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ON SETS OF POINTS OF APPROXIMATE SEMICONTINUITY IN EUCLIDEAN SPACES

In the whole paper we will work in the **n**-dimensional Euclidean space $\mathbb{R}^n$, where $n \geq 1$ is arbitrary but fixed. We denote by d the ordinary density topology on $\mathbb{R}^n$, i.e. a set A is d-open iff

$$\lim_{r \searrow 0} \frac{\lambda(B(x,r)\setminus A)}{\lambda(B(x,r))} = 0$$

holds for each $x \in A$.(Here λ denotes the (outer) Lebesque measure over $\mathbb{R}^n$ and $B(x,r) = \{ y \in \mathbb{R}^n;\ \| x - y \| \leq r \}$.)

For a given function $f:\mathbb{R}^n \longrightarrow \mathbb{R}$ we define the following sets:

$C(f) = \{ x \in \mathbb{R}^n;\ f$ is continuous at $x \}$,

$C_d(f) = \{ x \in \mathbb{R}^n;\ f$ is approximately continuous at $x \}$,

$S_d^+(f) = \{ x \in \mathbb{R}^n;\ \mathbf{ap}\ \mathbf{\lim\sup}_{t \longrightarrow x} f(t) \leq f(x) \}$, this is the set of points of approximate upper semicontinuity,

$T_d^+(f) = \{ x \in \mathbb{R}^n;\ \mathbf{ap}\ \mathbf{\lim\sup}_{t \longrightarrow x} f(t) < f(x) \}$, and similarly

$S_d^-(f) = \{ x \in \mathbb{R}^n;\ \mathbf{ap}\ \mathbf{\lim\inf}_{t \longrightarrow x} f(t) \geq f(x) \}$, and

$T_d^-(f) = \{ x \in \mathbb{R}^n;\ \mathbf{ap}\ \mathbf{\lim\inf}_{t \longrightarrow x} f(t) > f(x) \}$.

The prefix "ap" expresses that we consider lower and upper limits with respect to the topology d, see [5;Cor.6.22.]. Finally we put

$$\mathbf{CH}_d(f) = (C_d(f),\ S_d^+(f),\ T_d^+(f),\ S_d^-(f),\ T_d^-(f)) .$$

to be the d-characterizing quintuple of the function f.

In [2] Z.Grande stated

Problem A Let C,S^+,T^+ be subsets of the real line such that C is Lebesque measurable, $C \subset S^+\setminus T^+$, $T^+ \subset S^+$, and $S^+\setminus C$ has inner Lebesque measure zero. Must there exist a function $f:\mathbb{R}\to\mathbb{R}$ with $C = C_d(f)$, $S^+= S^+_d(f)$, and $T^+= T^+_d(f)$?

T.Natkaniec studied the question for which quintuples (C,S^+,T^+,S^-,T^-) of subsets of the real line a function $f:\mathbb{R}\to\mathbb{R}$ satisfying the equality $\mathbf{CH}_d(f) = (C,S^+,T^+,S^-,T^-)$ exists in his PhD-Thesis [7] and in his paper [8]. He formulated

Problem B Let $f:\mathbb{R}\to\mathbb{R}$ be arbitrary. Must there exist a set $D \subset \mathbb{R}$ of type G_δ such that $C_d(f) = D\setminus(T^+\cup T^-)$?

The goal of the presented paper is to derive a complete description of all "extended" d-characterizing sextuples

$$(C(f),\ C_d(f),\ S^+_d(f),\ T^+_d(f),\ S^-_d(f),\ T^-_d(f))$$

in $\mathbb{R}^n$. However, in **Corollary 9.a)** the description of the original quintuples $\mathbf{CH}_d(f)$ is given. Our main result is

1.Theorem Let $n \geq 1$ and let d be the ordinary density topology on $\mathbb{R}^n$. For a given sextuple (C_e,C,S^+,T^+,S^-,T^-) of subsets of $\mathbb{R}^n$ the following two statements are equivalent:

a) It holds (i) $S^+\cap S^-= C$

(ii) $T^+\subset S^+\setminus C$, $T^-\subset S^-\setminus C$ and both the sets T^+ and T^- have Lebesque measure zero.

(iii) Both the sets $S^+\setminus C$ and $S^-\setminus C$ have inner Lebesque measure zero.

(iv) There is some set $D \subset \mathbb{R}^n$ of type $F_{\sigma\delta}$ such that $C = D\setminus(T^+\cup T^-)$.

(v) C_e is of type G_δ and contained in C.

(vi) The closure of C_e contains each point $x \in \overline{C \setminus (T^+ \cup T^-)}$ at which C is of the second category (i.e. $B(x,r) \cap C$ is a second category set if $r > 0$).

b) There is some $f:\mathbb{R}^n \to \mathbb{R}$ such that $C(f) = C_e$ and

$$\mathbf{CH}_d(f) = (C, S^+, T^+, S^-, T^-).$$

From this theorem (essentially from **Corollary 9.a.**) negative answers to both **Problem A** and **Problem B** easily follow. Indeed, we choose a set $M \subset \mathbb{R}$ such that $\lambda(M) = 0$ and that M is not of type $G_{\delta\sigma}$. Then we put $C = \mathbb{R} \setminus M$, $S^+ = \mathbb{R}$, and $T^+ = \emptyset$, these sets clearly satisfy the assumption of **Problem A**. But if the required function f exists then we obtain from **Theorem 1** that $S^- = C$, $T^- = \emptyset$ and $D = C = \mathbb{R} \setminus M$ is of type $F_{\sigma\delta}$, a contradiction.(This idea occurs already in [9].) To disprove the conjecture of **Problem B** it suffices to set $C_e = \emptyset$, $C = S^+ = S^- = \mathbb{Q}$ (rationals), and $T^+ = T^- = \emptyset$. This sextuple satisfies the condition **a)** of **Theorem 1** , let f be a corresponding function from statement **b)**. According to **a.iv)** we obtain that $D = C_d(f) = \mathbb{Q}$, but $\mathbb{Q}$ is not of type G_δ.

