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Differentiability and Density Continuity

1 Introduction

The density topology [10,11] on $\mathbf{R}$ consists of all measurable subsets A of $\mathbf{R}$ such that, for every $x \in A$, x is a density point of A. It is a completely regular refinement of the natural topology. A function $f: \mathbf{R} \to \mathbf{R}$ is *density continuous* if and only if it is continuous as a selfmap of $\mathbf{R}$ equipped with the density topology. The class of density continuous functions was investigated by Ostaszewski [7,,8]. Bijections of the real line whose inverses are density continuous were studied by Bruckner [1] and Niewiarowski [6]. Ostaszewski [9] considered the class as a semigroup with composition as the operation. Ciesielski and Larson [2] showed that real-analytic functions are density continuous, and that the class of density continuous functions is not a linear space. Furthermore, there exist C^∞ functions which are not density continuous. Ciesielski, Larson, and Ostaszewski [4] proved that a typical continuous function is nowhere density continuous, and the class of sets of points of discontinuity of density continuous functions is that of nowhere dense F_σ subsets of $\mathbf{R}$.

Throughout this paper we are concerned with the relationship between density continuity and differentiability. In the process, we discuss the fact that any closed set can be made into the zero set of a C^∞ density continuous function, and we show that there is a nowhere approximately differentiable

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density continuous and continuous function. This example answers a problem posed by Ostaszewski [9].

The following theorem will be used several times in the sequel. It is a restatement of a theorem due to Ciesielski and Larson [2, Theorem 1].

Theorem 1 *If a function f is convex in a neighborhood of x, then f is density continuous at x.*

The following notation will be used:

$\mathbf{R}$ – the set of real numbers;

$\mathbf{N}$ – the set of natural numbers, $\mathbf{N} = \{1, 2, 3 \ldots\}$;

$|A|$ – the Lebesgue measure of a measurable set $A \subset \mathbf{R}$;

A^c – the complement of a set A;

$\overline{d}(A, x)$, $\underline{d}(A, x)$, $d^+(A, x)$, $d^-(A, x)$, $d(A, x)$ – the upper, lower, right-hand, left-hand, and ordinary (respectively) densities of a set $A \subset \mathbf{R}$ at a point $x \in \mathbf{R}$;

C^∞ – the set of all smooth (i.e., infinitely many times differentiable) functions $f : \mathbf{R} \to \mathbf{R}$;

$dist(x, F)$ – the distance of an $x \in \mathbf{R}$ from a set $F \subset \mathbf{R}$;

$\mathrm{supp}(f) = \{x \in \mathbf{R}: f(x) \neq 0\}$; and

$f^{(k)}(x)$ – the k'th derivative of f at x. In particular, $f^{(0)}(x) = f(x)$.

2 The zero-set of a density continuous function

Ciesielski and Larson [3] were able to show that the complements of level sets for the density continuous functions do not form a subbase for the density topology (i.e., the density topology is not generated; see also the discussion of this problem in [9]). This implies, that the structure of level sets, and the zero sets in particular, of density continuous functions, is not very rich. However, we were able to prove that closed sets are the zero sets of density continuous, continuous functions with certain differentiability properties.

Theorem 2 *Given a closed set F, there is a function $f \in C^{\infty}$ which is density continuous such that $F = \operatorname{supp}(f)^c$.*

In order to prove this theorem, we need two lemmas. The first has already been shown by Ciesielski, Larson and Ostaszewski [4, Lemma 1].

Lemma 1 *Suppose that I_n and J_n are sequences of intervals such that $I_n \subset J_n$ and I_n has the same center as J_n for all $n \in \mathbf{N}$. If*

$$Z_x = \bigcup_{\{n: x \notin J_n\}} I_n$$

and

$$\sum_{n \in \mathbf{N}} |I_n|/|J_n| < \infty,$$

then $d(Z_x, x) = 0$ for all $x \in \mathbf{R}$.

