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SYMMETRIC DERIVATES OF NON-MEASURABLE FUNCTIONS

One of the areas of divergence between the ordinary and symmetric derivative is in the Denjoy-Young-Saks Theorem. For the ordinary derivative we have:

THEOREM: (Denjoy-Young-Saks [3]) For any real function f and for almost every x, either $f'(x)$ exists (finite) or else $\underline{f}'(x)=-\infty$ and $\overline{f}'(x)=+\infty$.

For the symmetric derivative we have:

THEOREM: (Ezzell-Nymann [2]) For any measurable real function f and for almost every x, either $f^{sy}(x)$ exists (finite) or else $\underline{f}^{sy}(x)=-\infty$ and $\overline{f}^{sy}(x)=+\infty$.

That Ezzell and Nymann's theorem does not hold for arbitrary functions was shown by Uher [5]. Uher constructed a set U of full outer measure such that for all x in U, if f(x) is the characteristic function of U, then $\underline{f}^{sy}(x)=0$ and $\overline{f}^{sy}(x)=+\infty$. Uher asked if there is a function f for which the set $\{x|\ -\infty < \underline{f}^{sy}(x) < \overline{f}^{sy}(x) < +\infty\}$ has positive outer measure. We will show that in one sense, the symmetric version of the Denjoy-Young-Saks Theorem fails in the worst possible way. That is, any one of the following sets

$S_0 = \{x \mid -\infty = \underline{f}^{sy}(x) < \overline{f}^{sy}(x) = +\infty\}$;

$S_1 = \{x \mid -\infty < \underline{f}^{sy}(x) = \overline{f}^{sy}(x) < +\infty\}$;

$S_2 = \{x \mid -\infty < \underline{f}^{sy}(x) < \overline{f}^{sy}(x) = +\infty\}$;

$S_3 = \{x \mid -\infty = \underline{f}^{sy}(x) < \overline{f}^{sy}(x) < +\infty\}$;

$S_4 = \{x \mid -\infty < \underline{f}^{sy}(x) < \overline{f}^{sy}(x) < +\infty\}$;

$S_5 = \{x \mid -\infty < \underline{f}^{sy}(x) = \overline{f}^{sy}(x) = +\infty\}$;

$S_6 = \{x \mid -\infty = \underline{f}^{sy}(x) = \overline{f}^{sy}(x) < +\infty\}$;

may have full outer measure for a particular function. Further, S_0 and any other S_i can simultaneously have full outer measure. The combinations $\{S_0,S_2,S_5\}$ or $\{S_0,S_3,S_6\}$ may also simultaneously have full outer measure. No other pairs are possible by the following theorem.

<u>THEOREM</u>: Let f be a real function and let $S \in T = \{S_1,\ S_2 \cup S_5,\ S_3 \cup S_6,\ S_4\}$. Then almost every outer density point of S is in $S \cup S_0$.

<u>NOTATION</u>: Let A be a set. Then cA denotes the complement of A, χ_A denotes the characteristic function of A, $\lambda(A)$ and $\lambda^*(A)$ denote the Lebesgue measure (when it exists) and the outer Lebesgue measure of A respectively. $D^*(A)$ denotes the outer density points of A which is taken to mean $\{x \mid \lim_{\epsilon \to 0^+} \lambda^*(A \cap (x-\epsilon, x+\epsilon))/2\epsilon = 1\}$. If f is a real function and x is a real number then $\overline{f}^{sy}(x)$, $\underline{f}^{sy}(x)$, and $f^{sy}(x)$ denote the upper symmetric derivate, lower symmetric derivate, and symmetric derivative of f at x respectively. We say that d is a symmetric derived number of f at x iff there is some sequence of numbers $h_n \to 0$ such that $\lim_{n\to\infty} (f(x+h_n)-f(x-h_n))/2h_n = d$.

EXAMPLES: Let H be a Hamel basis for the reals (with full outer measure) and let G consist of all reals with first Hamel basis coefficient zero. Uher's example [5] uses the set U consisting of all reals x, whose Hamel basis representation uses only basis elements less than or equal to x. The following examples show that both S_0 and S_i, $1 \le i \le 6$, can have full outer measure.

1) S_0 and S_1	$f(x)=\chi_G(x)$. Then $S_1=G$ and $S_0=cG$ (see also [2] Example 1).
2) S_0 and S_2	$f(x)=\chi_U(x)$. Then $S_2=U$ and $S_0=cU$ (see [5]).
3) S_0 and S_3	$f(x)=-\chi_U(x)$. Then $S_3=U$ and $S_0=cU$.
4) S_0 and S_4	$f(x)=x\cdot\chi_G(x)$. Then $S_4=G$ and $S_0=cG$.
5) S_0 and S_5	$f(x)=n-2k$ where n is the number of basis elements in the representation of x, and k is the number of these basis elements greater than x. (The symmetric derivative is infinite on H. In fact, if $h \in H$ is the center of some non-degenerate interval (a,b), then $f(b)-f(a) \ge 1$.)
6) S_0 and S_6	f(x) is the negative of example 5.
7) S_0, S_2, and S_5	Partition the Hamel basis H into two subsets J, and K, each of full outer measure. Let $f(x)=n-2k$ where n is the number of basis elements in the representation of x which are in J, and k is the number of these basis elements which are greater then x. ($S_0=cH-\{0\}$, $S_2=K\cup\{0\}$ and $S_5=J$)
8) S_0, S_3, and S_6	f(x) is the negative of example 7.