However, we can not directly turn to the proof of **Theorem 1**; we must at first derive some helpful technical statements. We begin with some notation,agreements and facts. In the sequel topological notations referring to the topology d will be qualified by the prefix "d" to distinguish them from those pertaining to the Euclidean topology, for example $\overline{}^{d}$ resp. $\mathbf{Der}_d$ denotes the closure resp. the set of cluster points in the topology d . Further, in what follows words like measure, measurable and so on pertain to the Lebesque measure λ . For $x \in \mathbb{R}^n$, and $A \subset \mathbb{R}^n$ we put

$$\mathbf{dist}(x,A) = \mathbf{inf}(\{1\} \cup \{ r \,;\, B(x,r) \cap A \neq \emptyset \}).$$

If f maps $\mathbb{R}^n$ into $\mathbb{R}$ and if $x \in \mathbb{R}^n$ then we define

$$\operatorname{osc}(f,x) = \lim_{r \searrow 0} \sup\{ |f(y) - f(z)|;\ y,z \in B(x,r) \}.$$

We note that the function $x \longrightarrow \operatorname{osc}(f,x) \in [0,\infty]$ is upper semicontinuous.

In our approach the complete **Lusin-Menchoff** property of the topology d will play the key rôle (however, compare with **Remark 10**). It says the following. Whenever $P \subset \mathbb{R}^n$, F and F_d are disjoint subsets of P, and F resp. F_d is closed resp. d-closed in P then there exist disjoint subsets G and G_d of P such that $F \subset G_d$, $F_d \subset G$ and G resp. G_d is open resp. d-open in P, see [5; 3.18. and 6.34(B)a.]. This property reminds normality and it really ensures the existence of "sufficiently nice" extensions of certain functions. A comprehensive treatment of these questions can be found in [5]. We will use

2.Fact Let $P \subset \mathbb{R}^n$ be arbitrary. Assume that $S \subset P$ is both d-closed and of type G_δ in P. If $g: S \longrightarrow [-1, 1]$ is both approximately continuous and a **Baire one** function on S then there exists an approximately continuous function $f: P \longrightarrow [-1, 1]$ such that

i) $f(x) = g(x)$ if $x \in S$

ii) f is continuous on $P \setminus \overline{S}$

iii) f is continuous at those points of S at which g is.

Indeed, this statement quite easily follows from [5; Theo.3.30.] and [5; Exer.3.E.21.(b)].

Further we recall the following important fact which seems not to be known very well.

3.Fact Let $A \subset \mathbb{R}^n$ be arbitrary. Then there exists a sequence A_i,

$i \geq 0$, of mutually disjoint subsets of A with $\lambda(A) = \lambda(A_i)$ for any i.

This is mentioned in [10] and based on a result of Lusin in [6]. The paper [6] is really remarkable since it works only with the **axiom of choice** and offers an approach independent on and more general than Vitali's or Bernstein's constructions. Further on, in part b) of **Lemma 4** we use an idea from [1].

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4.Lemma Suppose that $M \subset U \subset \mathbb{R}^n$, $\lambda(M) = 0$ and U is open.

a) Then we can find a *d*-closed G_δ-set F such that $M \subset \mathbf{Int}\, F \subset F \subset U$ and that $F \cup \overline{M}$ is closed.

b) For each $\varepsilon > 0$ there is an open set V such that $M \subset V \subset U$ and that for each $x \in M$ some $r \in (0,\varepsilon)$ satisfying the inequality $\lambda\big(B(x,r) \cap (U\setminus V)\big) \geq (1-\varepsilon)\cdot\lambda(B(x,r))$ exists.

Proof

a) Because M is *d*-closed, the **Lusin-Menchoff** property ensures the existence of an open set U_1 with $M \subset U_1 \subset \overline{U_1}^d \subset U$ We put $W = \{\, x \,;\, \mathbf{dist}(x,M) < \mathbf{dist}(x,\mathbb{R}^n\setminus U_1) \,\}$. Then $M \subset W \subset U_1$, W is open, and $\overline{W} \subset \{\, x \,;\, \mathbf{dist}(x,M) \leq \mathbf{dist}(x,\mathbb{R}^n\setminus U_1)\} \subset \overline{M} \cup U_1$. Therefore we can choose a G_δ-set S such that $\lambda(S\setminus[(\overline{W}\setminus\overline{M}) \cup W]) = 0$ and $\overline{M} \cap \overline{(\overline{W}\setminus\overline{M}) \cup W}^d \subset S \subset U \cap \overline{M}$. Consequently S is *d*-closed. Now the required set is defined by $F = S \cup (\overline{W}\setminus\overline{M})$. Then F is of type G_δ, $M \subset W \subset \mathbf{Int}\, F \subset F \subset U$, and $F \cup \overline{M} = \overline{W \cup M}$ is closed. Since

$$\overline{F}^d \subset S \cup \overline{(\overline{W}\setminus\overline{M})}^d \subset S \cup \Big[\overline{(\overline{W}\setminus\overline{M})}^d \cap \overline{M}\ \Big] \cup \Big[\overline{(\overline{W}\setminus\overline{M})}^d \setminus\overline{M}\ \Big] \subset S \cup (\overline{W}\setminus\overline{M}) = F\ ,$$

we conclude that F is also *d*-closed.

b) For any $x \in U$ we define $r_x = \min\{\frac{\varepsilon}{2}, \mathbf{dist}(x, \mathbb{R}^n \setminus U)\}$. Clearly, if $\delta \in (0, \frac{\varepsilon}{2}]$ then there is some open set $V(\delta)$ with $\{x \in M ; r_x \geq 4\delta\} \subset V(\delta) \subset \{x \in U ; r_x > 2\delta\}$ and $\lambda(V(\delta)) \leq \delta \cdot \lambda(B(0,\delta))$. This choice of $V(\delta)$ ensures that for each $x \in U$ $\lambda\big(B(x,r_x) \cap V(\delta)\big) \leq \delta \cdot \lambda(B(x,r_x))$ holds. Indeed, if $r_x \geq \delta$ then $\delta \cdot \lambda(B(x,r_x)) \geq \delta\, \lambda(B(x,\delta)) \geq \lambda(V(\delta))$ and in case $r_x < \delta$ we have $B(x,r_x) \cap V(\delta) \subset B(x,\delta) \cap \{y \in U ; r_y > 2\delta\} = \emptyset$.

