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On a Conjecture of Stoilow Concerning Sets of Lebesgue Measure Zero

1 Introduction

Sets of measure zero will be called nullsets. Although these sets, from a measure theoretic perspective are all the same size, nullsets may have very different properties with respect to their cardinal, topological and metric properties. This explains why, from the beginning of this century, various classifications of nullsets have been proposed. (See for example Hausdorff dimension [12], Borel classification [2], [3], [4], [5], [6], Fréchet rarefaction [7] etc.)

In 1919, S. Stoilow [15] proposed a classification of nullsets and studied it further in [16] and [17]. (Also see [18, pp. 100-108].) Stoilow's classification (StC for short) has the advantage that it does not depend on the method of defining nullsets, but, as Stoilow [15] himself remarked, presents other shortcomings. For instance, Stoilow dimension (to be defined) depends on the way in which the elements of the set are written in a certain scale. As we will see in the second section of the present paper, there are sets of infinite Stoilow dimension with respect to a certain scale which have Stoilow dimension one with respect to another scale. However, StC seems to be an interesting and still unexplored hierarchy of nullsets.

In the sequel $\mathbf{N}$ is the set of nonnegative integers, I is the unit interval $[0,1]$ and $\mathbf{N}_k$ is the set of all positive integers not less than k, for any positive integer k. Let $b \in \mathbf{N}_2$, and let $A \subset I$ be a nullset. Define $A_1 = A$ and, for every $n \in \mathbf{N}_2$, let $A_n \subset I^n$ be the set of all n-tuples $(x_1, \ldots, x_n)$ with the property that

$$0.x_1^{(1)}x_1^{(2)}\ldots x_1^{(n)}x_2^{(1)}x_2^{(2)}\ldots x_2^{(n)}\ldots \in A,$$

where

$$x_i = 0.x_1^{(i)}x_2^{(i)}\ldots x_2^{(i)}\ldots x_k^{(i)}\ldots$$

for all $i = 1, 2, \ldots, n$. All above numbers are represented in the scale b, the representations being considered infinite. The construction of the sets A_n is the

same as that used in the standard proof of the fact that I^n and I have the same power.

We say that A has Stoilow dimension one with respect to the scale b if at least one of the two projections $proj_1\ A_2$ or $proj_2\ A_2$ on the first or, respectively, the second axis is not a linear nullset. Inductively, we say that A has Stoilow dimension n with respect to the scale b and we write $St(A,b) = n$ if $St(A,b) > n-1$ (that is, all the projections of sets $A_2, \ldots, A_n$ are linear nullsets) and at least one of the projections $proj_1 A_{n+1}, \ldots, proj_{n+1} A_{n+1}$ is not a linear nullset. If $St(A,b) = n$, then A is said to be in the n-th class of StC. Those sets for which $St(A,b) \neq n$ for every $n \in \mathbf{N}_1$ are called sets of infinite Stoilow dimension with respect to the scale b. For example, all countable sets are nullsets of infinite Stoilow dimension with respect to the scale b. In his first paper on this subject, Stoilow [15] wrote about sets of infinite Stoilow dimension with respect to the scale b, "Ils semblent être tous dénombrables". However, the classical Cantor ternary set has infinite Stoilow dimension with respect to the scale 3 as follows from Lemma 2.2 below, but has the power of the continuum. Thus it is natural to ask, "how large may sets be which have infinite Stoilow dimension with respect to the scale b?" The Cantor ternary set (which is the first natural counterexample to Stoilow's conjecture) is a nowhere dense, closed set in I and its Hausdorff dimension is $(\log 2)/(\log 3)$. (See Section 2 for complete references to this last result.)

The first Theorem of the present paper will prove that for any $\varepsilon > 0$ and for every sufficiently large scale b there exists a set which is a counterexample to Stoilow's conjecture with respect to b and its Hausdorff dimension is greater than $1 - \varepsilon$. In Section 3 we will construct, for every scale $b \in \mathbf{N}_2$, a dense set in I which has the power of the continuum and has infinite Stoilow dimension with respect to b. In the case $b = 2$, using some results due to T. Šalát [14], we will show that the above dense counterexample to Stoilow's conjecture may be taken to be a F_σ-set; that is, a countable union of closed sets. In the last section we will show that StC is effective, that is every class from this classification is nonempty and we will give a counterexample to show that StC does not possess the Darboux property in the sense of S. Marcus [10].

