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THE GAUSS-GREEN THEOREM

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1. The Gauss-Green formula. The Gauss-Green theorem is the set of assumptions under which the following formula holds:

$$\int_A \operatorname{div} v = \int_{\partial A} v \cdot n_A .$$

2. History. The full-fledged Gauss-Green theorem cannot be formulated by means of the Lebesgue integral, because derivatives need not be Lebesgue integrable. This was recognized already by Lebesgue. In 1912 and 1914, it lead to the development of the Denjoy- Perron integral, which gives the unrestricted *fundamental theorem of calculus*, i.e., the Gauss-Green theorem in dimension one. In spite of many efforts, no substantial progress was made in the higher dimensional case for nearly 70 years. Only in the eighties, variations of the generalized Riemann integral of Henstock and Kurzweil were shown to integrate the divergence of *any* differentiable vector field. These new averaging methods were devised by Mawhin, Kurzweil, Jarnik, and others including the author. Unfortunately, all of them have deficiencies varying in the degree of severity. Some lack certain basic properties one expects of any integral worth its name (I like to call them *gadgets* rather than integrals), others are either coordinate bound or cannot handle vector fields with many singularities, and all of them place unnatural restrictions on the domains of integration.

2. Objectives. I want to show that all of these deficiencies can be removed, and that a well-behaved coordinate free extension of the Lebesgue integral can be defined under assumptions *approaching the limits of generality*. As the papers on the subject tend to be *highly technical*, my primary goal is to expose the *intuition* behind the technical formalism, without compromising the level of useful generality.

3. The setting. Throughout, $\mathbf{R}^m$ with $m \geq 1$ is fixed, and all functions are real-valued. Let $A \subset \mathbf{R}^m$ and $x \in \mathbf{R}^m$. We define:

(1) $d(A) =$ diameter of A;

(2) $\lambda(A) = |A| = m$-dimensional outer Lebesgue measure of A;

(3) $\mathcal{H}(A) = (m-1)$-dimensional outer Hausdorff measure of A;

(4) $$\lim_{\varepsilon \to 0+} \frac{|A \cap U(x,\varepsilon)|}{|U(x,\varepsilon)|} = \begin{cases} 1 & \text{means that } x \text{ is a } \textit{density point} \text{ of } A; \\ 0 & \text{means that } x \text{ is a } \textit{dispersion point} \text{ of } A; \end{cases}$$

(5) $\operatorname{int}^* A =$ *essential interior* of $A =$ the set of all density points of A;

(6) $\operatorname{cl}^* A =$ *essential closure* of $A =$ the set of all nondispersion points of A;

(7) $\partial^* A = \operatorname{cl}^* A - \operatorname{int}^* A =$ *essential boundary* of A.

4. BV sets. For domains of integration we want the largest family of sets for which the *surface area* and the *exterior normal* can be defined. This is the family BV of sets with *bounded variation* introduced in the fifties by Caccioppoli, De Giorgi, and independently by Mařik. A bounded measurable set $A \subset \mathbf{R}^m$ with characteristic function χ_A is called a BV set if the distributional gradient $\nabla \chi_A$ is a vector-valued Borel measure of bounded variation $|\nabla \chi_A| = \sigma_A$, called the *surface measure* of A. A bounded set $A \subset \mathbf{R}^m$ is a BV set if and only if $\mathcal{H}(\partial^* A) < +\infty$. For a BV set A we define:

(1) $\|A\| = \sigma_A(\partial A) = \mathcal{H}(\partial^* A) =$ *perimeter of* A;

(2) $n_A =$ *Federer exterior normal* of A defined $\mathcal{H}$-almost everywhere on $\partial^* A$.