Since for any $x \in U$ $\lambda(B(x,r_x) \setminus U) = 0$, the set $V = \bigcup_{k=1}^{\infty} V(2^{-k}\varepsilon)$ satisfies $\lambda\big(B(x,r_x) \cap (U \setminus V)\big) = \lambda(B(x,r_x)) - \lambda\big(B(x,r_x) \cap V\big) \geq (1-\varepsilon)\, \lambda(B(x,r_x))$ $\square$

5.Proposition Let $M \subset \mathbb{R}^n$ be a set of type G_δ satisfying $\lambda(M) = 0$. Then

a) there exist d-open sets M_1 and M_2 of type F_σ such that $M_1 \cup M_2 = \mathbb{R}^n \setminus M$ and that for $i = 1;2$ $M_i \setminus \overline{M}$ is open and $\mathbf{Der}_d(\mathbb{R}^n \setminus M_i) \supset M$.

b) there exists a function $f : (\mathbb{R}^n \setminus M) \longrightarrow [-1, 1]$ such that f is approximately continuous on its domain and is continuous on $\mathbb{R}^n \setminus \overline{M}$ and that for each $x \in M$ the following is true:

$$\mathbf{ap}\lim_{t \to x}\mathbf{inf}\, f(t) = -1 \quad \text{and} \quad \mathbf{ap}\lim_{t \to x}\mathbf{sup}\, f(t) = 1.$$

Proof

a) Let $M = \bigcap_{k=1}^{\infty} U_k$, where the $\ldots U_k \supset U_{k+1} \supset \ldots$ are open. We put $G_0 = \mathbb{R}^n$ and now we assume that for some $k \geq 0$ the sets $G_0, G_1, \ldots, G_k$ and $F_0, \ldots, F_{k-1}$ have already been chosen. According to **Lemma 4.a)** we can choose a d-closed set F_k of type G_δ such that $F_k \cup \overline{M}$ is closed and that $M \subset \mathbf{Int}\, F_k \subset F_k \subset G_k \cap U_k$. Furthermore, we can find G_{k+1} open such that $M \subset G_{k+1} \subset \mathrm{Int}\, F_k$ and for any $x \in M$ there is some $r \in (0, \frac{1}{k+1})$ with

$$\lambda\big(B(x,r) \cap (F_k \setminus G_{k+1})\big) > (1/2)\cdot\lambda(B(x,r)) \qquad (*)$$

We proceed in this way and finally we define $M_1 = \bigcup_{k=0}^{\infty}(G_{2k} \setminus F_{2k+1})$ and $M_2 = \bigcup_{k=0}^{\infty}(G_{2k+1} \setminus F_{2k+2})$. It is obvious that M_1 and M_2 are d-open and of type F_σ, moreover $M_1 \cup M_2 = \bigcup_{k=0}^{\infty}(G_k \setminus G_{k+1}) = \mathbb{R}^n \setminus M$. For $i = 1, 2$ the set $M_i \setminus \overline{M} = \bigcup_{k=0}^{\infty}\big(G_{2k+(i-1)} \setminus (F_{2k+1} \cup \overline{M})\big)$ is open. Since $(F_{2k} \setminus G_{2k+1}) \cap M_2$ and $(F_{2k-1} \setminus G_{2k}) \cap M_1$ are empty if $k \geq 1$, it follows from $(*)$ that for any $x \in M$ and $i = 1, 2$

$$\limsup_{r \searrow 0} \frac{\lambda\big(B(x,r) \cap (\mathbb{R}^n \setminus M_i)\big)}{\lambda(B(x,r))} \geq \frac{1}{2}.$$

Therefore $M \subset \mathbf{Der}_d(\mathbb{R}^n \setminus M_1) \cap \mathbf{Der}_d(\mathbb{R}^n \setminus M_2)$; part a) is proved.

b) We put $P = \mathbb{R}^n \setminus M$ and $F_1 = P \setminus M_1$, $F_2 = P \setminus M_2$, where the M_i are taken from statement a). Now we define $S = F_1 \cup F_2$ and the function $g : S \longrightarrow [-1, 1]$ by

$$g(x) = \begin{cases} 1 & \text{if } x \in F_1 \\ -1 & \text{if } x \in F_2 \end{cases} \qquad \text{(of course } F_1 \cap F_2 = \emptyset \text{).}$$

Both F_1 and F_2 are d-closed and of type G_δ in P, moreover $F_1 \setminus \overline{M}$ and $F_2 \setminus \overline{M}$ are closed in $P \setminus \overline{M}$. Obviously, P, S, and g fulfill the assumptions of **Fact 2**. Now any function f from the conclusion of this statement has all properties required for it. Indeed, because g is continuous at any point $x \in S \cap (P \setminus \overline{M})$ and because $P \setminus \overline{S} \supset (P \setminus \overline{M}) \setminus S$, f is continuous on the set $P \setminus \overline{M} = \mathbb{R}^n \setminus \overline{M}$ and the statement for the upper and lower approximate limits immediately follows from $\mathbf{Der}_d F_1$ $\mathbf{Der}_d F_2 \supset M$, $f(F_1) = \{-1\}$, and $f(F_2) = \{1\}$. $\square$

<u>**6.Proposition**</u> Let (C, S^+, S^-) be a triple of subsets of $\mathbb{R}^n$ satisfying

i) $C = S^+ \cap S^-$ is d-open and of type F_σ, and

ii) $S^+\setminus C$ and $S^-\setminus C$ have inner measure zero.

