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ON THE GENERALIZED RIEMANN INTEGRAL DEFINED BY MEANS OF SPECIAL PARTITIONS

Let m be a fixed positive integer, and let $\mathbb{R}^m$ be the product of m copies of the set $\mathbb{R}$ of all real numbers. In $\mathbb{R}^m$ we use the metric induced by the norm $|x| = \max\{|\xi_1|,\dots,|\xi_m|\}$ for $x = (\xi_1,\dots,\xi_m)$ in $\mathbb{R}^m$; note that this metric, which is convenient for our purposes, differs from the usual Euclidean metric in $\mathbb{R}^m$. If $E \subset \mathbb{R}^m$, then $d(E)$ and $|E|$ denote, respectively, the diameter and the outer Lebesgue measure of E; moreover, if E has a positive diameter, we set $r(E) = |E|/[d(E)]^m$.

An *interval* is a compact nondegenerate interval in $\mathbb{R}^m$. A *special partition* (cf. [P, Remark 7.4]) of an interval A is a collection $P = \{(C_1,x_1),\dots,(C_p,x_p)\}$ where $C_1,\dots,C_p$ are intervals whose interiors are disjoint and whose union is A, and x_i is a *vertex* of C_i, $i = 1,\dots,p$. If $r(C_i) > \epsilon$ for an $\epsilon > 0$ and $i = 1,\dots,p$, we call P a special *ϵ-partition* of A. If δ is a positive function on A and $d(C_i) < \delta(x_i)$ for $i = 1,\dots,p$, we say that P is a *δ-fine* special partition of A. An easy compactness argument shows that a δ-fine special 1-partition of an interval $A \subset \mathbb{R}^1$ exists for any positive function δ on A (see [H, Theorem 1]). However, in case of $m \geq 2$, the existence of δ-fine special ϵ-partitions has been an open problem for several years. A partial solution was given by Z. Buczolich, who proved the following theorem (see [B]).

THEOREM (Buczolich). There is a positive constant $\kappa < 1$ such that for each interval $A \subset \mathbb{R}^2$ and each positive function δ on A we can find a δ–fine special κ–partition of A.

The proof is quite involved, and it is not clear whether it can be generalized to higher dimensions. Recently, A. Mkhalfi attempted to solve the problem in any dimension (see [M, Lemma 2]), but it appears that there is a gap in his argument. Thus, to my knowledge, the problem is still open when $m \geq 3$. For this reason, throughout the remainder of this note, we assume that $m \leq 2$.

DEFINITION. Let f be a real–valued function on an interval A. We say that f is s–*integrable* on A if there is a real number I with the following property: given an $\epsilon > 0$, we can find a positive function δ on A such that

$$\left| \sum_{i=1}^{p} f(x_i)\,|C_i| - I \right| < \epsilon$$

for each δ–fine special ϵ–partition $\{(C_1,x_1),\dots,(C_p,x_p)\}$ of A.

It follows from the Theorem that the number I in the Definition is determined uniquely by the s–integrable function f; it is called the s–*integral* of f over A, denoted by $S(f,A)$. A standard completeness argument (see [H, Theorem 4] or [P, Proposition 3.4]) shows that the s–integrability of f over A implies the s–integrability of f over each interval $B \subset A$; the function $B \mapsto S(f,B)$ is called the *indefinite* s–*integral* of f in A. We use the letters s and S to indicate that the integral arises from special partitions.

Clearly, the s–integral generalizes the integral from [P, Definition 3.1]; for $m = 1$, these integrals actually coincide. It is also easy to establish that the s–integral has all the properties listed in [P, Sections 3–6], with the exception of [P, Proposition 4.10 and Corollary 4.11] which are *false* for the s–integral when $m = 2$. In particular, the space of all s–integrable functions on an interval A is a linear space on which the s–integral is a nonnegative linear functional, and the indefinite s–integral is an additive function of intervals. To illustrate the use of special partitions, we prove the additivity of the indefinite s–integral.

PROPOSITION. Let an interval A be the union of intervals B_1, B_2 whose interiors are disjoint, and let f be a function on A which is s–integrable on B_1 and B_2. Then f is s–integrable on A and $S(f,A) = S(f,B_1) + S(f,B_2)$.

PROOF. If $P = \{(C_1,x_1),\dots,(C_p,x_p)\}$ is a partition of a subinterval of A, we set $\sigma(P) = \Sigma_{i=1}^{p} f(x_i)|C_i|$. Choose an $\epsilon > 0$, and find positive functions δ_j on B_j, $j = 1,2$, so that $|\sigma(P_j) - S(f,B_j)| < \epsilon/2$ for every δ_j–fine special $(\epsilon/2)$–partition P_j of B_j. With no loss of generality, we may assume that for each $x \in B_1 - B_2$ the number $\delta_1(x)$ is smaller than the distance from x to B_2, and symmetrically, for each $x \in B_2 - B_1$ the number $\delta_2(x)$ is smaller than the distance from x to B_1. Now let

$$\delta(x) = \begin{cases} \delta_1(x) & \text{if } x \in B_1 - B_2 , \\ \delta_2(x) & \text{if } x \in B_2 - B_1 , \\ \min\{\delta_1(x),\delta_2(x)\} & \text{if } x \in B_1 \cap B_2 , \end{cases}$$

and choose a δ–fine special ϵ–partition P of A. Since P is a special partition, it follows from our choice of δ that $P_j = \{(C,x) \in P : C \subset B_j\}$, $j = 1,2$, is a δ_j–fine special ϵ–partition of B_j. As $\epsilon > \epsilon/2$, we have

$$|\sigma(P) - [S(f,B_1) + S(f,B_2)]| \le |\sigma(P_1) - S(f,B_1)| + |\sigma(P_2) - S(f,B_2)| < \epsilon$$

and the Proposition is proved.

