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# A CRITERION FOR MEASURABILITY OF COUNTABLE-TO-ONE FUNCTIONS

Let $X$ be a subset of $\mathbf{R}$ and denote by $\mathcal{B}(X) = \{B \cap X : B$ Borel subset of $\mathbf{R}\}$ the $\sigma$-field of all measurable subsets of $X$. Given subsets $X$ and $Y$ of $\mathbf{R}$, a function $f : X \to Y$ is measurable if $f^{-1}(C) \in \mathcal{B}(X)$ for each $C \in \mathcal{B}(Y)$. If $f$ is a one-one correspondence, and both $f$ and $f^{-1}$ are measurable, then $f$ is a Borel-isomorphism (or generalised homeomorphism as in [1]), and $X$ and $Y$ are Borel-isomorphic. The following result is well known [1; p. 434]:

1.1 Lemma: Let $X$ be a subset of $\mathbf{R}$, and let $f : X \to \mathbf{R}$ be measurable. Then $f$ extends to a measurable function $g : \mathbf{R} \to \mathbf{R}$.

A subset of $\mathbf{R}$ is analytic if it is the image of a Borel subset of $\mathbf{R}$ under a measurable map. A measurable subset of an analytic set is again analytic. For basic facts about these sets, vide [1].

1.2 Theorem: Let $f : X \to Y$ be a one-one correspondence between subsets $X$ and $Y$ of $\mathbf{R}$. Suppose that $X$ is analytic. In order that $f$ be a Borel-isomorphism, it is necessary and sufficient that for each $A \subseteq X$, the sets $A$ and $f(A)$ be Borel-isomorphic.

Proof: Necessity is obvious. Suppose now that $f$ has the indicated property. Given any $A \in \mathcal{B}(X)$, we know that $A$ and $X - A$ are analytic. Thus $f(A)$ and $f(X-A) = Y - f(A)$ are analytic. By Lusin's first separation theorem [1; p. 485], there is some Borel subset $B$ of $\mathbf{R}$ such that $f(A) \subseteq B$ and $Y - f(A) \subseteq \mathbf{R} - B$. It follows that $f(A) \in \mathcal{B}(Y)$. We have shown that $f^{-1}$ is measurable. Since $Y = f(X)$ is analytic, a symmetrical argument shows that $f$ is measurable.

Q.E.D.

Under the continuum hypothesis (CH), the condition of analyticity is not needed. This will follow from

<u>1.3 Theorem (CH)</u>: Let $f : X \to \mathbf{R}$ be a countable-to-one function defined on a subset $X$ of $\mathbf{R}$. In order that $f$ be measurable, it is necessary and sufficient that for each $A \subseteq X$, the set $f(A)$ be a measurable image of $A$.

<u>Proof</u>: Necessity is obvious. To prove sufficiency, we show the contrapositive. Suppose that $f$ is not measurable. List all measurable, countable-to-one functions on $X$ as $f_0\ f_1 \ldots f_\alpha \ldots \alpha < \omega_1$. We construct the elements of a set $A \subseteq X$ by transfinite induction: suppose that the points $x_\beta$ have been chosen for all $\beta < \alpha$, where $\alpha$ is a countable ordinal. Choose $x_\alpha$ from the set

$$\{x \in X : f(x) \neq f_\alpha(x)\} - f^{-1}\{f_\beta(x_\beta) : \beta < \alpha\}$$
$$- f_\alpha^{-1}\{f(x_\beta) : \beta < \alpha\},$$

whose uncountability is easily seen. Finally, put $A = \{x_\alpha : \alpha < \omega_1\}$. We assert that $f(A)$ is not a measurable image of $A$. Were it so, there would be some measurable function $g : A \to R$ with $g(A) = f(A)$. By lemma 1.1, $g$ is the restriction of some one of the functions $f_\alpha$. Then $x_\alpha \in A$, but we shall demonstrate that $g(x_\alpha) = f_\alpha(x_\alpha)$ is not a member of $f(A)$. For suppose $f_\alpha(x_\alpha) = f(x_\beta)$ for some $\beta < \omega_1$. It is easy to check that this violates the conditions under which $x_\alpha$ and $x_\beta$ were chosen.

Q.E.D.

1.4 Corollary (CH): Let $f : X \to Y$ be a one-one correspondence between arbitrary subsets $X$ and $Y$ of $R$. In order that $f$ be a Borel-isomorphism, it is necessary and sufficient that for each $A \subseteq X$, the sets $A$ and $f(A)$ be Borel-isomorphic.

In Theorem 1.3, the hypothesis that $f$ be countable-to-one cannot be eliminated. To see this, let $f$ be the indicator function of a non-Borel subset of $X = R$. I do not know whether the assumption of CH is necessary in the previous results.

[1] Kuratowski, K., Topology, Vol. I, Academic Press-PWN, New York-Warsaw 1966

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