζ in X such that

$$\int_U |\mu - \zeta| < c. \tag{2.2.2}$$

Indication of proof: The argument is so similar to previous ones (see [2]), albeit in AB($\mathbb{R}$)(F), that we need only make a few salient remarks and leave the rest of the routine details to the reader. We treat the first alternative. The second follows similarly.

Suppose that V and W are mutually exclusive sets of F and $0 < c$. There are ρ_1, ρ_2 and ρ_3 in X such that

$$0 \le \mu(V) - \rho_1(V) < c/3,\ 0 \le \mu(W) - \rho_2(W) < c/3 \text{ and } 0 \le \mu(V\cup W) - \rho_3(V\cup W) < c/3. \tag{2.2.3}$$

By Theorem 2.1, our previous differential equivalence replacement remarks and our hypothesis, we have the following existence, equivalence and inclusion remark:

$$X \text{ contains } \int\max\{\rho_1,\max\{\rho_2,\rho_3\}\} = \int\max\{\rho_1,\rho_2,\rho_3\} \text{ in } A(\mathbb{R})(F). \tag{2.2.4}$$

Let $\delta = \int\max\{\rho_1,\rho_2,\rho_3\}$. Clearly, if Y is V, W or V∪W and ρ' is ρ_1,ρ_2, or ρ_3, respectively, then

$$0 \le \mu(Y) - \delta(Y) \le \mu(Y) - \rho'(Y) < c/3, \tag{2.2.5}$$

so that

$$|\mu(V) + \mu(W) - \mu(V\cup W)| = |\mu(V) + \mu(W) - \delta(V) - \delta(W) + \delta(V\cup W) - \mu(V\cup W)| \le$$
$$|\mu(V) - \delta(V)| + |\mu(W) - \delta(W)| + |\delta(V\cup W) - \mu(V\cup W)| < 3c/3 = c. \tag{2.2.6}$$

Therefore μ is in $A(\mathbb{R})(F)$; the remainder of the inequality and inclusion assertions follow very easily.

We end this section by stating a Bochner – Radon – Nikodym type theorem that we shall use in section 3.

<u>Theorem 2.B.1</u> (see [3]). If α is in $rB(F)$, μ is in $AB(\mathbb{R})(F)^+$, $\int_U \alpha\mu$ exists and $0 < c$, then there is $D << \{U\}$ such that if $H << E << D$, a is an α – function on E and b is an α – function on H, then

$$\sum_E \sum_{H(I)} |a(I) - b(J)|\mu(J) < c. \tag{2.B.1.1}$$

3. Some existence, continuity, inclusion and uniformity theorems.

We begin this section by stating a previous integral existence theorem.

<u>Theorem 3.A.1</u> [1,4]. If γ is in $rB(F)$, each of μ, η and $\mu - \eta$ is in $AB(\mathbb{R})(F)^+$ and $\int_U \gamma\mu$ exists, then $\int_U \gamma\eta$ exists.

We now prove Theorems 3.1 and 3.2, as stated in the introduction.

<u>Proof of Theorem 3.1</u>: Clearly $\int_U \alpha(\xi_2 - \xi_1)$ exists by linearity, and each of $\xi_2 - \xi_1$, $\rho - \xi_1$ and $\xi_2 - \xi_1 - (\rho - \xi_1)$ is in $AB(\mathbb{R})(F)^+$. Therefore, by Theorem 3.A.1, $\int_U \alpha(\rho - \xi_1)$ exists, so that by linearity, $\int_U \alpha\rho$ exists.

<u>Proof of Theorem 3.2</u>: $\rho_1 - \rho_2$ is in $A(\mathbb{R})(F)$. If V is in F, then

$|\rho_1(V) - \rho_2(V)| = \max\{\rho_1(V),\rho_2(V)\} - \min\{\rho_1(V),\rho_2(V)\} \le \xi_2(V) - \xi_1(V) \le \xi_2(U) - \xi_1(U).$

$$(3.2.1)$$

Therefore $\rho_1 - \rho_2$ is in AB($\mathbb{R}$)(F).

If I is in F, then, clearly,

$$|T(\rho_1)(I) - T(\rho_2)(I)| = |\int_I \alpha(\rho_1 - \rho_2)| \le M\int_I |\rho_1 - \rho_2|. \tag{3.2.2}$$

This implies that $T(\rho_1) - T(\rho_2)$ is in AB($\mathbb{R}$)(F) and that $M\int|\rho_1 - \rho_2| - \int|T(\rho_1) - T(\rho_2)|$ is in AB($\mathbb{R}$)(F)$^+$.