Lemma 2 *Suppose that f is a function which is convex upward on $\mathbf{R}$ and increasing on $[0, \infty)$, with $f(0) = 0$. If $h_0 > 0$, $0 < \rho < 1$, and $H \subset (0, h_0)$ is a measurable set such that*

$$|H \cap (0, h)| > h\rho > 0, \ \forall h \in (0, h_0), \tag{1}$$

then

$$|f^{-1}(H) \cap (0, t)| \geq t\rho > 0, \ \forall t \in (0, f^{-1}(h_0)).$$

Proof. Choose $t \in (0, f^{-1}(h_0))$. Let $h = f(t)$. There is a nonnegative, nonincreasing function g on $(0, h)$ such that

$$f^{-1}(k) = \int_0^k g(l)dl \text{ for each } k \in (0, h).$$

Denote

$$v(s) = \sup\{k \in (0, h) \colon g(k) \geq s\}.$$

Assume (1). Then

$$\begin{aligned}
|f^{-1}(H)\cap(0,t)| &= \int_{H\cap(0,h)} g(l)dl \\
&= \int_{H\cap(0,h)}\left(\int_0^{g(l)} dr\right) dl \\
&= \int_0^{g(0+)}\int_{H\cap(0,v(r))} dl\, dr \\
&\geq \rho\int_0^{g(0+)} v(r)dr \\
&= \rho\int_0^h g(t)dt = \rho t.
\end{aligned}$$

This proves Lemma 2, so we move on to the proof of the theorem.

Without loss of generality, we may assume that $F \subset (0,1)$ and that $(0,1)\backslash F$ has an infinite number of components denoted by $I_n = (a_n, b_n)$, $n \in \mathbf{N}$.

For each n, let $c_n = (a_n + b_n)/2$ and choose an $\varepsilon_n \in (0, |I_n|4^{-n})$. In each I_n choose $f_n \in C^\infty$ such that $\text{supp}(f_n) = I_n$, $f_n^{(k)}(a_n) = f_n^{(k)}(b_n) = 0$ for all $k \geq 0$, f_n is convex upward on $(a_n, c_n - \varepsilon_n)$ and $(c_n + \varepsilon_n, b_n)$, f_n is convex downward on $(c_n - \varepsilon_n, c_n + \varepsilon_n)$ and

$$\sup_{0\leq k\leq n} \|f_n^{(k)}\|_\infty < 1/n. \tag{2}$$

Define

$$f(x) = \sum_{n\geq 1} f_n(x). \tag{3}$$

Using (2), (3) and the disjointness of the I_n we see that $f^{(k)}$ converges uniformly for all $k \geq 0$. This implies that $f \in C^\infty$. It is also clear that $\text{supp}(f) = F^c$.

To show that f is density continuous, we note Theorem 1 implies f is density continuous on each I_n, right density continuous at each a_n and left density continuous at each b_n.

Choose any $x \in F \setminus \{a_n : n \geq 0\}$, let $\rho \in (0,1)$ and let H be a density neighborhood of 0. There exists an $h_0 > 0$ such that whenever $0 < h < h_0$, then $|H \cap (0,h)| > h\rho$. From the choice of x, there is a $\delta > 0$ such that whenever $n \in S = \{n : I_n \cap (x, x+\delta) \neq \emptyset\}$, then $\|f_n\|_\infty < h_0$.

From Lemma 2 it is clear that for all $n \in S$,

$$|f^{-1}(H)\cap((a_n, c_n-\varepsilon_n)\cup(c_n+\varepsilon_n, b_n))| \geq \rho|((a_n, c_n-\varepsilon_n)\cup(c_n+\varepsilon_n, b_n))|. \quad (4)$$

Lemma 1 and the choice of ε_n shows that

$$d\left(\bigcup_{n\geq 1}(c_n - \varepsilon_n, c_n + \varepsilon_n), x\right) = 0. \quad (5)$$

Since $F \subset f^{-1}(H)$, we see from (4) and (5) that

$$d^+(f^{-1}(H), x) \geq \rho.$$

The arbitrarity of ρ and x shows that f is right density continuous everywhere. A similar argument establishes left density continuity.

The proof is the theorem is finished. Note that a part of the above proof can be also used to show the following corollary.