Then there exists $f:\mathbb{R}^n \longrightarrow [-1, 1]$ continuous on **Int** C such that $\mathbf{CH}_d(f) = (C, C \cup S^+, \emptyset, C \cup S^-, \emptyset)$ and that moreover

$$\mathbf{ap\ lim\ inf}_{t \longrightarrow x} f(t) = -1 \quad \text{and} \quad \mathbf{ap\ lim\ sup}_{t \longrightarrow x} f(t) = 1 \qquad (**)$$

holds whenever $x \notin C$.

<u>**Proof**</u> Since C is of type F_σ, we can find a set $M \subset \mathbb{R}^n\setminus C$ of type G_δ with $\{x \in \mathbb{R}^n\setminus C \ ; \ x \notin \mathbf{Der}_d(\mathbb{R}^n\setminus C)\} \subset M$ and $\lambda(M) = 0$. We choose a function $\tilde{f}$ according to **Proposition 5.b)**. **Fact 3** ensures the existence of mutually disjoint sets K_k, $k \geq 1$, satisfying $\bigcup_{k=1}^{\infty} K_k = \mathbb{R}^n\setminus(S^+\cup S^-)$ and $\mathbf{Der}_d K_k = \mathbf{Der}_d[\mathbb{R}^n\setminus(S^+\cup S^-)]$ for $k \geq 1$ Indeed, we need only to guarantee that for any k, $m \geq 1$

$$\lambda(K_k \cap \{ x \ ; \ m-1 \leq \|x\| < m \}) = \lambda\big(\{ x \ ; \ m-1 \leq \|x\| < m \}\setminus(S^+\cup S^-)\big)$$

holds. Now we define the function $f:\mathbb{R}^n \longrightarrow [-1, 1]$ by

$$f(x) = \begin{cases} \tilde{f}(x) & \text{if } x \in C \\ 1 & \text{if } x \in S^+\setminus C \\ -1 & \text{if } x \in S^-\setminus C \\ (-1)^k(1-\frac{1}{k}) & \text{if } x \in K_k \end{cases}$$

Clearly $\overline{M} \cap \mathbf{Int}\ C = \emptyset$, hence f is continuous on **Int** C. Since C is d-open, we have $C_d(f) \supset C$. Therefore, in order to finish the proof it suffices to show that the statement **(**)** holds for any $x \in \mathbb{R}^n\setminus C$. If $x \notin \mathbf{Der}_d(\mathbb{R}^n\setminus C)$ then $x \in M \cap \mathbf{Int}_d(\{x\} \cup C)$ and **(**)** follows from the choice of $\tilde{f}$ and f. Because $S^+\setminus C$ has inner measure zero, the set $\mathbb{R}^n\setminus C$ is a measurable hull of $\mathbb{R}^n\setminus S^+$. Hence, for any $k \geq 1$ we obtain

$$\mathbf{Der}_d(\mathbb{R}^n\setminus C) = \mathbf{Der}_d(\mathbb{R}^n\setminus S^+) = \mathbf{Der}_d(S^-\setminus C) \cup \mathbf{Der}_d[\mathbb{R}^n\setminus(S^+\cup S^-)]$$
$$= \mathbf{Der}_d(S^-\setminus C) \cup \mathbf{Der}_d K_k$$

Similarly, we have $\mathbf{Der}_d(\mathbb{R}^n\setminus C) = \mathbf{Der}_d(S^+\setminus C) \cup \mathbf{Der}_d K_k$ if $k \geq 1$. This shows that **(**)** holds also in case $x \in \mathbf{Der}_d(\mathbb{R}^n \setminus C)$. $\square$

7.Lemma

a) For each open set $G \subset \mathbb{R}^n$ there exists an approximately continuous $f:\mathbb{R}^n \longrightarrow [-1, 1]$ such that f is continuous on $\mathbb{R}^n \setminus \partial G$ and that for each $x \in \partial G$ $\liminf_{t \longrightarrow x} f(t) = -1$ and $\limsup_{t \longrightarrow x} f(t) = 1$ hold.

b) For each d-open set $D \subset \mathbb{R}^n$ of type F_σ there exists an approximately continuous map $f:\mathbb{R}^n \longrightarrow [0, 1]$ which is continuous on $\mathbf{Int}\ D$ and satisfies $D = \{ x ; f(x) > 0 \}$.

Proof

a) It is well known that one can find a countable set $A \subset G$ with $\mathbf{Der}\ A = \partial G$. The **Lusin-Menchoff** property guarantees the existence of an open set U fulfilling $A \subset U \subset \overline{U}^d \subset G$. Because $A \cap \mathbf{Der}\ A = \emptyset$,for each $x \in A$ there is an $r_x > 0$ such that $B(x,r_x) \subset U$ and $B(x,2r_x) \cap A = \{x\}$. Then $B(x,r_x) \cap B(y,r_y) = \emptyset$ whenever $x,y \in A$ and $x \neq y$. Therefore, the formula

$$f(x) = \begin{cases} 0 & \text{if } x \notin \bigcup \{B(y,r_y) \ ; \ y \in A\} \\ \cos\left(\dfrac{3\pi \cdot \|x-y\|}{2 \cdot r_y}\right) & \text{if } x \in B(y,r_y) \text{ and } y \in A \end{cases}$$

defines a function on $\mathbb{R}^n$. Since the set $\bigcup \{B(x,r_x) \ ; \ x \in \hat{A}\}$ is closed in G whenever $\hat{A} \subset A$, it is quite easy to show that f has all required properties.

b) A well-known theorem of Zahorski yields the existence of an approximately continuous and upper semicontinuous map $\hat{g}:\mathbb{R}^n \longrightarrow [0, 1]$ such that $D = \{ x ; \hat{g}(x) > 0 \}$ (see, for example, [5; Cor.3.14]). We set $P = \mathbb{R}^n$, $S = \mathbb{R}^n \setminus \mathbf{Int}\ D$, and $g = \hat{g}/_S$ According to **Fact 2** there exists an approximately continuous $\hat{f}:\mathbb{R}^n \longrightarrow [0, 1]$ which is continuous on $\mathbf{Int}\ D$ and satisfies $\hat{f}(x) = g(x)$ for $x \notin \mathbf{Int}\ D$. Now one easily verifies that the function f defined by

$$f(x) = \min \{ 1 , \operatorname{dist}(x,\mathbb{R}^n\setminus\operatorname{Int} D) + \hat{f}(x) \}$$

exhibits all properties ascribed to it.$_\square$

8.Proposition Assume that the sets C_e, D, C, S^+, T^+, S^-, and T^- fulfill the conditions **a)** of **Theorem** 1. Then

a) there exists a nonincreasing sequence of d-open sets D_k, $k \geq 0$, of type F_σ such that $D_0 = \mathbb{R}^n$, $\bigcap_{k=0}^{\infty} D_k = D$ and $C_e \subset \bigcap_{k=0}^{\infty} \operatorname{Int} D_k \subset \overline{C_e \cup T^+ \cup T^-}$.