The proof of Theorem 1 depends on the following lemma:

<u>LEMMA</u>: If $\overline{f}^{sy}<0$ on a set A, then for almost every outer density point, x, of A, if d is a positive symmetric derived number at x, f also has a symmetric derived number at x which is less than or equal to -3d.

Proof: Let $A_n=\{x\in A \mid (f(x+h)-f(x-h))/2h < 0 \text{ for all } 0<h<1/n\}$. Then $A=\cup A_n$ and $\lambda^*(A_n)$ approaches $\lambda^*(A)$ so $\lambda(D^*(A_n))$ approaches $\lambda(D^*(A))$. It suffices then to show that every outer density point of any A_n satisfies the desired condition. Fix n and let x be an outer density point of A_n. Let d>0 be a symmetric derived number at x. We show the case of d finite, the argument for d infinite being similar. Pick a sequence $\{h_k\}$ of positive numbers less than 1/2n and approaching zero so that $\lim (f(x+h_k)-f(x-h_k))/2h_k = d$ and the relative outer measure of A_n in the interval $(x-h_k,x+h_k)$ is more than $1-1/(k+1)$. We can then pick a_k and b_k in $A_n\cap(x-h_k/3-2h_k/(k+1),x-h_k/3)$ and $A_n\cap(x+h_k/3,x+h_k/3+2h_k/(k+1))$ respectively. Reflect $x-h_k$ about a_k and then about b_k to get $c_k=2b_k-(2a_k-(x-h_k))$ and reflect $x+h_k$ about b_k and then about a_k to get $d_k=2a_k-(2b_k-(x+h_k))$. Observe that $d_k<x<c_k$, $(c_k+d_k)/2=x$, and $f(c_k)-f(d_k)<-(f(x+h_k)-f(x-h_k))$ since a_k and b_k are in A_n. Also, $c_k-d_k=4b_k-4a_k-2h_k<4(2h_k/3 + 4h_k/(k+1)) - 2h_k=2h_k/3 + 16h_k/(k+1)$. Thus, $(f(c_k)-f(d_k))/(c_k-d_k) < -(f(x+h_k)-f(x-h_k)) / (2h_k(1/3+8/(k+1)))$ which approaches -3d as k approaches infinity. This finishes the proof of the lemma.

The proof of the theorem will follow from this lemma. It can be shown that no pair of sets from T share a common outer density point. We will show this, for example, for the pair $\{S_1,S_4\}$. We choose this pair since it is the most involved. Suppose then that S_1 and S_4 share a common outer density point and therefore each has relative outer measure greater than 1/2 in some interval I. For each positive rational number q, let $A_q=\{x\in I \mid q<\overline{f}^{sy}(x)-\underline{f}^{sy}(x)\}$. Then for some q, $D^*(A_q)$ has relative measure greater than 1/2. Fix such a q and for each rational number r let $B_r=\{x\in I \mid r < f^{sy}(x) < r+q/3\}$. Then for some r, $D^*(A_q)$ must intersect $D^*(B_r)$. Fix such an r. Then there must also be an element a in $A_q \cap D^*(B_r)$. Let δ denote $\overline{f}^{sy}(a)-\underline{f}^{sy}(a)$, and γ denote $(\overline{f}^{sy}(a)+\underline{f}^{sy}(a))/2$. If $r\leq\gamma-\delta/6$ then we may apply the lemma to $f(x)-(\gamma+\delta/6)x$ and get that $\underline{f}^{sy}(a) \leq (\gamma+\delta/6)-3(\overline{f}^{sy}(a)-(\gamma+\delta/6)) = 4(\gamma+\delta/6)-3\overline{f}^{sy}(a)$ and so $\overline{f}^{sy}(a)-\underline{f}^{sy}(a)\geq 4(\overline{f}^{sy}(a)-(\gamma+\delta/6))=4(\delta/3)$. Similarly, if $r\geq\gamma+\delta/6$ then by applying the lemma to $(\gamma+\delta/6)x-f(x)$ we also get that $\overline{f}^{sy}(a)-\underline{f}^{sy}(a)\geq 4(\delta/3)$. But this contradicts that a is in A_q.

For the other pairs from T, the proof follows easily from the lemma and is omitted. Suppose that $S\in T$ and $A=D^*(S)-S-S_0$ has positive outer measure. Then A must intersect some other $S'\in T$ in a set of positive outer measure, and therefore $A\cap S'$ has an outer density point. But any outer density point of A is an outer density point of S, contradicting the lemma. Therefore A must have measure zero and the theorem is proved.

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