Of course, many interesting questions concerning StC remain open. Some of them are briefly discussed in Section 4.

2 Counterexamples with large Hausdorff dimension

It is shown in [9] that a linear set of positive Lebesgue measure has Hausdorff dimension 1. Hence the Hausdorff dimension differs from one only for nullsets. The main result of this section is the existence of a set of infinite Stoilow dimension with respect to the scale b having the power of the continuum and Hausdorff dimension greater than $1-\varepsilon$. Such a set is a counterexample to Stoilow's conjecture. Here $\varepsilon > 0$ is fixed and b is a sufficiently large integer. In fact we may pick the above set to have Hausdorff dimension "very near" to a number $h, 0 < h < 1$, given in advance. More precisely, we may state the following result.

Theorem 2.1 *Let h_1 and h_2 be two given real numbers in Int I with $h_1 < h_2$ and n a positive integer. Then, for every sufficiently large integer b, there exists a set A having the power of the continuum and infinite Stoilow dimension with respect to b^n such that*

$$h_1 < \dim \text{ Hausdorff } (A) < h_2.$$

The proof of this Theorem will be based upon an auxiliary result which has its own interest.

We begin with some notation. Let $d \in \{0, 1, \ldots, b-1\}$ be a fixed digit in the scale $b, b \geq 3$. We denote by $C(b; d)$ the generalized Cantor set of all numbers from I with the property that in their representatons in the scale b, d is a missing digit. Then $C(b; d_1, d_2, \ldots, d_k) := C(b; d_1) \cap C(b; d_2) \cap \ldots \cap C(b; d_k)$ is the set of all numbers with $d_1, \ldots, d_k$ missing in their representations in the scale b, where $b \in \mathbf{N}_3, k$ is a positive integer smaller than b and $d_1, d_2, \ldots, d_k$ are k fixed digits in the scale b.

As we claimed in the Introduction, the classical Cantor ternary set $C(3; 1)$ has infinite Stoilow dimension with respect to the scale 3. A more general result is contained in the following lemma.

Lemma 2.2 *Let n and k be two positive integers, $b \in \mathbf{N}_3$ with $b > k$ and let $d_1, \ldots, d_k$ be k fixed digits in the scale b. Then the set $C(b; d_1, \ldots, d_k)$ has infinite Stoilow dimension with respect to b^n.*

Proof. A first remark (derived directly from the definitions) is the following.

If $B \subseteq A$, then $St(B, b) \geq St(A, b)$. In particular if $B \subseteq A$ and the Stoilow dimension of A is infinite, then the Stoilow dimension of B is also infinite.

Thus it is sufficient to prove Lemma 2.2 for $k = 1, d_1 = d$ and $C := C(b; d)$. The representation of a number in the scale b^n may be obtained from the corresponding representation in the scale b, by considering "blocks" of n digits and interpreting every block as a digit in the scale b^n. Because d is a missing digit in the scale b, for every element of C, d has the same property for every element from $proj_i C_t$, with t and i fixed, $1 \leq i \leq t$. Therefore, in the representation of an element from $proj_i C_t$ in the scale b, the block $dd \ldots d$ (n times) does not occur. Hence the projection $proj_i C_t$ is included in $C(b^n; d(b^n - 1)/(b - 1))$ which is a nullset. Thus $proj_i C_t$ is itself a set of Lebesgue measure zero and this shows that C has infinite Stoilow dimension.

Remark 2.3 *The Stoilow dimension of the sets $C(b; d_1, \ldots, d_k)$ may change if we use other scales. For instance, $Q := C(4; 1, 2)$ has infinite Stoilow dimension with respect to $4^n, n \geq 1$, by Lemma 2.2, but $St(Q, 2) = 1$! Indeed, the elements of Q have representations in the scale 2 of the form*

$$0.x_1x_1x_2x_2x_3x_3 \ldots \text{ where } x_j = 0 \text{ or } 1 \quad (j = 1, 2, \ldots)$$

since very element of Q has a representation in the scale 4 only with digits 0 and 3 and the "block" 00 in the scale 2 is the digit 0 in the scale 4 and 11 in the scale 2 is 3 in the scale 4. Thus $proj_1 Q_2$ contains every number of the form $0.x_1x_2x_3 \ldots$ in the scale 2. That is, $proj_1 Q_2 = I$ which is not a nullset. Hence $St(Q, 2) = 1$. This example shows the importance of the scale in the behavior of the Stoilow dimension of a set.