b) there is a function $f:\mathbb{R}^n \to [-1, 1]$ such that $C(f)\setminus\overline{T^+ \cup T^-} = C_e\setminus\overline{T^+ \cup T^-}$, $C(f) \supset C_e$ and $\mathbf{CH}_d(f) = (D, S^+ \cup D, \emptyset, S^- \cup D, \emptyset)$.

Proof Let $D = \bigcap_{k=0}^{\infty} \hat{D}_k$, where the sets $\hat{D}_k$ are F_σ-sets and hence of type G_δ in the topology d (Each measurable set can be written as the set difference of a G_δ-set and a set of measure zero, hence it is of d-type G_δ.). Therefore $\hat{D}_k = \bigcap_{i=0}^{\infty} U(k,i)$ with $U(k,i)$ d-open. For each pair (k,i) there is a d-open set $D(k,i)$ of type F_σ satisfying $\hat{D}_k \subset D(k,i) \subset U(k,i)$, indeed, we select the sets $D(k,i)$ to fulfill $\lambda\big(U(k,i)\setminus D(k,i)\big) = 0$. Consequently we can write $D = \bigcap_{k=0}^{\infty} \tilde{D}_k$, where the $\tilde{D}_k$, $k \geq 0$, form a nonincreasing sequence of d-open sets of type F_σ. Further on, we can choose a nonincreasing sequence of open sets G_k such that $G_0 = \mathbb{R}^n$, $\bigcap_{k=0}^{\infty} G_k = C_e$ and $\bigcap_{k=0}^{\infty} \overline{G_k} = \overline{C_e}$, the last equality holds if we guarantee $G_k \subset \{ x ; \operatorname{dist}(x,C_e) < \frac{1}{2\cdot k} \}$ for $k \geq 1$. Next, since the set $\mathbb{R}^n\setminus(\overline{T^+ \cup T^-} \cup D)$ is of type $G_{\delta\sigma}$, it has the **Baire** property. Consequently we can find some G_δ-set $\hat{S} \subset \mathbb{R}^n\setminus(\overline{T^+ \cup T^-} \cup D)$ such that $\left(\mathbb{R}^n\setminus(\overline{T^+ \cup T^-} \cup D)\right)\setminus\hat{S}$ is a first category set, see [4; § 11.IV.]. Now we select a G_δ-set $S \subset \hat{S}$

with $\lambda(S) = 0$ and $\overline{S} \supset \hat{S}$. We define $D_k = (\tilde{D}_k \setminus S) \cup G_k$ for $k \geq 0$. Since S is d-closed, each D_k is d-open. Clearly, the sets D_k are of type F_σ, $\bigcap_{k=0}^{\infty} D_k = D$ and $C_e \subset \bigcap_{k=o}^{\infty} \mathbf{Int}\, D_k$. To finish the proof of part **a)** we fix an arbitrary $x \in \mathbb{R}^n \setminus \overline{(T^+ \cup T^- \cup C_e)}$. Then we can find $k_0 \geq 1$ with $x \notin \overline{G_{k_0}}$. According to the condition **1.a.vi)** $B(x,r) \setminus D$ and hence also $B(x,r) \cap \hat{S}$ are second category sets whenever $r > 0$. This shows that $x \in \overline{S \setminus G_{k_0}}$ and implies $x \notin \mathbf{Int}\big(\mathbb{R}^n \setminus (S \setminus G_{k_0})\big) \supset \mathbf{Int}\big((\tilde{D}_{k_0} \cup G_{k_0}) \setminus (S \setminus G_{k_0})\big) = \mathbf{Int}\, D_{k_0}$. We have just proved $\bigcap_{k=0}^{\infty} \mathbf{Int}\, D_k \subset \overline{T^+ \cup T^- \cup C_e}$ and turn to part **b)**.

Fix any $k \geq 1$. According to **Proposition 6** we can select a function $f_k : \mathbb{R}^n \longrightarrow [-1, 1]$ continuous on $\mathbf{Int}\, D_k$ such that $\mathbf{CH}_d(f_k) = (\ D_k,\ D_k \cup S^+,\ \emptyset,\ D_k \cup S^-,\ \emptyset\ )$ and that for each point $x \notin D_k$ both $\mathbf{ap\ lim\ inf}_{t \longrightarrow x} f_k(t) = -1$ and $\mathbf{ap\ lim\ sup}_{t \longrightarrow x} f_k(t) = 1$ hold. Further on, **Lemma 7** ensures the existence of two approximately continuous maps $g_k : \mathbb{R}^n \longrightarrow [0, 1]$ and $h_k : \mathbb{R}^n \longrightarrow [-1, 1]$ such that g_k is continuous on $\mathbf{Int}\, D_{k-1}$, $D_{k-1} = \{\ x\ ;\ g_k(x) > 0\ \}$, h_k is continuous on $\mathbb{R}^n \setminus \partial G_k$, and $\mathbf{osc}(h_k, x) = 2$ if $x \in \partial G_k$. We define

$$\tilde{f}_k(x) = \frac{1}{2}\left(f_k(x) \cdot g_k(x) + h_k(x) \cdot \mathbf{dist}(x, \mathbb{R}^n \setminus G_{k-1})\right)$$

and $f(x) = \sum_{k=1}^{\infty} 2^{-k} \cdot \tilde{f}_k(x)$. One easily verifies that each $\tilde{f}_k$ fulfills