Two BV sets which differ by a set of measure zero have the same perimeter. If $A \in BV$, then for each C^∞ vector field v in a neighborhood of $\operatorname{cl} A$ we have

$$\int_A \operatorname{div} v \, d\lambda = \int_{\partial^* A} v \cdot n_A \, d\mathcal{H} .$$

The *regularity* of a BV set A is given by

$$r(A) = \begin{cases} \frac{|A|}{d(A)\|A\|} & \text{if } d(A)\|A\| > 0, \\ 0 & \text{otherwise,} \end{cases}$$

and if $E \subset \mathbb{R}^m$, we set $BV_A = \{B \in BV : B \subset A\}$.

To get an idea about the *complexity* of BV sets, consider the iterations of *"caviar"* (i.e., a countable union of balls whose total perimeter is finite) and *"swiss cheese"* (i.e., a ball from which countably many balls of finite total perimeter were removed). We need *all* BV sets because the family BV has an important *compactness property* needed in variational problems of geometric measure theory (e.g., minimal surfaces) and conservation laws.

5. Convergence. If $\{B_n\}$ is a sequence in BV and B is a BV set, we write $B_n \to B$ whenever $B_n \subset B$, $n = 1, 2, \ldots$, $\sup \|B_n\| < +\infty$, and $\lim |B - B_n| = 0$.

6. The germ of the Gauss-Green theorem. The following lemma is the *infinitesimal* form of the Gauss-Green theorem.

LEMMA 1. *Let $U \subset \mathbb{R}^m$ be open, $x \in U$, and let v be a continuous vector field in U which is differentiable at x. Then given $\varepsilon > 0$, there is a $\delta > 0$ such that*

$$\left| div\, v(x)|B| - \int_{\partial^* B} v \cdot n_B \, d\mathcal{H} \right| < \varepsilon |B|$$

for each $B \in BV_U$ with $x \in clB$, $d(B) < \delta$, and $r(B) > \varepsilon$.

Now let v be differentiable in U, and let A be a subset of U. Since we *like to believe* that in this case the function $F : B \mapsto \int_{\partial^* B} v \cdot n_B \, d\mathcal{H}$, $B \in BV_A$, is an *"indefinite integral"* of $\operatorname{div} v$ in A, it is reasonable to define an indefinite integral of an arbitrary function f on A by abstracting the essentials from Lemma 1.

(1) The indefinite integral of f should be an additive function F which is *continuous* in the following sense:
$$B_n \to B \quad \Rightarrow \quad \lim F(B_n) = F(B).$$
This property corresponds to the *continuity* of v.
(2) The expression $|f(x)|B| - F(B)|$ should be bounded by a ***nonnegative additive function***, say M, to avoid the accumulation of errors.
(3) It should be possible to make the function M ***arbitrarily small***.
(4) As we want to allow for *singularities* of v (i.e., for points where v is not differentiable) the approximation $|f(x)|B| - F(B)| < M(B)$ should be required only outside an *exceptional subset* of A.

7. Variational integral. Let $A \in BV$, and let f and F be functions on $\mathrm{cl}^* A$ and BV_A, respectively. Given $\varepsilon > 0$ and a set $T \subset \mathbb{R}^m$, an *ε-majorant* of the pair (f, F) in A mod T is a nonnegative additive function M on BV_A satisfying the following conditions:

(i) $M(A) < \varepsilon$;
(ii) $\forall x \in \mathrm{cl}^* A - T, \exists \delta > 0 \ni \quad |f(x)|B| - F(B)| < M(B) \quad \forall B \in BV_A \ni x \in \mathrm{cl} B, d(B) < \delta, r(B) > \varepsilon.$

DEFINITION 1. Let $A \in BV$ and let f be a function on $\mathrm{cl}^* A$. We say that f is ***v-integrable*** ("v" for *variationally*) in A if there is a continuous additive function F on BV_A satisfying the following condition:

$$\forall \varepsilon > 0, \exists T \subset \mathbb{R}^m \text{ of } \mathcal{H}\text{-}\sigma\text{-finite measure} \ni (f, F) \text{ has an } \varepsilon\text{-majorant in } A \bmod T.$$

Some remarks to Definition 1.