The next theorem involves Σ – boundedness and the functionals L and G, as defined and discussed in section 2. In the next section we shall put this theorem to use, employing the functional L; the functional G would serve as well.

<u>Theorem 3.3</u>. If τ is a function from F into [0;1] and each of η_1 and η_2 is in $[\xi_1;\xi_2]$, then

$$\tau\eta_1 + (1-\tau)\eta_2 \tag{3.3.1}$$

is Σ – bounded on U with respect to {U} and each of

$$\int L(\tau\eta_1 + (1-\tau)\eta_2) \text{ and } \int G(\tau\eta_1 + (1-\tau)\eta_2) \tag{3.3.2}$$

is in $[\int\min\{\eta_1,\eta_2\},\int\max\{\eta_1,\eta_2\}]$ and therefore, clearly in $[\xi_1;\xi_2]$.

Proof: Let $\zeta_1 = \int\min\{\eta_1,\eta_2\}$ and $\zeta_2 = \int\max\{\eta_1,\eta_2\}$. If V is in F and D << {V} (and therefore a subset of a refinement of {U}), then

$$\zeta_1(V) = \Sigma_D \zeta_1(I) = \Sigma_D[\tau(I)\zeta_1(I) + (1-\tau(I))\zeta_1(I)] \le \Sigma_D[\tau(I)\eta_1(I) + (1-\tau(I))\eta_2(I)] \le \Sigma_D[\tau(I)\zeta_2(I) + (1-\tau(I))\zeta_2(I)] = \Sigma_D \zeta_2(I) = \zeta_2(V). \tag{3.3.3}$$

It therefore follows that $\tau\eta_1 + (1-\tau)\eta_2$ is Σ – bounded on U with respect to {U} (letting U = V), and for each V in F,

$$\zeta_1(V) \le G(\tau\eta_1 + (1-\tau)\eta_2)(V) \le L(\tau\eta_1 + (1-\tau)\eta_2)(V) \le \zeta_2(V), \tag{3.3.4}$$

so that for each W in F,

$$\zeta_1(W) = \int_W \zeta_1 \le \int_W G(\tau\eta_1 + (1-\tau)\eta_2) \le \int_W L(\tau\eta_1 + (1-\tau)\eta_2) \le \int_W \zeta_2 = \zeta_2(W). \tag{3.3.5}$$

Therefore each of $\int L(\tau\eta_1 + (1-\tau)\eta_2)$ and $\int G(\tau\eta_1 + (1-\tau)\eta_2)$ is in $[\int \min\{\eta_1,\eta_2\}, \int \max\{\eta_1,\eta_2\}]$ and therefore in $[\xi_1;\xi_2]$.

We end this section with a uniformity theorem that we shall use in proving Theorem 4.6.

Theorem 3.4. If $0 < c$, then there is $D << \{U\}$ such that if $E << D$, a is an $\alpha-$ function on E and ρ is in $[\xi_1;\xi_2]$, then

$$\Sigma_E |a(I)\rho(I) - \int_I \alpha\rho| < c.$$

Proof: By Theorem 2.B.1 and differential equivalence there is $D << \{U\}$ such that if $E << D$, a is an $\alpha-$ function on E, $H << E$ and b is an $\alpha-$ function on H, then

$$\Sigma_E \Sigma_{H(I)} |a(I) - b(J)|\,|\xi_2(J) - \xi_1(J)| < c/4 \tag{3.4.1}$$

and

$$\Sigma_E |a(I)\xi_1(I) - \Sigma_{H(I)} b(J)\xi_1(J)| < c/4. \tag{3.4.2}$$

So suppose that $E << D$, a is an $\alpha-$ function on E and ρ is in $[\xi_1;\xi_2]$. It easily follows that there is $H << E$ and an α–function b on H such that if I is in E, then