Now we are in a position to prove Theorem 2.1.

<u>**Proof of Theorem 2.1**</u>. The set $C(b; d_1, \ldots, d_k)$ has infinite Stoilow dimension with respect to b^n (Lemma 2.2) and has the power of the continuum. On the other hand, its Hausdorff dimension is $(\log(b - k))/\log b$. (See [8], [19], [9].)

For sufficiently large integers b we have $b^{h_2 - h_1} > 1 + b^{-h_1}$ since the left-hand member of the above inequality tends to infinite while the right-hand one tends to 1 when b tends to infinity. Hence $b^{h_2} - b^{h_1} > 1$ for all large b and because $0 < b^{h_1} < b^{h_2} < b$, there exists an integer $k < b$ such that $b^{h_1} < b - k < b^{h_2}$. This yields $h_1 < (\log(b - k))/\log b < h_2$. Therefore, for any k-tuple $(d_1, \ldots, d_k)$ of digits in the scale b, we have $h_1 <$ dim Hausdorff $(C(b; d_1, \ldots, d_k)) < h_2$ which completes the proof.

3 Dense sets as counterexamples to Stoilow's conjecture

The counterexamples to Stoilow's conjecture given in Theorem 2.1 are based on generalized Cantor sets. Thus even if they may have large Hausdorff dimension for sufficiently large scales, from the density point of view they are rather thin. However, for <u>all</u> scales $b \in \mathbf{N}_2$, we may construct counterexamples to Stoilow's conjecture dense in I. It is worthwhile mentioning the influence of Šalát's work [13,14] on the results of this section.

Theorem 3.1 *Let $b \in \mathbf{N}_2$. There exists a set $S = S(b) \subset I$ with the following properties:*

(i) S <u>has infinite Stoilow dimension with respect to</u> b;

(ii) S <u>has the power of the continuum</u>;

(iii) S <u>is dense in</u> I.

Proof. Let $S = S(b)$ be the set of all numbers x from I with the property that the infinite series $\sum_{n\geq 1} e_n(x)/n$ converges. Here $e_n(x) = 0$ if the n-th digit in the representation of x in the scale b is even and $e_n(x) = 1$ if the n-th digit is odd. Firstly, we show that S is a nullset. In 1964, T. Šalát [13] proved that if (a_n) is a decreasing sequence of positive numbers with $\liminf_{n\to\infty} na_n > 0, \lim_{n\to\infty} a_n = 0$ and if $e_t \in \{0,1\}$ such that $\limsup_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} e_t > 0$, then the infinite series $\sum_{t\geq 1} e_t a_t$ diverges. We will apply this theorem for $a_t = 1/t$ and $e_t = e_t(x)$. Since the strong law of large numbers implies that, for almost all $x \in I, \lim_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} e_t(x) > 0$ according to the aforementioned result of Šalát, S is a nullset. We note that this argument is similar to the one due to Galambos [9, pp. 123-124] in the case $b = 2$.

Now we show that S has infinite Stoilow dimension with respect to b. To this end consider the set

$$S_n = \{(x_1, \ldots, x_n) : x_i = 0.x_1^{(i)} \ldots x_p^{(i)} \ldots ; y \in S\},$$

where $y = 0.x_1^{(1)}x_1^{(2)} \ldots x_1^{(n)}x_2^{(1)}x_2^{(2)} \ldots x_2^{(n)} \ldots$ and let $x_k \in proj_k S_n$. We recall that all numbers are represented in the scale b. From the definition of the set S we have the convergence of the series $\sum_{t\geq 1} e_t(y)/t$. This shows that the series

$$\sum_{t\geq 1} e_{x_1^{(k)}+(t-1)_n}(y)/(x_1^{(k)} + (t-1)n)$$

has the same nature as $\sum_{t\geq 1} e_t(y)/t$ (being a subseries of a convergent series with non-negative terms). Hence the above series converges and this shows that $proj_k S_n \subseteq S$ for any positive integer n and all k with $1 \leq k \leq n$. Thus $proj_k S_n$ are all nullsets and S has infinite Stoilow dimension with respect to the scale b.