(1) We can switch the first two quantifiers and write: $\exists T \ni \forall \varepsilon$.
(2) F is determined *uniquely* by $f \lceil (A - N)$ where $|N| = 0$. This follows *nontrivially* by elaborating on ideas of Besicovitch. Thus we can integrate functions defined only *almost everywhere* in A; for $\mathrm{cl}^* A$ differs from A by a set of measure zero. We call F the *indefinite v-integral* of f in A. The number $I_v(f, A) = F(A)$ is called the *v-integral* of f over A.

(3) There are no *topological* restrictions imposed on the exceptional set T. We shall see from Theorem 1 that in terms of the measure $\mathcal{H}$, the set T is as large as possible.

(4) In a different context variational integrals were considered previously by Henstock, who introduced the name. His integrals can be defined by means of either *additive* or *superadditive* ε-majorants. It appears that in our situation, the uniqueness of the indefinite v-integral *depends* on the additivity of ε-majorants.

7. The Gauss-Green theorem. A vector field v on an open set $U \subset \mathbf{R}^m$ is *almost differentiable* at $x \in U$ if

$$\limsup_{y \to x} \frac{|v(y) - v(x)|}{|y - x|} < +\infty .$$

By Stepanoff's theorem, a vector field v almost differentiable everywhere in $E \subset U$ is differentiable almost everywhere in E.

A vector field v on an *arbitrary* set $E \subset \mathbf{R}^m$ is almost differentiable at $x \in E$ if it is extendable to a vector field w on an open neighborhood of E such that w is almost differentiable at x according to the above definition. If E is *measurable* and v is almost differentiable (in particular, differentiable) everywhere in E, then the differential Dv is determined *uniquely* almost everywhere in E.

The following theorem is a relatively easy consequence of Lemma 1.

THEOREM 1. *Let $A \in BV$ and let $T \subset \mathbf{R}^m$ be of $\mathcal{H}$-σ-finite measure. Suppose that v is a continuous vector field on $\mathrm{cl}A$ which is almost differentiable on $\mathrm{cl}^*A - T$. Then $\mathrm{div}\, v$ is v-integrable in A and*

$$I_v(\mathrm{div}\, v, A) = \int_{\partial^* A} v \cdot n_A \, d\mathcal{H} .$$

The Cantor function shows that Theorem 1 is false if T is not of $\mathcal{H}$-σ-finite measure.

8. The additivity of the v-integral. The v-integral is *rather* well-behaved.

(1) $f \mapsto I_v(f, A)$ is a nonnegative linear functional.

(2) If f is integrable in a BV set A, it is integrable in each $B \in BV_A$, and the function $B \mapsto I_v(f, B)$ is additive and continuous on BV_A.

(3) Each v-integrable function is almost everywhere a derivative of its indefinite v-integral; in particular, it is measurable.

(4) f is Lebesgue integrable if and only if both f and $|f|$ are v-integrable, in which case the Lebesgue and variational integrals of f have the same value.

(5) The monotone and dominated convergence theorems hold for the v-integral.

(6) The v-integral is invariant with respect to *Lipschitzian* change of coordinates, and hence it can be lifted to *rectifiable sets* as well as to C^1 *manifolds*.

But not *quite*, since the additivity is deficient!