$$|\int_I \alpha\rho - \Sigma_{H(I)} b(J)\rho(J)| < c/4N, \tag{3.4.3}$$

where N is the number of elements of E, so that

$$\Sigma_E |\int_I \alpha\rho - \Sigma_{H(I)} b(J)\rho(J)| < c/4 \tag{3.4.4}$$

Therefore

$$
\begin{aligned}
&\Sigma_E |a(I)\rho(I) - \int_I \alpha\rho| = \Sigma_E |a(I)(\rho(I) - \xi_1(I)) + a(I)\xi_1(I) - \Sigma_{H(I)} b(J)\xi_1(J) + \Sigma_{H(I)} b(J)\xi_1(J) \\
&- \Sigma_{H(I)} b(J)\rho(J) + \Sigma_{H(I)} b(J)\rho(J) - \int_I \alpha\rho| = \\
&\Sigma_E |[\Sigma_{H(I)} a(I)(\rho(J) - \xi_1(J)) - \Sigma_{H(I)} b(J)(\rho(J) - \xi_1(J))] + [a(I)\xi_1(I) - \Sigma_{H(I)} b(J)\xi_1(J)] + \\
&[\Sigma_{H(I)} b(J)\rho(J) - \int_I \alpha\rho]| \le \\
&\Sigma_E \Sigma_{H(I)} |a(I) - b(J)||\rho(J) - \xi_1(J)| + \Sigma_E |a(I)\xi_1(I) - \Sigma_{H(I)} b(J)\xi_1(J)| + \\
&\Sigma_E |\Sigma_{H(I)} b(J)\rho(J) - \int_I \alpha\rho| < \\
&\Sigma_E \Sigma_{H(I)} |a(I) - b(J)|(\xi_2(J) - \xi_1(J)) + c/4 + c/4 < 3c/4 < c.
\end{aligned}
\tag{3.4.5}
$$

4. Closure, maximum value, minimum value and convergence properties of T.

Proof of Theorem 4.1: There is a function β from F into {0,1} such that if I is in F, then

$$
\beta(I) = \begin{cases} 1 \text{ if } \int_I \alpha\eta_1 \le \int_I \alpha\eta_2 \\ 0 \text{ otherwise} \end{cases} \tag{4.1.1}
$$

Note that if I is in F, then

$$
\beta(I)\int_I \alpha\eta_2 + (1 - \beta(I))\int_I \alpha\eta_1 = \max\{\int_I \alpha\eta_2, \int_I \alpha\eta_1\}. \tag{4.1.2}
$$

By Theorem 3.3, $\beta\eta_2 + (1-\beta)\eta_1$ is Σ − bounded on U with respect to {U} and

$$
\int L(\beta\eta_2 + (1-\beta)\eta_1) \tag{4.1.3}
$$

is in $[\xi_1;\xi_2]$, so that it follows, by Theorem 3.1 and differential equivalence, that for each V in F, the following existence and equality holds:

$$
\int_V [\alpha \int L(\beta\eta_2 + (1-\beta)\eta_1)] = \int_V [\alpha L(\beta\eta_2 + (1-\beta)\eta_1)]. \tag{4.1.4}
$$

Now, suppose that V is in F and 0 < c. There is D << {V} such that if H << D, then

$$\Sigma_H |\max\{\int_I \alpha\eta_1, \int_I \alpha\eta_2\} - \int_I \max\{\alpha\eta_1, \alpha\eta_2\}| < c/8 \qquad (4.1.5)$$

and if a is an $\alpha-$function on H and X is either η_1, η_2 or $L(\beta\eta_2 + (1-\beta)\eta_1)$, then

$$\Sigma_H |a(J)X(I) - \int_J \alpha X| < c/8. \qquad (4.1.6)$$

By statement 5) of Theorem 2.A.1 there is H << D such that

$$\Sigma_H |L(\beta\eta_2 + (1-\beta)\eta_1)(J) - (\beta(J)\eta_2(J) + (1-\beta(J))\eta_1(J))| < c/[8(1+M)]$$

(4.1.7)

Suppose that b is an $\alpha-$function on H. Leaving some minor inequality observations to the reader, we see that