Since when in $\sum_{t\geq 1} 1/t$; we omit all digits which contain a 9 in their representation in the scale ten, the resulting series converges, all numbers which, in the scale b, have odd digits only in positions which in the scale ten miss the digit 9 are elements of S. Thus S has the power of the continuum.

Now we show that S is dense in I. It is sufficient to prove that for every pair of positive integers m and n with $0 \leq m \leq b^n - 1$, $S \cap (m/b^n, (m+1)/b^n]$ is nonempty. If the b-adic (finite) expansion of m/b^n is $m/b^n = \sum_{k=1}^{n} \varepsilon_k b^{-k}$, then every number $x \in (m/b^n, (m+1)/b^n]$ is characterized by the fact that its b-adic expansion $x = \sum_{k=1}^{\infty} \varepsilon_k(x) b^{-k}$ satisfies $\varepsilon_k(x) = \varepsilon_k$ for $k = 1, \ldots, n$. Then the number x_0 given by its b-adic expansion $x_0 = \sum_{k-1}^{\infty} \varepsilon_k(x_0) b^{-k}$ where $\varepsilon_k(x_0) = \varepsilon_k$ for $k = 1, \ldots, n, \varepsilon_k(x_0) = 1$ for $k > n$ and $k = j^2$ (with a certain suitable j) and $\varepsilon_k(x_0) = 0$ for $k > n$ and $k \neq i^2$ ($i = 1, 2, 3, \ldots$), is a number from the interval $(m/b^n, (m+1)/b^n]$. On the other hand, the convergence of the series $\sum_{j=1}^{\infty} 1/j^2$ implies $x_0 \in S$.

The proof is now complete.

The set $S(2)$ (constructed above for an arbitrary scale b) has a very extensive literature. We refer the reader again to Šalát [13], [14] and the references cited therein. Using some of these previous results we may obtain further additional information on counterexamples to Stoilow's conjecture in the case $b = 2$. For instance, Theorem 1.8 by Šalát [14] tells us that $S = S(2)$ is a $F_{\sigma\delta}$ set in $(0, 1]$. However, Theorem 3.2 below shows that there is even a F_σ-set which is a counterexample to Stoilow's conjecture.

Theorem 3.2 *There exists a set $S^* \subset I$ with the following properties:*

(i) S^* has infinite Stoilow dimension with respect to the scale 2;

(ii) S^* has the power of the continuum:

(iii) S^* is dense in I;

(iv) S^* is a F_σ-set.

Proof. We define $S^* = S(2) \cup (\{k/2^n : 1 \leq k \leq 2^n, n \in \mathbf{N}\} \setminus \{1\})$, where $S(2)$ is the set given in the proof of Theorem 3.1. Since $S(2)$ has infinite Stoilow dimension with respect to 2, the union of $S(2)$ with every countable set has the same property. Thus (i) is true and for the same reasons (ii) and (iii) are

satisfied. Now we show that S^* is a F_σ-set. To this end it suffices to construct a function f whose set of discontinuities is S^* because a well known result asserts that such a set is a F_σ-set. Let f be the real function defined on I in the following way: if $x \in S(2)$; i.e., the number $g(x) := \sum_{n=1}^{\infty} e_n(x)/n$ is finite, then we put $f(x) = (g(x))/(1+g(x))$; if $x \notin S(2)$, we define $f(x) = 1$. Then Theorem 2.1 of Šalát [14] says that the set of discontinuities of f is S^*. Hence Theorem 3.2 is proved.

Remark 3.3 *It is interesting to note that Šalát [14] proved that the function f used above also has many other remarkable properties. For example f is a strong locally recurrent function, it has the Darboux property and is of the second Baire class. (For definitions see Šalát [14].)*

4 Miscellaneous remarks

A natural question which arises is the following. Is StC <u>effective</u>, i.e. are all classes from StC non empty? The examples which show that the answer is affirmative are constructed by a modification of the generalized Cantor sets $C(b;d)$ in order to obtain a set with a given Stoilow dimension.

