Real Analysis Exchange Vol 14 (1988-89)

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DARBOUX CONTINUOUS FUNCTIONS ON NON ARCHIMEDEAN VALUED FIELDS

1. INTRODUCTION: It is well known that the only non-discrete locally-compact valued fields other than the fields **R** of real numbers and **C** of complex numbers are the local fields which are the p-adic fields Q_p, their finite extensions and the field of formal Laurent series over the finite field Z_p. Analysis over the fields **R** or **C** has been studied extensively. To have a complete picture therefore it is but natural to consider analysis over valued fields other than **R** or **C**. The valuation of any such field satisfies the following stronger form of the usual triangle inequality.

$$\left| x + y \right| \leq \text{Max } (|x|, |y|)$$

which is known as the ultrametric inequality and the valued fields in which the valuation satisfies the above inequality are known as non-archimedean valued fields. In the sequel K denotes a non-trivial non-archimedean complete valued field. This ultrametric inequality causes fascinating deviations from the classical analysis (over **R** or **C**). For example a series Σa_n with a_n in K converges if and only if the n^{th}-term a_n tends to zero in K. Each disc in K is open and closed and each point of a disc is a centre. Geometrically any triangle in K is isosceles, the non-archimedean valued fields cannot be ordered and so on. These and other deviations make the search for analogues and for differences of classical results in non-archimedean analysis interesting. In this note we prove analogues of Bolzano's theorem and Intermediate value theorem in the non-archimedean case for the Darboux Continuous functions.

Of the many properties of real valued continuous functions defined on a compact interval in the real line the Intermediate value property is well-known which itself is an offshoot of Bolzano's theorem (p.85, Apostol [1]). The proofs of those theorems like that of the Mean value Theorem make essential use of the fact that the real field is ordered. On the other hand in the p-adic fields or more generally non-archimedean valued fields there is no order compatible with the algebra and the topological structure of the field (see p.128, van Rooij [2]). It is therefore necessary to find a suitable analogue of the notion of betweenness before attempting at an analogue of the above two theorems for the non-archimedean case. In [3] and [4] Schikhof has made an attempt to define such a notion of betweenness, monotonic functions etc. in the non-archimedean analysis and we make use of some of these notions in this note.

2. **DEFINITIONS AND MAIN THEOREMS:** We now recall some definitions and results from Schikhof [3].

Let x,y be elements of K. The smallest ball in K containing x and y is denoted by [x,y]. It then follows easily that for all x,y in K, [x,y] = [y,x] and

$$z \in [x,y] \iff |z-x| \leq |x-y| \iff z = \alpha x+(1-\alpha)y$$

for some $\alpha \in K$, $|\alpha| \leq 1$. If $x \neq y$ then $\alpha = (z-y)/(x-y)$.

DEFINITION 1. A subset C of K is called convex if x and y in C implies $[x,y] \subset C$.

It is easy to see that the empty set, singletons, balls and the whole space K are convex sets and they are the only convex sets of K.

DEFINITION 2. Let X be a subset of K. A set $C \subset X$ is called convex in X (or relatively convex) if x,y in C implies that $[x,y] \cap X \subset C$ or equivalently C is the intersection of X with a convex subset of K.

The relation $\sim^*$ defined on K^*-the non-zero elements of K by $x \sim y$ if 0 does not belong to [x,y] is an equivalence relation. It is easy to see that $K^+ = \{x \in K : |1-x| < 1\}$ is a multiplicative subgroup of the commutative group K^* and

(1) $0 \notin [x,y] \iff |x-y| < |x| \iff \left|xy^{-1}-1\right| < 1 \iff \left|x^{-1}y-1\right| < 1.$

$x \sim y$ implies $|x| = |y|$ but not conversely.

It may be noted that the relation $\sim$ defined on the non-zero reals R^* in the above manner where $[x,y]$ is the smallest closed interval containing x and y is also an equivalence relation and $x \sim y$ means x and y have the same sign i.e. both positive or both negative. The equivalence relation partitions R^* into two equivalence classes namely positive and negative real numbers. In such a case $x \nsim y$ implies that $xy < 0$. But in the non-archimedean case there are more than two equivalence classes defined by the equivalence relation $\sim$.

One usually calls a real vlaued function defined on a closed interval having the intermediate value property as Darboux function or Darboux continuous function. This definition of Darboux function requires that the image of a connected set be connected. The only connected subsets fo the real line are intervals which are also convex sets. Motivated by this, Darboux continuity is defined in the non-archimedean case as follows:

DEFINITION 3. Let X be a subset of K and $f: X \longrightarrow K$ be a function defined on X with values in K. f is called weakly Darboux continous if for every relatively convex set C in X the set f(C) is convex in f(X). f is called Darboux continuous if for every relatively convex set C in X, the set f(C) is convex in K.