$$|[\int_V \alpha L(\beta\eta_2 + (1-\beta)\eta_1)]_1 - [\int_V \max\{\alpha\eta_1, \alpha\eta_2\}]_2| \leq$$
$$|[\,]_1 - \Sigma_H b(J)L(\beta\eta_2 + (1-\beta)\eta_1)(J)| + |\Sigma_H b(J)L(\beta\eta_2 + (1-\beta)\eta_1)(J) -$$
$$\Sigma_H b(J)(\beta(J)\eta_2(J) + (1-\beta(J))\eta_1(J))| + |\Sigma_H b(J)(\beta(J)\eta_2(J) + (1-\beta(J))\eta_1(J)) -$$
$$\Sigma_H \max\{\int_J \alpha\eta_1, \int_J \alpha\eta_2\}| + |\Sigma_H \max\{\int_J \alpha\eta_1, \int_J \alpha\eta_2\} - [\,]_2| <$$
$$c/8 + \Sigma_H M|L(\beta\eta_2 + (1-\beta)\eta_1)(J) - (\beta(J)\eta_2(J) + (1-\beta(J))\eta_1(J))| + |\Sigma_H b(J)(\beta(J)\eta_2(J) +$$
$$(1-\beta(J))\eta_1(J)) - \Sigma_H (\beta(J)\int_J \alpha\eta_2 + (1-\beta(J))\int_J \alpha\eta_1)| + c/8 \leq$$
$$c/8 + Mc/[8(1+M)] + \Sigma_H [\beta(J)|b(J)\eta_2(J) - \int_J \alpha\eta_2| + (1-\beta(J))|b(J)\eta_1(J) - \int_J \alpha\eta_1|] +$$
$$c/8 <$$
$$c/8 + c/8 + \Sigma_H |b(J)\eta_2(J) - \int_J \alpha\eta_2| + \Sigma_H |b(J)\eta_1(J) - \int_J \alpha\eta_1| + c/8 < c/4 + c/8 + c/8 +$$
$$c/8 < c.$$

(4.1.8)

It therefore follows, once again with reference to differential equivalence, that

$$\int_V [\alpha \int L(\beta\eta_2 + (1-\beta)\eta_1)] = \int_V \alpha L(\beta\eta_2 + (1-\beta)\eta_1) = \int_V \max\{\alpha\eta_1, \alpha\eta_2\} =$$
$$\int_V \max\{\int \alpha\eta_1, \int \alpha\eta_2\}.$$

(4.1.9)

In a similar fashion it follows that there is an element ζ_2 in $[\xi_1;\xi_2]$ such that

$$\int \alpha\zeta_2 = \int \min\{\int \alpha\eta_1, \int \alpha\eta_2\}. \tag{4.1.10}$$

<u>Proof of Theorem 4.2</u>: Suppose that ρ is in $[\xi_1;\xi_2]$. If I is in F, then

$$\xi_1(I) \le \rho(I) \le \xi_2(I), \tag{4.2.1}$$

so that if x is in $\alpha(I)$, then either

$$x\xi_1(I) \le x\rho(I) \le x\xi_2(I) \text{ or } x\xi_2(I) \le x\rho(I) \le x\xi_1(I), \tag{4.2.2}$$

so that

$$\min\{x\xi_1(I), x\xi_2(I)\} \le x\rho(I) \le \max\{x\xi_1(I), x\xi_2(I)\}. \tag{4.2.3}$$

This clearly implies that each of $\int \max\{\alpha\xi_1,\alpha\xi_2\} - \int \alpha\rho$ and $\int \alpha\rho - \int \min\{\alpha\xi_1,\alpha\xi_2\}$ is in $AB(\mathbb{R})(F)^+$.

<u>Proof of Theorem 4.3</u>: The proof of Theorem 4.3 follows somewhat the pattern of that of Theorem 4.1. We first do the "minimal" extremal part.

By Theorem 4.1 there is ζ_1 and ζ_2, each in $[\xi_1;\xi_2]$, such that

$$\int \alpha\zeta_1 = \int \max\{\alpha\xi_1,\alpha\xi_2\} \text{ and } \int \alpha\zeta_2 = \int \min\{\alpha\xi_1,\alpha\xi_2\}. \tag{4.3.1}$$

There is a function τ from F into $[0;1]$ such that if I is in F, then

$$|\lambda(I) - [\tau(I)\int_I \alpha\zeta_1 + (1-\tau(I))\int_I \alpha\zeta_2]| = \inf\{|\lambda(I) - [t\int_I \alpha\zeta_1 + (1-t)\int_I \alpha\zeta_2]| : 0 \le t \le 1\}. \tag{4.3.2}$$

Let $L = L(\tau\zeta_1 + (1-\tau)\zeta_2)$.