$$\mathbf{CH}_d(\tilde{f}_k) = (\ D_k \cup (\mathbb{R}^n \setminus D_{k-1}),\ C_d(\tilde{f}_k) \cup S^+,\ \emptyset,\ C_d(\tilde{f}_k) \cup S^-,\ \emptyset\ )$$

and is continuous on at any point in $(\mathbf{Int}\, D_k) \setminus (\partial G_k \cap G_{k-1})$. Since f is the sum of a uniformly convergent series and since $C_d(\tilde{f}_k) \cup C_d(\tilde{f}_m) = \mathbb{R}^n$ for $k \neq m$, we immediately obtain that

$$\mathbf{CH}_d(f) = (\ \bigcap_{k=1}^{\infty} C_d(\tilde{f}_k),\ \bigcup_{k=1}^{\infty} \big(S_d^+(\tilde{f}_k) \setminus C_d(\tilde{f}_k)\big) \cup C_d(f),\ \emptyset,$$
$$,\ \bigcup_{k=1}^{\infty} \big(S_d^-(\tilde{f}_k) \setminus C_d(\tilde{f}_k)\big) \cup C_d(f),\ \emptyset\ )$$

$$= \left(\bigcap_{k=1}^{\infty} D_k \,,\, S^+ \cup \bigcap_{k=1}^{\infty} D_k \,,\, \emptyset \,,\, S^- \cup \bigcap_{k=1}^{\infty} D_k \,,\, \emptyset \right)$$

$$= (D \,,\, D \cup S^+ ,\, \emptyset \,,\, D \cup S^- ,\, \emptyset).$$

Because each $\tilde{f}_k$ is continuous on G_k, we conclude that really $C_e = \bigcap_{k=1}^{\infty} G_k \subset C(f)$. It remains to show that f is discontinuous at any $x_0 \in \mathbb{R}^n \setminus (\overline{T^+ \cup T^- \cup C_e})$. In case $x_0 \in \bigcap_{l=1}^{\infty} \operatorname{Int} D_l$ for the unique $k \geq 1$ with $x_0 \in G_{k-1} \setminus G_k$ also $x_0 \in \overline{C_e} \subset \overline{G_k}$ holds, hence we conclude $x_0 \in \partial G_k$ and $\mathbf{osc}\left(2^{-k-1} \cdot h_k \cdot \mathbf{dist}(\cdot, \mathbb{R}^n \setminus G_{k-1}), x_0\right) = 2^{-k} \cdot \mathbf{dist}(x_0, \mathbb{R}^n \setminus G_{k-1}) > 0$. It is easy to see that both $2^{-k-1} \cdot f_k \cdot g_k$ and $\sum_{l \neq k} 2^{-l} \cdot \tilde{f}_l$ are continuous at x_0, hence $\mathbf{osc}(f, x_0) = \mathbf{osc}\left(2^{-k-1} \cdot h_k \cdot \mathbf{dist}(\cdot, \mathbb{R}^n \setminus G_{k-1}), x_0\right) > 0$ If $x_0 \notin \bigcap_{l=1}^{\infty} \operatorname{Int} D_l$ we fix the unique $k \geq 1$ such that $x_0 \in \overline{\mathbb{R}^n \setminus D_k} \cap \operatorname{Int} D_{k-1}$. Suppose that there exists an open set U with $x_0 \in U \subset \{ x \,;\, g_k(x) > \frac{1}{2} g_k(x_0) \} \subset \operatorname{Int} D_{k-1}$ and $|f(x) - f(y)| < 2^{-k-3} g_k(x_0)$ if $x, y \in U$. Fix any $x_1 \in U \setminus D_k$. Since $f - 2^{-k-1} f_k g_k$ is approximately continuous at $x_1 \in D_{k-1} \setminus D_k$, we find a d-open set U_d with $x_1 \in U_d \subset U$ and

$$2^{-k-1} |f_k(x) \cdot g_k(x) - f_k(y) \cdot g_k(y)| < 2^{-k-2} g_k(x_0) \quad \text{if} \quad x, y \in U_d.$$

This implies $|f_k(x) \cdot g_k(x) - f_k(y) \cdot g_k(y)| < \frac{1}{2} \cdot g_k(x_0)$ if $x, y \in U_d$ and contradicts to the following facts: $g_k \geq \frac{1}{2} g_k(x_0)$ on U_d,

$$\liminf_{\substack{t \to x_1 \\ t \in U_d^1}} f_k(t) = -1 \quad \text{and} \quad \limsup_{\substack{t \to x_1 \\ t \in U_d^1}} f_k(t) = 1$$

Consequently $\mathbf{osc}(f, x_0) \geq 2^{-k-3} \cdot g_k(x_0) > 0$. □

II

After the foregoing preparations we return to the

<u>**Proof of Theorem 1**</u> At first we show that **b)** implies **a)**. Therefore, let $f: \mathbb{R}^n \to \mathbb{R}$ satisfy $C(f) = C_e$ and $\mathbf{CH}_d(f) = (C, S^+, T^+, S^-, T^-)$.

Then clearly $S^+\cap S^-= C$, $T^+\subset S^+\setminus C$, and $T^-\subset S^-\setminus C$. The statements $\lambda(T^+) = \lambda(T^-) = \lambda_*(S^+\setminus C) = \lambda_*(S^-\setminus C) = 0$ have already been proved in [2], but we can derive them also from the following facts holding even in general topological spaces and from the well-known statement that d-discrete and d-first category sets are of measure zero, see for example [5; Theo.6.9.]. Indeed, $T^+= \bigcup_{r\in\mathbb{Q}}\{ x ; f(x) > r > \mathbf{ap\ lim\ sup}_{t\to x} f(t) \}$ (and similarly T^-) is a countable union of d-discrete sets. Further on, because $\mathbf{Int}_d S^+\setminus\overline{C}^d\subset \bigcup_{r\in\mathbb{Q}} \partial_d\{ x \in \mathbf{Int}_d S^+; f(x) < r \}$ is d-open and of the d-first category, we see that $\mathbf{Int}_d(S^+\setminus C) = \emptyset$. To finish the proof of the first implication, we simultaneously prove **iv), v)** and **vi)**. For this purpose we denote for $k \ge 1$ and $q \in \mathbb{Q}$

$$F(k,q) = \left\{ x \in \mathbb{R}^n ; \mathbf{lim\ sup}_{r\searrow 0} \frac{\lambda[B(x,r)\setminus f^{-1}((q-k^{-1}, q+k^{-1}))]}{\lambda(B(x,r))} < \frac{1}{k} \right\}$$