As noted by Schikhof (p.13 [13]) a Darboux continuous function need not be continous and a continuous function need not be Darboux continuous. Now we prove the analogue of Bolzano's Theorem for the non-archimedean case.

THEOREM 1. Lex X be a subset of K and $f : X \longrightarrow K$ be Darboux continuous. Let a, b be in X and $f(x) \sim f(b)$. Then there exists a point c in $[a,b] \cap X$ such that $f(c) = 0$.

Proof: By definition $f(a) \sim f(b)$ implies that 0 belongs to $[f(a),f(b)]$. Since the ball $[a,b]$ is convex in K, $[a,b] \cap X$ is convex in X. Hence by Darboux continuity of f the set $f([a,b] \cap X)$ is convex in K and it contains f(a) and f(b). It follows that $[f(a),f(b)] \subset f([a,b] \cap X)$ because the ball $[f(a), f(b)]$ is the smallest convex set in K containing f(a) and f(b). This implies that 0 belongs to $f([a,b] \cap X)$ and hence there exists a point c in $[a,b] \cap X$ such that $f(c) = 0$.

REMARK. In non-archimedean analysis even continuous functions may not possess this property as the following example shows.

EXAMPLE. Consider the p-adic field Q_5 associated with the prime $p = 5$, with the normalised valuation $| \ |$ on Q_5. Define

$$A = \{ x \in Q_5 \mid |x-1| < \tfrac{1}{5} \}$$

$$B = \{ x \in Q_5 \mid |x-6| < \tfrac{1}{5^2} \}$$

On $A \cup B$ define f by

$$f(x) = \chi_A(x) + 2\, \chi_B(x)$$

where χ_A and χ_B are the characteristic functions of A and B respectively. Then for $x \in A \cup B$, $f(x) \neq 0$ and f is continuous,

$f(1) \nsim f(6)$, but for no point c in $A \cup B$ does $f(c) = 0$.

The following is the analogue of the Intermediate value Theorem (p.85, Apostol [1]) for the non-archimedean case.

THEOREM 2. Lex X be a subset of K and $f : X \longrightarrow K$ be Darboux continuous. Let a, b belong to X and $f(a) \nsim f(b)$. Define

$$S = \{ x \in K ; |x| < \min (|f(a)|, |f(b)|) \}$$

Then for every $\alpha \in S$, there exists a point c in $[a, b] \cap X$ such that $f(c) = \alpha$.

Proof: Without loss of generality we can assume that $|f(a)| \leq |f(b)|$ so that

$$S = \{ x \in K : |x| < |f(a)| \}$$

Since $f(a) \nsim f(b)$, it follows from the discussion following definition 2 above that

$$|f(a) - f(b)| \geq |f(a)| \text{ and } |f(b)|$$

$$\geq \max (|f(a)|, |f(b)|)$$

But by the ultra metric inequality of $|\ |$ we have

$$|f(a) - f(b)| \leq \max (|f(a)|, |f(b)|)$$

and therefore $f(a) \nsim f(b)$ implies that,

$$|f(a)-f(b)| = \max (|f(a)|, |f(b)|) = |f(b)|.$$

Now for any $x \in S$,

$$x \in S \implies |x| < |f(a)| \implies |x-f(a)| = |f(a)| \leq |f(b)|$$

$$= \max(|f(a)|, |f(b)|)$$

implying thereby that x is in $[f(a), f(b)]$ or equivalently $S \subset [f(a), f(b)]$. Since $f(b) \in [f(a), f(b)]$ but $f(b) \notin S$, S is a proper subset of $[f(a), f(b)]$. Further for any α in S, $|\alpha| < |f(a)| \leq |f(b)|$. Defining

$$F(x) = f(x) - \alpha$$

we see that F is Darboux continuous and $F(a) \not\sim F(b)$. Otherwise $F(a) \sim F(b)$ would imply that

$$|F(a)-F(b)| < |F(a)| \quad (= |F(b)|)$$

i.e.

$$|f(a)-f(b)| < |f(a)| \quad (= |f(b)|)$$

which implies that $f(a) \sim f(b)$ contradicting our hypothesis. Thus F satisfies all the conditions of Theorem 1 above and hence there exists a point c in $[a, b] \cap X$ such that $F(c) = 0$.

This completes the proof of Theorem 2.

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Received May 2, 1988