Again, by Theorem 3.3, $\int L$ is in $[\xi_1;\xi_2]$, so that for each V in F, by Theorem 3.1 and differential equivalence, we have the following existence and equality:

$$\int_V \alpha \int L = \int_V \alpha L. \tag{4.3.3}$$

Now, suppose that ρ is in $[\xi_1;\xi_2]$ and V is in F. Suppose that $0 < c$. There is $D << \{V\}$ such that if $H << D$, then

$$0 \le \int_V |\lambda - \int \alpha L| - \Sigma_H |\lambda(J) - \int_J \alpha L| < c/8, \tag{4.3.4}$$

and if b is an α − function on H and X is L, ζ_1 or ζ_2, then

$$\Sigma_H |\int_J \alpha X - b(J)X(J)| < c/8. \tag{4.3.5}$$

By Theorem 2.A.1, statement 5), there is H << D such that

$$\Sigma_H |L(J) - (\tau(J)\zeta_1(J) + (1 - \tau(J))\zeta_2(J))| < c/[8(1 + M)]. \tag{4.3.6}$$

Now, by Theorem 4.2, for each J in H,

$$\int_J \alpha\zeta_2 \leq \int_J \alpha\rho \leq \int_J \alpha\zeta_1, \tag{4.3.7}$$

so that there is t(J) in [0;1] such that

$$\int_J \alpha\rho = t(J)\int_J \alpha\zeta_1 + (1 - t(J))\int_J \alpha\zeta_2. \tag{4.3.8}$$

There is an α – function b on H. It follows that

$$\int_V |\lambda - \int \alpha\rho| \geq \Sigma_H |\lambda(J) - \int_J \alpha\rho| = \Sigma_H |\lambda(J) - (t(J)\int_J \alpha\zeta_1 + (1 - t(J))\int_J \alpha\zeta_2)| \geq$$
$$\Sigma_H |\lambda(J) - (\tau(J)\int_J \alpha\zeta_1 + (1 - \tau(J))\int_J \alpha\zeta_2)| = \Sigma_H |\lambda(J) - \int_J \alpha L + \int_J \alpha L - b(J)L(J) +$$
$$b(J)L(J) - b(J)(\tau(J)\zeta_1(J) + (1 - \tau(J))\zeta_2(J)) + b(J)(\tau(J)\zeta_1(J) + (1 - \tau(J))\zeta_2(J)) -$$
$$(\tau(J)\int_J \alpha\zeta_1 + (1 - \tau(J))\int_J \alpha\zeta_2)| \geq \Sigma_H |\lambda(J) - \int_J \alpha L| - \Sigma_H |\int_J \alpha L - b(J)L(J)| -$$
$$\Sigma_H |b(J)|\,|L(J) - (\tau(J)\zeta_1(J) + (1 - \tau(J))\zeta_2(J))| - \Sigma_H (\tau(J)|b(J)\zeta_1(J) - \int_J \alpha\zeta_1| +$$
$$(1 - \tau(J))|b(J)\zeta_2(J) - \int_J \alpha\zeta_2|) > \int_V |\lambda(J) - \int_J \alpha L| - c/8 - Mc/[8(1 + M)] - c/8 - c/8 \geq$$
$$\int_V |\lambda - \int \alpha L| - 4c/8. \tag{4.3.9}$$

Therefore

$$\int_V |\lambda - \int \alpha\rho| \geq \int_V |\lambda - \int \alpha L| = \int_V |\lambda - \int(\alpha\int L)|. \tag{4.3.10}$$

Now, the "maximal" extremal portion of this proof is, in certain respects, quite similar to the preceding work. We indicate modifications and leave routine details to the reader. ζ_1 and ζ_2 are as in the "minimal" extremal part.

If ρ is in $[\xi_1;\xi_2]$ and V is in F, then

$$|\lambda(V) - \int_V \alpha\rho| \leq \int_U |\lambda - \int \alpha\rho| \leq [\int_U |\lambda - \int \alpha\zeta_1|]_1 + \int_U |\int \alpha\zeta_1 - \int \alpha\rho| \leq [\,]_1 + M\int_U |\zeta_1 - \rho|$$
$$\leq [\,]_1 + M(\xi_2(U) - \xi_1(U)). \tag{4.3.11}$$

Note that it follows from Theorem 3.3 that if $0 \leq t \leq 1$, then $t\zeta_1 + (1 - t)\zeta_2$ is in $[\xi_1;\xi_2]$.

Thus there is a function ω from F into [0;1] such that if I is in F, then

$$|\lambda(I)-(\omega(I)\int_I\alpha\zeta_1+(1-\omega(I))\int_I\alpha\zeta_2)| = \sup\{|\lambda(I)-(t\int_I\alpha\zeta_1+(1-t)\int_I\alpha\zeta_2)| : 0\le t\le 1\} \tag{4.3.12}$$

Let $L = L(\omega\zeta_1+(1-\omega)\zeta_2)$.