It is well known (and easy to show) that each $F(k,q)$ is of type F_σ. Now we define $F(k) = \bigcup_{q\in\mathbb{Q}} F(k,q)$ and then $D = \bigcap_{k=1}^{\infty} F(k)$. Obviously, D is of type $F_{\sigma\delta}$. Moreover, it is the set of all points at which the approximate limit of f exists. This quite directly follows from the fact that $\mathbf{ap\ lim}_{t\to x} f(t)$ does not exist *iff* $\mathbf{ap\ lim\ inf}_{t\to x} f(t) < \mathbf{ap\ lim\ sup}_{t\to x} f(t)$ *iff* there exist $a < b$ such that both $f^{-1}((-\infty, a))$ and $f^{-1}((b, \infty))$ have positive upper density at x. Consequently, $C = D\setminus(T^+\cup T^-)$. For $k \ge 1$ we put $G_k= \{ x \in \mathbb{R}^n; \mathbf{osc}(f,x) < \frac{3}{k} \}$. Then each G_k is an open set and $C \supset C_e= \bigcap_{k=1}^{\infty} G_k$. Moreover, for $k \ge 1$ $\bigcup_{q\in\mathbb{Q}} \mathbf{Int}\ F(k,q)\setminus\overline{T^+\cup T^-} \subset G_k$ holds. Indeed, from the **density theorem** $\lambda[(\mathbf{Int}\ F(k,q))\setminus f^{-1}((q-k^{-1},q+k^{-1}))] = 0$ if $q \in \mathbb{Q}$ follows. Hence, for $x \in \mathbf{Int}\ F(k,q)$ $\mathbf{ap\ lim\ inf}_{t\to x} f(t) \ge q-k^{-1}$ and $\mathbf{ap\ lim\ sup}_{t\to x} f(t) \le q+k^{-1}$ holds. Since $x \notin T^+\cup T^-$ implies the estimate

$\mathbf{ap\ lim\ inf}_{t\to x} f(t) \le f(x) \le \mathbf{ap\ lim\ sup}_{t\to x} f(t)$, we obtain $\mathbf{osc}(f,x) \le \frac{2}{k}$ if

$x \in \mathbf{Int}\ F(k,q)\setminus\overline{T^+\cup T^-}$. Next, we define S to be the set of all points $x \in C\setminus\overline{T^+\cup T^-}$ at which C is of the second category. Since $\left(C\setminus\overline{T^+\cup T^-}\right)\setminus S$ is a first category set, the statement **vi)** easily follows if we show that $\overline{\bigcup_{q\in\mathbb{Q}} \mathbf{Int}\ F(k,q)} \supset S$ for any $k \geq 1$. But indeed, if $x \in S$, $k \geq 1$, and $0 < r < \mathbf{dist}(x, T^+\cup T^-))$ then $B(x,r) \cap C = B(x,r) \cap D \subset \bigcup_{q\in\mathbb{Q}} B(x,r) \cap F(k,q)$. Consequently, there is some $\hat{q} \in \mathbb{Q}$ such that the F_σ-set $B(x,r) \cap F(k,\hat{q})$ is of the second category and, therefore, has nonempty interior.

It remains to show that **b)** follows from **a)**. Let $(C_e, C, S^+, T^+, S^-, T^-)$ (and D) satisfy the conditions **a)**. We choose a function $\tilde{f}:\mathbb{R}^n \longrightarrow [-1,1]$ according to **Proposition 8.b)**. Then we select a map $\Delta:\mathbb{R}^n \longrightarrow [-1,1]$ such that $T^+ = \{\, x\ ;\ \Delta(x) > 0 \,\}$, $T^- = \{\, x\ ;\ \Delta(x) < 0 \,\}$ and that for arbitrary $x \in \mathbf{Der}(T^+\cup T^-)$ the statement $\lim\limits_{t\longrightarrow x} \Delta(t) = 0$ holds iff $x \in C_e$. The existence of such functions is mentioned (in a more general setting) in [3], but since this paper is still in print, we shortly outline the construction. Let $\mathbb{R}^n = U_0 \supset U_1 \supset \dots$ be a sequence of open sets such that $\bigcap_{k=0}^{\infty} U_k = C_e \cap \overline{T^+\cup T^-}$. For any positive integer k let T_k^+ and T_k^- satisfy $T_k^+ \subset T^+\cap U_k$, $T_k^- \subset T^-\cap U_k$, and moreover $\mathbf{Der}\ T_k^+ = \mathbf{Der}(T^+\cap U_k)\setminus U_k$, and $\mathbf{Der}\ T_k^- = \mathbf{Der}(T^-\cap U_k)\setminus U_k$. We define $\Delta_k:\mathbb{R}^n \longrightarrow [-1,1]$ by

$$\Delta_k(x) = \begin{cases} 1 & \text{if } x \in T_k^+ \cup (T^+\setminus U_k) \\ -1 & \text{if } x \in T_k^- \cup (T^-\setminus U_k) \\ 0 & \text{else}\ . \end{cases}$$

Then $\mathbf{Der}\ \Delta_k^{-1}(\{1\}) = (\mathbf{Der}\ T^+)\setminus U_k$ and $\mathbf{Der}\ \Delta_k^{-1}(\{-1\}) = (\mathbf{Der}\ T^-)\setminus U_k$. Consequently it suffices to set $\Delta = \sum_{k=1}^{\infty} 2^{-k}\cdot\Delta_k$. Finally we put $f = \frac{1}{2}\cdot(\tilde{f} + \Delta)$.