The purpose of this note is to show by example that for $m = 2$ the indefinite s–integral is neither *continuous* nor *bounded* function of intervals. This is the point where the s–integral differs significantly from the integral defined in [P, Definition 3.1] (cf. [P, Proposition 4.10]).

EXAMPLE. Let $m = 2$, $A = [-1,1]^2$, $B = [0,1]^2$, and for $n = 1,2,\dots$, let

$$B_{+n} = [3\cdot 2^{-n-1}, 2^{-n+1}] \times [0, 2^{-n^2}] \quad \text{and} \quad B_{-n} = [0, 2^{-n^2}] \times [3\cdot 2^{-n-1}, 2^{-n+1}] .$$

For $x \in A$, set $f(x) = \pm 2^{n^2+n-1}/n$ if $x \in B_{\pm n}$, and $f(x) = 0$ otherwise. Finally, let $\mathbf{0} = (0,0)$.

We show first that f is s–integrable on B, and that $S(f,B) = 0$. To this end choose an $\epsilon > 0$, and find positive integers j and k so that $2^{-j} < \epsilon$ and $j/k < \epsilon$. Next define a positive function δ on B which satisfies the following conditions:

(1) $\delta(\mathbf{0}) < 2^{-k}$;

(2) $\delta(x) < |x|$ for each $x \in B - \{\mathbf{0}\}$;

(3) $\delta(x) < 2^{-n-1}$ for each x in the boundary of $B_{\pm n}$, $n = 1,2,\dots$;

(4) $\delta(x)$ is smaller than the distance from x to the boundary of $B_{\pm n}$ for each $x \in B - \{\mathbf{0}\}$ which does not lie on the boundary of $B_{\pm n}$, $n = 1,2,\dots$.

Now if $P = \{(C_1,x_1),\dots,(C_p,x_p)\}$ is a δ–fine special ϵ–partition of B, then $0 \in \{x_1,\dots,x_p\}$ by condition (2). We may assume that $x_1 = \mathbf{0}$. As P is a special partition, it follows from conditions (3) and (4) that each of the intervals $C_2,\dots,C_p$ is either contained in $B_{\pm n}$ for *some* integer $n \geq 1$, or is disjoint from $B_{\pm n}$ for *all* integers $n \geq 1$. Thus $f(x_i)|C_i| = S(f,C_i)$ for $i = 2,\dots,p$. Since $r(C_1) > 2^{-j}$, using the additivity of the s–integral and condition (1), it is not difficult to deduce that

$$\left|\sum_{i=1}^{p} f(x_i)|C_i|\right| = \left|\sum_{i=2}^{p} S(f,C_i)\right| \leq \sum_{n=k+1}^{k+j} \frac{1}{n} \leq j/k < \epsilon .$$

The same argument shows that $S(f,[0,x]^2) = 0$ for each $x \in (0,1]$, a fact which proves to be important.

Since $f = 0$ outside B, we see that f is s–integrable in A, and $S(f,A) = 0$. We set $F(x,y) = S(f,[-1,x]\times[-1,y])$ for each $(x,y) \in (-1,1]^2$, and show that F is *unbounded* in a neighborhood of $\mathbf{0}$. Indeed,

$$F(2^{-q+1},2^{-q^2}) = \sum_{n=q}^{q^2-1} S(f,B_{+n}) = \sum_{n=q}^{q^2-1} \frac{1}{n} \geq \int_q^{q^2} \frac{dt}{t} = \log q ,$$

and similarly, $F(2^{-q^2},2^{-q+1}) \leq -\log q$, $q = 1,2,\dots$. In particular, F is *discontinuous* at $\mathbf{0}$.

Two conclusions can be drawn from the Example.

(a) The Alexiewicz norm $\|f\| = \sup\{S(f,[-1,x]\times[-1,y]) : (x,y) \in A\}$ *cannot* be defined in the space of s–integrable functions on an interval A (cf. [O, Definition 2]).

(b) The s–integral is a *proper* extension of the integral defined in [P, Definition 3.1]. Indeed, if the function f of the Example were integrable in the latter sense, then by [P, Proposition 4.10], the function F of the Example would be continuous.

In view of (a), it appears that the behavior of the integral from [P, Definition 3.1] is superior to that of the s–integral. We feel that the added generality of the s–integral established in (b) is of little value, and that the technical complexity of [P, Definition 3.1] may be unavoidable for obtaining an integral with desirable properties.

REMARK. Conclusions (a) and (b) hold also for the GM–integral of [M, Definition 4], which is easily seen to coincide with the s–integral.

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