For ω in place of τ we state inequalities and equations (4.3.3) through (4.3.8), immediately preceding the ω counterpart of (4.3.4) with the following:

$$0\le\int_V|\lambda-\int\alpha\rho|-\Sigma_H|\lambda(J)-\int_J\alpha\rho|<c/8 \tag{4.3.13}$$

Now, the ω counterpart of inequality (4.3.9) begins as follows:

$$\int_V|\lambda-\int\alpha\rho|<c/8+\Sigma_H|\lambda(J)-\int_J\alpha\rho| = c/8+\Sigma_H|\lambda(J)-(t(J)\int_J\alpha\zeta_1+(1-t(J))\int_J\alpha\zeta_2)|$$

$\le c/8+\Sigma_H|\lambda(J)-(\omega(J)\int_J\alpha\zeta_1+(1-\omega(J))\int_J\alpha\zeta_2)| =$ .... continues as in (4.3.9), but with "$\le$" instead of "$\ge$" and "+" between the sums instead of "–" and culminates with "$\le\int_V|\lambda-\int\alpha L|+5c/8$".

(4.3.14)

Therefore

$$\int_V|\lambda-\int\alpha\rho|\le\int_V|\lambda-\int\alpha L| = \int_V|\lambda-\int(\alpha\int L)|. \tag{4.3.15}$$

Proof of Theorem 4.4: There is a function τ from F into [0;1] such that if I is in F, then

$$\lambda(I)=\tau(I)\int_I\alpha\eta_1+(1-\tau(I))\int_I\alpha\eta_2. \tag{4.4.1}$$

Much as before, let $L = L(\tau\eta_1+(1-\tau)\eta_2)$, so that by Theorem 3.3, $\int L$ is in $[\xi_1;\xi_2]$, in particular in $[\int\min\{\eta_1;\eta_2\};\int\max\{\eta_1;\eta_2\}]$, so that by Theorem 3.1 and differential equivalence, the following existence and equality holds:

$$\int_V(\alpha\int L)=\int_V\alpha L. \tag{4.4.2}$$

Suppose that V is in F and $0 < c$. There is $D << \{U\}$ such that if $H << D$, b is an α – function H and X is η_1, η_2 of L, then

$$\Sigma_H |\int_J \alpha X - b(J)X(J)| < c/8. \tag{4.4.3}$$

By statement 5) of Theorem 2.A.1, there is H << D such that

$$\Sigma_H |L(J) - (\tau(J)\eta_1(J) + (1 - \tau(J))\eta_2(J)| < c/[8(1 + M)]. \tag{4.4.4}$$

Now,

$$|\lambda(V) - \int_V \alpha L| = |\Sigma_H [\tau(J)\int_J \alpha\eta_1 + (1 - \tau(J))\int_J \alpha\eta_2] -$$

$$\Sigma_H [\tau(J)b(J)\eta_1(J) + (1 - \tau(J))b(J)\eta_2(J)] + \Sigma_H [b(J)[\tau(J)\eta_1(J) + (1 - \tau(J))\eta_2(J)] -$$

$$\Sigma_H b(J)L(J) + \Sigma_H b(J)L(J) - \int_V \alpha L| \leq$$

$$\Sigma_H [\tau(J)|\int_J \alpha\eta_1 - b(J)\eta_1(J)| + (1 - \tau(J))|\int_J \alpha\eta_2 - b(J)\eta_2(J)| +$$

$$\Sigma_H |b(J)||\tau(J)\eta_1(J) + (1 - \tau(J))\eta_2(J) - L(J)| + \Sigma_H |b(J)L(J) - \int_J \alpha L| <$$

$$c/8 + c/8 + Mc/[8(1 + M)] + c/8 < c/2 < c. \tag{4.4.5}$$

Therefore

$$\lambda(V) = \int_V \alpha L = \int_V (\alpha \int L). \tag{4.4.6}$$

<u>Proof of Corollary 4.4</u>: It clearly follows from the hypothesis, Theorem 4.2 and Theorem 4.1 that the hypothesis of Theorem 4.4 is satisfied, and therefore the conclusion.