Proposition 4.1 *Let b and n be two positive integers and let d be a fixed digit in the scale b. Define and set $K(b,d,n)$ of all numbers from I which have representations in the scale b such that d appears in no position of the form $nq+r$ with $q \in \mathbf{N}, 0 < r < n$. Then $St(K(b,d,n);b) = n-1$.*

Proof. Let $K := K(b,d,n)$. Because digits in the positions of the form nq may attain any digit of the scale b, the n-th projection $proj_n K_{n+1}$ is equal to I. Thus $St(K,b) \leq n-1$. Let us assume that there exists a positive integer k with $1 \leq k \leq n-1$ such that $St(K,b) = k$. Thus there is a number i with $1 \leq i \leq k+1$ such that $proj_i K_{k+1}$ is not a nullset. However, every member of $proj_i K_{k+1}$ avoids the occurrence of the block dd in the representation in the scale b. Indeed the digit d appears in the representation of a number in the scale b, only in a position of the form ns by the construction of K. Thus the next digit in the representation is in the position $ns+k+1$ and because $k+1 \leq n-1$, this digit is not d. Hence, passing again to representations in the scale $b^2, d(b^2-1)/(b-1)$ is a missing digit. This implies $proj_i K_{k+1} \subseteq C(b^2, d(b^2-1)/(b-1))$ which shows that $proj_i K_{k+1}$ is a nullset. This contradiction leads to the desired equality $St(K,b) = n-1$.

Remark 4.2 *The above examples from Proposition 4.1 permit us to show that Stoilow's hierarchy does not have the <u>Darboux property</u>.*

According to S. Marcus [10], a (decreasing) family $\{F_i\}$ of sets with $F_i \cap F_j = \emptyset$ for $i \neq j$ has the Darboux property if for any $i < j-1$, for every $A_i \in F_i, A_j \in F_j$ with $A_i \supseteq A_j$ and for every k with $i < k < j$ there exists $A_k \in F_k$ such that $A_i \supseteq A_k \supseteq A_j$. Taking for F_n the n-th class of StC, $i = 2, j = 5, k = 3$ and $A_2 = K(2,1,3), A_5 = K(2,1,6)$ we have $A_2 \supseteq A_5$. However, if $A_3 \in F_3$ is a set with $A_3 \subseteq A_2$ it follows that each number from A_3 contains the digit 1 in the representation in the scale 2 only in positions of the form $3q$. Hence, all four projections (in $\mathbf{R}^4$) have 11 as a missing block, as a argument similar to the proof of Proposition 4.1 shows. Thus, considering representations in the scale 4, we get that all four projections are nullsets, which contradicts $A_3 \in F_3$.

The fact that StC does not have the Darboux property shows that StC behaves somewhat irregularly with respect to the relation of set inclusion.

Of course, there are many other topics for further study. Firstly, we state the following

Problem 4.3 *Let $b \in \mathbf{N}_2$. It is true that there exists a counterexample to Stoilow's conjecture with respect to the scale b which is of the second Baire category in I?*

According to a result of Banerjee and Lahiri [1] the set $S(2)$ from Theorem 3.1 is of first category in I. In [11] Solomon Marcus suggested investigating an analogous classification for sets of the first category and for sets of Jordan measure zero, for instance. The answer to analogous Stoilow's conjecture for these classifications may be obtained by means of some examples presented above.

Section 2 contains examples of uncountable sets which have infinite Stoilow dimension with respect to an infinite sequence of scales of the form b^n. However, little is known for multiplicatively-independent scales, like 2 and 3 for instance. Even the answer to the following question seems to be unknown.

Problem 4.4 *Is there an uncountable set of infinite Stoilow dimension with respect to all scales $b \geq 2$?*

Also it would be interesting to investigate analogues of Theorem 2.1, in which Hausdorff dimension is replaced by other classifications of nullsets. Let us mention here the following problem raised in [11]:

Problem 4.5 (S. Marcus). *What relations exist between Stoilow classification of sets of measure zero, on the one hand, and Borel and Fréchet classifications of the same sets, on the other hand?*

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