Corollary 1 *Let F be a closed subset of $\mathbf{R}$. Then the function $f(x) = dist(x, F)$ is density continuous.*

Theorem 3 *Let F be a closed subset of $\mathbf{R}$ which is of measure zero. There exists a density continuous function f such that* $\text{supp}(f) = F$ *and f is not approximately differentiable at any point of F, while being differentiable elsewhere.*

Proof. If the complement of F has finitely many components then F is finite and the result is trivial. Therefore, without loss of generality, we may assume that $F \subset (0,1)$ and that $(0,1)\setminus F$ has an infinite number of components, denoted by $I_n = (a_n, b_n),\ n \in \mathbf{N}$.

Let F' be the set of all accumulation points of F. We also define

$$G = (0,1)\setminus F = \bigcup_{n\in\mathbf{N}}(a_n, b_n),$$

$$c_n = (a_n + b_n)/2, n \in \mathbf{N},$$

$$a'_n = c_n - (c_n - a_n)/2, n \in \mathbf{N},$$

$$b'_n = c_n + (b_n - c_n)/2, n \in \mathbf{N},$$

$$G' = \bigcup_{n\in\mathbf{N}} (a'_n, b'_n),$$

and

$$C = \{c_n : n \in \mathbf{N}\}.$$

Note that every point of F is a density point of G, and

$$\overline{d}(G', x) \geq \frac{1}{4}$$

for every $x \in F$.

Put $g(x) = 0$, if $x \in F$. To define g on G we will proceed by induction.

Let $m_0 = 0$. The induction step is as follows. For $i \in \mathbf{N}$, let $\mathcal{V}_i$ be a finite class of closed intervals, not necessarily disjoint, of positive length not exceeding i^{-2} such that

$$F' \subset \bigcup \mathcal{V}_i \subset [0,1] \setminus \bigcup_{n=1}^{m_i} (a_n, b_n).$$

In addition to the above we will also require that for every $J \in \mathcal{V}_i$, $n \in \mathbf{N}$ either $J \cap (a_n, b_n) = \emptyset$ or $(a_n, b_n) \subset J$.

As F is of measure zero, we can find $m_{i+1} > m_i$ such that for every $J \in \mathcal{V}_i$

$$\left| J \cap \bigcup_{n=1}^{m_{i+1}} (a_n, b_n) \right| \geq \frac{1}{2}|J|. \tag{6}$$

Inequality (6) implies that

$$\begin{aligned} \left| J \cap \bigcup_{n=m_i+1}^{m_{i+1}} (a'_n, b'_n) \right| &= \frac{1}{2} \left| J \cap \bigcup_{n=m_i+1}^{m_{i+1}} (a_n, b_n) \right| \\ &= \frac{1}{2} \left| J \cap \bigcup_{n=1}^{m_{i+1}} (a_n, b_n) \right| \geq \frac{1}{4}|J| \end{aligned} \tag{7}$$

for every $J \in \mathcal{V}_i$.

For $n = m_i + 1, \ldots, m_{i+1}$ define $g(c_n) = i^{-1}$, and let g be linear in the corresponding intervals $[a_n, c_n]$, $[c_n, b_n]$. This completes the definition of g.

We will show now that g is not approximately differentiable at any point of F.

If $x \in F \setminus F'$ then x is an isolated point of F and as such equals $a_n = b_m$ for some $n, m \in \mathbf{N}$. Clearly the right-hand derivative of g at x is positive, while the left-hand derivative is negative, so that g is not approximately differentiable at x.

If $x \in F'$, then for every $i \in \mathbf{N}$ there exists a $J \in \mathcal{V}_i$ such that $x \in J$. For every $z \in J \cap \bigcup_{m_i+1}^{m_{i+1}} (a'_n, b'_n)$

$$\left| \frac{g(z) - g(x)}{z - x} \right| = \frac{g(z)}{|z - x|} \geq \frac{\frac{1}{2} i^{-1}}{i^{-2}} = \frac{1}{2} i.$$

This, combined with (7) implies that g is not approximately differentiable at x.

An argument analogous to the one of Theorem 2 shows that g is density continuous. However, it is not differentiable at any of the points c_n, $n \in \mathbf{N}$. We will modify g to obtain a function possessing all the desired properties.