Since $\underset{t \to x}{\textbf{ap lim}}\ \Delta(t) = 0$ everywhere and since also $\underset{t \to x}{\textbf{ap lim sup}}\ \tilde{f}(t) = \tilde{f}(x)$ if $x \in S^+ \cup D$, and $\underset{t \to}{\textbf{ap lim inf}}\ \tilde{f}(t) = \tilde{f}(x)$ if $x \in S^- \cup D$, we immediately conclude that $\mathbf{CH}_d(f) = (D\setminus(T^+ \cup T^-), (D \cup S^+)\setminus T^-, T^+, (D \cup S^-)\setminus T^+, T^-) =$ (C, S^+, T^+, S^-, T^-). Further on, because for each $x \in C_e \cup \left(\mathbb{R}^n \setminus \overline{T^+ \cup T^-}\right)$ $\underset{t \to x}{\textbf{lim}}\ \Delta(t) = \Delta(x) = 0$, it follows that $C(f)\setminus\overline{T^+ \cup T^-} = C(\tilde{f})\setminus\overline{T^+ \cup T^-} =$ $C_e\setminus\overline{T^+ \cup T^-}$ and $C(f) \supset C_e$. Hence, the proof of **Theorem 1** will be finished if we show that f is discontinuous at any point $x \in \overline{T^+ \cup T^-} \setminus C_e$. We may assume that $x \in \mathbf{Der}(T^+ \cup T^-)$ because the case $x \in T^+ \cup T^- \subset \mathbb{R}^n\setminus C_d(f)$ is trivial. From $T_d^+(\tilde{f}) = T_d^-(\tilde{f}) = \emptyset$ we derive that $\mathbf{osc}(f,t) \geq \frac{1}{2}\,|\Delta(t)|$ for any $t \in T^+ \cup T^-$. Since this inequality holds also if $t \notin T^+ \cup T^-$ we get $\mathbf{osc}(f,x) \geq \underset{t \to x}{\textbf{lim sup}}\ \mathbf{osc}(f,t) \geq \frac{1}{2}\cdot \underset{t \to x}{\textbf{lim sup}}\ |\Delta(t)| > 0$, i.e. f is discontinuous at x. $\square$

We remark that it is quite easy to show that **a.vi)** is equivalent to

a.vi') The closure of C_e contains each point in $C\setminus\overline{T^+ \cup T^-}$ at which C is residual.

From **Theorem 1** we immediately obtain the following

9.Corollary

a) For a given quintuple (C, S^+, T^+, S^-, T^-) of subsets of $\mathbb{R}^n$ the following two statements are equivalent:

i) The conditions **a.i)...a.iv)** from **Theorem 1** hold .

ii) There is a function $f:\mathbb{R}^n \to \mathbb{R}$ such that $\mathbf{CH}_d(f) = (C,S^+,T^+,S^-,T^-)$.

b) For a set $M \subset \mathbb{R}^n$ the following is equivalent:

i) M is a measurable set with empty interior.

ii) There is a function $f:\mathbb{R}^n \to \mathbb{R}$ such that $M = C_d(f)\setminus C(f)$.

Proof

a) According to **Theorem 1** we need only to show that **a)** implies the existence of a set C_e such that the conditions **1.a.i)...a.vi)** hold. Since $\overline{C\setminus T^+ \cup T^-} = \overline{D\setminus T^+ \cup T^-}$ has the **Baire** property, we can choose a G_δ-set $C_e \subset \overline{C\setminus T^+ \cup T^-}$ such that $\left(\overline{C\setminus T^+ \cup T^-} \right)\setminus C_e$ is a first category set. Obviously, C_e is dense at any $x \in \overline{C\setminus T^+ \cup T^-}$ at which C is of the second category.

b) Since $\text{Int}(C_d(f)\setminus C(f)) \subset \overline{C_d(f)\setminus C(f) \cup T^+ \cup T^-}$ the implication ii)$\Rightarrow$i) follows from the statements **1.a.iv),v)** and **vi)**. Conversely, let $M \subset \mathbb{R}^n$ be a measurable set with $\text{Int } M = \emptyset$. We can choose a G_δ-set D satisfying $M \subset D$, $\lambda(D\setminus M) = 0$ and $\overline{D\setminus M} = \mathbb{R}^n$. Then the sextuple $(\emptyset, M, D, D\setminus M, M, \emptyset)$ fulfils the conditions **1.a)**. $\square$

10.Remark In our approach we mainly used topological methods. More special properties of the measure λ or of the topology d were used only at some few places (mainly **Fact 3** and **Lemma 4**).Therefore, it seems to be highly probable that this approach applies also to other "reasonable" (**Lusin-Menchoff** property !) density topologies, examples may be found in [5: 6.11 & 6.34(B)] . Here we restricted our attention to the familiar topology d since this keeps the whole matter clearer and avoids undue technical complications. However, since it looks hopeful to study by this approach also the question of characterizing quintuples for other types of fine topologies (for instance r.- and a.e.- modifications, see [5;7.A & 7.B]), we make the following, perhaps useful, remark. For our purpose it suffices to know only that d has the **Lusin-Menchoff** property, its completeness is superfluous.Indeed, we use **Fact 2** , the only statement requiring the **Lusin - Menchoff** property of some induced fine topology , only in cases where P is of type F_σ . But in [5; Exec.3.B.1.] it is shown that the **Lusin-Menchoff** property is hereditary with respect to F_σ-subsets.

Finally, it seems to be a more difficult problem to describe all pairs consisting of the *d*-characterizing quintuple and the *euclidean*-characterizing quintuple of any real function on $\mathbb{R}^n$. The notion of an "extended" *d*-characterizing quintuple is , of course, only a first step in this direction. The same problem appears for pairs of *qualitative*- and *euclidean*-characterizing quintuples, see [3]. In both cases the description of all pairs consisting of the "fine topological"-characterizing quintuple and the *euclidean* $(C^+(f),S^+(f),S^-(f))$ - triple would already be very interesting.

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