PROPOSITION. *Let B and C be disjoint BV sets, $A = B \cup C$, and let f be a function on A which is v-integrable in B and C. Then f is v-integrable in A whenever $\mathrm{cl}B = \mathrm{cl}^*B$ and $\mathrm{cl}C = \mathrm{cl}^*C$.*

PROOF: Extend f arbitrarily to cl^*A, and choose an $\varepsilon > 0$. Let F_B and F_C be the indefinite v-integrals of f in B and C, respectively, and let M_B and M_C be, respectively, the $(\varepsilon/2)$-majorants of (f, F_B) and (f, F_C) in A mod T_B and A mod T_C where T_B and T_C are sets of $\mathcal{H}$-σ-finite measure. For $D \in BV_A$ set

$$F(D) = F_B(B \cap D) + F_C(C \cap D) \qquad \text{and} \qquad M(D) = M_B(B \cap D) + M_C(C \cap D),$$

and let $T = T_B \cup T_C \cup (\partial^* B) \cup (\partial^* C)$. To show that M is an ε-majorant of (f, F) in A mod T, observe that each $x \in \mathrm{cl}^*A - T$ belongs to int^*B or int^*C by the choice of T. By our assumption, x is either in $\mathrm{cl}^*B - \mathrm{cl}C$ or in $\mathrm{cl}^*C - \mathrm{cl}B$. Thus in a neighborhood of x either $F = F_B$ and $M = M_B$ or $F = F_C$ and $M = M_C$, and the Proposition follows.

In the previous proof, severe difficulties would have arisen if x had been in $(\mathrm{cl}B) \cap (\mathrm{cl}C)$ because the regularity of $D \in BV_A$ provides no information about the regularities of $B \cap D$ and $C \cap D$. In fact, an example of Buczolich shows that without the assumptions about $\mathrm{cl}B$ and $\mathrm{cl}C$ the Proposition is *false*. Thus the v-integral is still a mere *gadget*.

An *easy way* to fix the additivity is to say that f is *w-integrable* ("w" for *weakly*) in A whenever there is a division $\{A_1, \ldots, A_n\}$ of A such that f is v-integrable in A_i for $i = 1, \ldots, n$. But we shall do *better* by following the ideas of Mařík.

9. The continuous integral. We say that the family $\mathcal{C} \subset BV$ is *closed* whenever

$$(C_n \in \mathcal{C},\ n = 1,2,\ldots,\ C \in BV, \text{ and } C_n \to C) \quad \Rightarrow \quad C \in \mathcal{C}.$$

The intersection of closed subfamilies of BV is closed, and we define the *closure* $\mathrm{Cl}\mathcal{E}$ of a family $\mathcal{E} \subset BV$ as the intersection of all closed subfamilies of BV containing $\mathcal{E}$. Similar to the construction of *Baire functions*, the closure of $\mathcal{E} \subset BV$ is obtained by a *transfinite* iteration of limits. A *kernel* of a BV set A is any family $\mathcal{K} \subset BV_A$ with $A \in \mathrm{Cl}\mathcal{K}$.

DEFINITION 2. A function f on a BV set A is *c-integrable* ("c" for *continuously*) in A whenever there is a continuous additive function F on BV_A such that the family of all $B \in BV_A$ for which F is the indefinite v-integral of f in B is a kernel of A.

This resembles the *constructive* definition of the Denjoy integral. Indeed, the c-integral is obtained by forming *"improper"* v-integrals and iterating the process *transfinitely*. Consequently, there are *no* "improper" c-integrals, the *good* properties of the v-integral are inherited, and it follows from the next lemma, due to Miranda and Tamanini, that the deficiency in the Proposition disappears.

LEMMA 2. *Each BV set A contains BV subsets A_n such that $\mathrm{cl}A_n = \mathrm{cl}^* A_n$, $n = 1,2,\ldots$, and $A_n \to A$.*