Choose a sequence of intervals K_n centered at c_n, such that

$$|K_n| / (b_n - a_n) < 2^{-n}$$

and modify g on each K_n to be differentiable on (a_n, b_n) and convex downward on K_n. In light of Theorem 1 and Lemma 1, this will not change its density continuity. Call the modified function f. Since the modification took place on a set of density 0 at every point of F, f has all the desired properties.

3 A nowhere approximately differentiable continuous density continuous function

Theorem 4 *There exists a continuous, density continuous function which is nowhere approximately differentiable.*

Proof. Let $I = [0, 1]$. Malý [5] constructs a density continuous function $f: I \to I$ such that there exists a set A of measure zero with $|f(A)| = 1$. The function f is actually the x-coordinate of a Peano area-filling curve. We will recall Malý's construction and show that the function f is nowhere approximately differentiable.

First, let $g\colon [0,9] \to [0,3]$ be defined as follows: $g(0) = 0, g(1) = 1, g(2) = 0, g(3) = 1, g(4) = 2, g(5) = 1, g(6) = 2, g(7) = 3, g(8) = 2, g(9) = 3$, and let g be linear in each interval of the form $[i, i+1]$, where $0 \leq i \leq 8$. Define

$$f_1(x) = \frac{1}{3} g\left(\frac{1}{9}x\right).$$

For every $n \in \mathbf{N}$, $k \in \mathbf{N}$, $0 \leq k < 9^n$, $0 \leq t \leq 9^{-n}$ let

$$f_{n+1}\left(\frac{k}{9^n} + t\right) = f_n\left(\frac{k}{9^n}\right) + \frac{1}{3} g\left(9^{n+1}t\right)\left(f_n\left(\frac{k+1}{9^n}\right) - f_n\left(\frac{k}{9^n}\right)\right).$$

Malý [5] shows that the sequence $\{f_n\}_{n\in\mathbf{N}}$ converges uniformly to a continuous, density continuous function $f\colon I \to I$.

It remains to show that f is approximately differentiable nowhere.

To do this, we note the following facts, which are obvious from the construction.

(a) f_n is linear on each interval of the form $[j9^{-n}, (j+1)9^{-n}]$.

(b) $f_n([j9^{-n}, (j+1)9^{-n}]) = [k3^{-n}, (k+1)3^{-n}]$ for all j with $0 \leq j < 9^n$ and some k with $0 \leq k < 3^n$.

Choose $x \in [0,1]$ and $n \in \mathbf{N}$. There is a j, $0 \leq j < 9^n$, such that $x \in J = [j9^{-n}, (j+1)9^{-n}]$. Let $f(x) \in [\alpha, \beta] = [k3^{-n}, (k+1)3^{-n}]$. Assume that f_n is increasing on J and $f(x) \leq \frac{1}{2}(\alpha + \beta)$ (the other cases are handled similarly). Consider the interval

$$K = \left[\left(j + \frac{2}{3}\right) 9^{-n}, (j+1)\, 9^{-n}\right].$$

If $y \in K$ then

$$f(y) \in \left[\left(k + \frac{2}{3}\right) 3^{-n}, (k+1)3^{-n}\right]$$

and

$$\frac{f(y) - f(x)}{y - x} \geq \frac{\left(k + \frac{2}{3}\right) 3^{-n} - \left(k + \frac{1}{2}\right) 3^{-n}}{(j+1)\, 9^{-n} - j9^{-n}} = \frac{\frac{1}{6} 3^{-n}}{9^{-n}} = \frac{1}{6} 3^n.$$

This implies

$$9^n \left|\left\{y \in (x, x + 9^{-n}) : \frac{f(y) - f(x)}{y - x} \geq \frac{1}{6} 3^n\right\}\right| \geq \frac{1}{3}.$$

In general, for the above, and other cases of the behavior of f on J,

$$9^n\left|\left\{y\in\left(x-9^{-n},x+9^{-n}\right):\left|\frac{f(y)-f(x)}{y-x}\right|\geq\frac{1}{6}3^n\right\}\right|\geq\frac{1}{12}$$

for all n. It follows from that f is not approximately differentiable at x.

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