<u>Proof of Theorem 4.5</u>: Suppose that each of ρ_1 and ρ_2 is in $[\xi_1;\xi_2]$ and for i = 1,2, $\int\alpha\rho_i$ is in $[\int\alpha\eta_1;\int\alpha\eta_2]$. We shall show that if P is max or min, then $\int\alpha P\{\rho_1,\rho_2\}$ is in $[\int\alpha\eta_1;\int\alpha\eta_2]$. Suppose that V is in F and $0 < c$. There is D << {V} such that if $E \subseteq H$ << D, a is an α − function on E and i = 1,2, then

$$\Sigma_E |a(I)\rho_i(I) - \int_I \alpha\rho_i| < c/3 \text{ and } \Sigma_E |a(I)\max\{\rho_1(I),\rho_2(I)\} - \int_I \alpha\max\{\rho_1,\rho_2\}| < c/3 \tag{4.5.1}$$

Let $D_1 = \{I : I \text{ in } D, \rho_2(I) \leq \rho_1(I)\}$ and $D_2 = D - D_1$. There is an α − function b on D. Now,

$$\int_V \alpha\eta_1 = \Sigma_{D_1} \int_I \alpha\eta_1 + \Sigma_{D_2} \int_I \alpha\eta_1 \leq [\Sigma_{D_1} \int_I \alpha\rho_1 + \Sigma_{D_2} \int_I \alpha\rho_2]_1 \leq \Sigma_{D_1} \int_I \alpha\eta_2 + \Sigma_{D_2} \int_I \alpha\eta_2 = \int_V \alpha\eta_2 \tag{4.5.2}$$

and

$$|[\,]_1 - \int_V \alpha\max\{\rho_1,\rho_2\}| = |\Sigma_{D_1} (\int_I \alpha\rho_1 - b(I)\rho_1(I)) + \Sigma_{D_2} (\int_I \alpha\rho_2 - b(I)\rho_2(I)) + \Sigma_{D_1} b(I)\max\{\rho_1(I),\rho_2(I)\} + \Sigma_{D_2} b(I)\max\{\rho_1(I),\rho_2(I)\} - \int_V \alpha\max\{\rho_1,\rho_2\}| \leq \Sigma_{D_1} |\int_I \alpha\rho_1 - b(I)\rho_1(I)| + \Sigma_{D_2} |\int_I \alpha\rho_2 - b(I)\rho_2(I)| + |\Sigma_D b(I)\max\{\rho_1(I),\rho_2(I)\} - \int_V \alpha\max\{\rho_1,\rho_2\}| < c/3 + c/3 + c/3 = c. \tag{4.5.3}$$

Therefore

$$-c < \int_V \alpha\max\{\rho_1,\rho_2\} - [\,]_1 < c, \tag{4.5.4}$$

so that adding appropriate terms in (4.5.2), we have

$$\int_V \alpha\eta_1 - c < \int_V \alpha\max\{\rho_1,\rho_2\} < \int_V \alpha\eta_2 + c. \tag{4.5.5}$$

Therefore

$$\int_V \alpha\eta_1 \leq \int_V \alpha\max\{\rho_1,\rho_2\} \leq \int_V \alpha\eta_2. \tag{4.5.6}$$

Therefore each of $\int\alpha\eta_2 - \int\alpha\max\{\rho_1,\rho_2\}$ and $\int\alpha\max\{\rho_1,\rho_2\} - \int\alpha\eta_1$ is in $AB(\mathbb{R})(F)^+$, and, in a similar fashion, each of $\int\alpha\eta_2 - \int\alpha\min\{\rho_1,\rho_2\}$ and $\int\alpha\min\{\rho_1,\rho_2\} - \int\alpha\eta_1$ is in $AB(\mathbb{R})(F)^+$.

It therefore follows from Theorem 2.2 that there is μ_1 and μ_2, each in $[\xi_1;\xi_2]$, such that $\mu_2 - \mu_1$ is in $AB(\mathbb{R})(F)^+$,

$$\{\rho : \rho \text{ in } [\xi_1;\xi_2],\ \int\alpha\rho \text{ in } [\int\alpha\eta_1;\int\alpha\eta_2]\}_1 \subseteq [\mu_1;\mu_2], \tag{4.5.7}$$

and, if $0 < c$, then there is ζ_1 and ζ_2, each in $\{\ \}_1$, such that if $i = 1,2$, then

$$\int_U |\mu_i - \zeta_i| = |\mu_i(U) - \zeta_i(U)| < c/(1 + M), \tag{4.5.8}$$

so that, by Theorem 3.2,

$$\int_U \left| \int \alpha\mu_i - \int \alpha\zeta_i \right| \leq M \int_U |\mu_i - \zeta_i| \leq Mc/(1 + M) < c. \tag{4.5.9}$$

It therefore follows routinely that if i = 1,2, then $\int \alpha\mu_i$ is in $[\int \alpha\eta_1;\int \alpha\eta_2]$, so that μ_i is in $\{\ \}_1$.