PROOF: Fix an integer $n \geq 1$ and let $\varphi(B) = \|B\| - n|B|$ for each BV set B. Construct a *minimizing sequence* $\{B_k\}$ in BV_A so that

$$\lim_{k\to\infty} \varphi(B_k) = \inf_{B \in BV_A} \varphi(B).$$

Then the sequences $\{|B_k|\}$ and $\{\varphi(B_k)\}$ are bounded, and so is $\{\|B_k\|\}$. It follows from the *compactness* property of BV that there is a subsequence $\{C_k\}$ of $\{B_k\}$ and a BV set A_n such that

$$\lim_{k\to\infty} |(A_n - C_k) \cup (C_k - A_n)| = 0$$

Subtracting from A_n a set of measure zero, we may assume that A_n is a subset of A, and by the *semicontinuity* of perimeters,

$$\varphi(A_n) = \|A_n\| - n|A_n| \leq \liminf_{k\to\infty} \|C_k\| - n \lim_{k\to\infty} |C_k| \leq \limsup_{k\to\infty} \varphi(C_k) = \inf_{B \in BV_A} \varphi(B).$$

Thus A_n minimizes the functional φ on BV_A. Generally BV sets which are *solutions of variational problems* have additional nice properties (e.g., they have smooth or analytic boundaries). Using this principle, one can show (*nontrivially*) that $\mathrm{cl}A_n = \mathrm{cl}^* A_n$. Since

$$\|A_n\| - n|A_n| \leq \|A\| - n|A|,$$

we see that $\|A_n\| \leq \|A\|$ and $n|A - A_n| \leq \|A\|$. This implies that $A_n \to A$ and the lemma is proved.

10. The integral. If $m \geq 2$, then by weakening the continuity requirement, a further extension of the c-integral is possible.

(1) If $\{B_n\}$ is a sequence in BV and B is a BV set, we write $B_n \xrightarrow{*} B$ whenever $B_n \subset B$, $n = 1,2,\ldots$, and $\lim \|B - B_n\| = 0$.
(2) An additive function F of BV sets is *bounded* if

$$B_n \xrightarrow{*} B \quad \Rightarrow \quad \lim F(B_n) = F(B).$$

If $A \in BV$ and v is a bounded $\mathcal{H}$-measurable vector field on $\mathrm{cl}A$, then $B \mapsto \int_{\partial^* B} v \cdot n_B \, d\mathcal{H}$ is a bounded additive function on BV_A.

DEFINITION 2. A function f on a BV set A is *integrable* in A whenever there is a bounded additive function F on BV_A and a sequence $\{A_n\}$ of BV sets such that $A_n \xrightarrow{*} A$ and F is the indefinite c-integral of f in A_n, $n = 1,2,\ldots$.

Intuitively, the integral is just an *"improper"* c-integral, and thus it inherits all the good properties of the c-integral (except for the continuity, of course).

THEOREM 2. *Let $A \in BV$, and let $S, T \subset \mathbf{R}^m$ be such that $\mathcal{H}(S) = 0$ and T has $\mathcal{H}$-σ-finite measure. Suppose that v is a bounded vector field on clA which is continuous on $clA - S$ and almost differentiable on $cl^* A - T$. Then $div\, v$ is integrable in A and the Gauss-Green formula holds.*

We note again that no *topological* restrictions are imposed on the exceptional set S. The *Heaviside* function shows that Theorem 2 is false if $\mathcal{H}(S) > 0$.

11. Conclusion. While the problem of integrating the divergence of a differentiable vector field has been nicely solved, a *major challenge* remains: among many averaging methods available we have to select those (preferably just *one*) of *universal appeal*. I do not believe this can be done by inventing new and "more powerful" integrals or averaging gadgets. Rather, we should identify the properties, *axioms* if you want, which will determine *the integral* uniquely independent of a particular definition. Without drawing any parallels between the *importance* of the two fields, one can say that the present state of integration is somewhat analogous to that of homology theory prior to the introduction of Eilenberg-Steenrod axioms. I feel that in search for the axioms, it would be overly optimistic to expect in higher dimensions the same level of generality which the classical Denjoy-Perron integral achieves in dimension one. Reasons for this may be that only in dimension one the following is true:

(1) Each set of $\mathcal{H}$-σ-finite measure is *countable*.
(2) The subtle notion of differentiability is reduced to the existence of a *single limit*.

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