Now, suppose that ρ is in $[\mu_1;\mu_2]$. If I is in F and a(I) is in α(I), then either

$$\min\{a(I)\mu_1(I),a(I)\mu_2(I)\} \leq a(I)\mu_1(I) \leq a(I)\rho(I) \leq a(I)\mu_2(I) \leq \max\{a(I)\mu_1(I),a(I)\mu_2(I)\}$$

or

$$\min\{a(I)\mu_1(I),a(I)\mu_2(I)\} \leq a(I)\mu_2(I) \leq a(I)\rho(I) \leq a(I)\mu_1(I) \leq \max\{a(I)\mu_1(I),a(I)\mu_2(I)\}. \tag{4.5.10}$$

It therefore follows that if V is in F, then

$$\int_V \alpha\eta_1 \leq \int_V \min\int \alpha\mu_1,\int \alpha\mu_2\} \leq \int_V \alpha\rho \leq \int_V \max\int \alpha\mu_1,\int \alpha\mu_2\} \leq \int_V \alpha\eta_2, \tag{4.5.11}$$

so that ρ is in $\{\ \}_1$.

Therefore $[\mu_1;\mu_2] \subseteq \{\ \}_1$, so that

$$[\mu_1;\mu_2] = \{\rho : \rho \text{ in } [\xi_1;\xi_2], \int \alpha\rho \text{ in } [\int \alpha\eta_1;\int \alpha\eta_2]\}, \tag{4.5.12}$$

so that

$$T([\mu_1;\mu_2]) \subseteq [\int \alpha\eta_1;\int \alpha\eta_2]. \tag{4.5.13}$$

Now, suppose that λ is in $[\int \alpha\eta_1;\int \alpha\eta_2]$. By Theorem 4.4 there is an element ζ of $[\xi_1;\xi_2]$ such that

$$\lambda = \int \alpha\zeta \tag{4.5.14}$$

Since $\int \alpha\zeta$ is in $[\int \alpha\eta_1;\int \alpha\eta_2]$, it follows by (4.5.12) that ζ is in $[\mu_1;\mu_2]$. Therefore

$$[\int \alpha\eta_1;\int \alpha\eta_2] \subseteq T([\mu_1;\mu_2]). \tag{4.5.15}$$

Therefore

$$T([\mu_1;\mu_2]) = [\int \alpha\eta_1;\int \alpha\eta_2]. \tag{4.5.16}$$

We end this paper by proving Theorem 4.6, as stated in the introduction.

Proof of Theorem 4.6: The argument, as is usually the case for this type of theorem, uses a uniformity of integrability condition. In this case, Theorem 3.4 gives such a condition and our demonstration is brief.

First, showing that ζ is in $[\xi_1;\xi_2]$ is quite routine and we leave the details to the reader.

Now, suppose that $0 < c$. By Theorem 3.4, there is $D << \{U\}$ such that if $E << D$, a is an α − function on E and ρ is in $[\xi_1;\xi_2]$, then

$$|[\Sigma_E a(I)\rho(I)] - T(\rho)(U)| \leq \Sigma_E |a(I)\rho(I) - \int_I \alpha\rho| < c/3. \tag{4.6.1}$$

There is an α − function b in D. There is a positive integer N such that if m is a positive integer $\geq N$, then

$$\Sigma_D |\zeta(I) - \eta_m(I)| < c/[3(1 + M)], \tag{4.6.2}$$

so that

$$|T(\zeta)(U) - T(\eta_m)(U)| \leq |T(\zeta)(U) - \Sigma_D b(I)\zeta(I)| + |\Sigma_D b(I)\zeta(I) - \Sigma_D b(I)\eta_m(I)| + |\Sigma_D b(I)\eta_m(I) - T(\eta_m)(U)| \leq c/3 + \Sigma_D |b(I)||\zeta(I) - \eta_m(I)| + c/3 < c/3 + Mc/[3(1 + M)] + c/3 < c. \tag{4.6.3}$$

Therefore

$$T(\eta_m)(U) \to T(\zeta)(U) \text{ as } n \to \infty. \tag{4.6.4}$$

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