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THE HOMEOMORPHIC TRANSFORMATION OF c-SETS INTO SUPERDENSE SETS

Sets having the property that each point of the set is a bilaterial condensation point of the set are called c-<u>sets</u>. A d-<u>set</u> is a set having metric density 1 at each of its points. In [1] Gorman proves that for every F_σ c-set, $E \subseteq I = [0,1]$, there is a homeomorphism f, of I onto itself such that f(E) is an F_σ d-set. The proof Gorman gives relies heavily on both the Lebesgue Density Theorem and the fact that every measurable set contains an F_σ subset of the same measure. In [2] Lukeš, Malý, and Zajíček define the notion of superdensity and show that this notion does not support a density theorem. Specifically, if $E \subseteq I$, then x is a point of <u>superdensity</u> of E if

$$\lim_{h\to 0^+} \frac{m((x-h,x+h)\cap(I\setminus E))}{h^2} = 0,$$

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where m denotes Lebesgue measure. Lukeš, Malý, and Zajíček then exhibit a set for which it is not true that almost every point of the set is a point of superdensity of the set. It was natural, then, that the present authors during a discussion of superdensity asked whether Gorman's Theorem remains true if the notion of d-set is replaced by the appropriate superdensity analogue. It is the purpose of this note to show that it does. To this end, let $p: I \to I$ be a nondecreasing function with $\lim_{h \to 0^+} p(h) = 0$. Then E is said to be p-dense at x if

$$\lim_{h \to 0^+} \frac{m((x-h,x+h) \cap (I \setminus E))}{p(h)} = 0,$$

and E is a p-set if it is p-dense at each of its points. Our result is then the following.

<u>THEOREM HW</u>. <u>Let</u> E <u>be</u> <u>an</u> F_σ c-<u>set</u> <u>in</u> I, <u>and</u> <u>suppose</u> $p: I \to I$ <u>is</u> <u>nondecreasing</u> <u>and</u> $\lim_{h \to 0^+} p(h) = 0$. <u>Then</u> <u>there</u> <u>is</u> <u>a</u> <u>homeomorphism</u> $H: I \to I$ <u>that</u> <u>transforms</u> E <u>into</u> <u>an</u> F_σ p-<u>set</u>.

In light of Gorman's LEMMAs 3, 4, 5, and 6 and the proof of his theorem, the entire proof of THEOREM HW rests on the existence of a non empty, nowhere dense F_σ p-set and the remainder of the paper is devoted to constructing such a set.

Let $\{\alpha_n\}$ be a sequence of positive numbers such that

$\sum_{n=1}^{\infty} 2^n \alpha_n < 2$ and $\alpha_m \geq \sum_{n=m+1}^{\infty} 2^n \alpha_n$ for $m = 1,2, \ldots$, and

$$\sum_{n=1}^{\infty} 2(2^n-1)p^{-1}(n^2\alpha_n) < \infty.$$

(eg. Let $\alpha_1 = \frac{1}{4}$ and $\alpha_n = \min\{\frac{\alpha_{n-1}}{4^n}, \frac{p(4^{-n-1})}{n^2}\}$ if $n > 1$)

We use this sequence to construct a certain Cantor set (nowhere dense and perfect), C; the desired set is then an appropriate F_σ subset of C. The complement of C is defined inductively as follows.

1. Let I_1 be the open interval of length α_1 centered at $\frac{1}{2}$.
2. Let I_2 and I_3 be open intervals of length α_2 which are centered at the midpoints of the components of $I \setminus I_1$.

In general, if intervals I_k, $k = 2^{i-1}, 2^{i-1}+1, \ldots, 2^i-1$ have been defined at the i^{th} stage, we let I_k, $k = 2^i, 2^i+1, \ldots, 2^{i+1}-1$ denote those open intervals of length α_i which are centered at the midpoints of the components of $I \setminus \bigcup_{k=1}^{2^i-1} I_k$. We let $G = \bigcup_{k=1}^{\infty} I_k$ and $C = I\setminus G$.

Now, if n is fixed and J is an interval, we let J(n) denote that open interval of length $m(J) + 2p^{-1}(n^2\alpha_n)$ which is concentric with J. Then let

$$B_n = \bigcup_{k=1}^{2^n-1} (I_k(n)\setminus I_k) \quad \text{and}$$

$$B = \lim\sup B_n \equiv \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} B_m$$

It follows that $m(B_m) \leq 2(2^m-1)p^{-1}(m^2\alpha_m)$ and consequently,

$$m(B) \leq \operatorname*{limit}_{n\to\infty} \sum_{m=n}^{\infty} 2(2^m-1)p^{-1}(m^2\alpha_m).$$

Because the sequence $\{\alpha_n\}$ satisfies $\sum_{n=1}^{\infty} 2(2^n-1)p^{-1}(n^2\alpha_n) < \infty$ we have $m(B) = 0$. Now, let $C_n = C\setminus B_n$ and define

$$D = \liminf_{n\to\infty} C_n = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} C_m = C\setminus B.$$

It is easy to see that C_n is closed for each n and that D is an F_σ subset of C. Also, as $m(B) = 0$, we have that $m(C) = m(D)$. It is also easy to see that D contains no endpoint of an interval contiguous to C. Let $x_o \in D$. Then $x_o \in \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} C_m$ so there is an N such that $x_o \in \bigcap_{n=N}^{\infty} C_n$. Thus, if $h < 2p^{-1}(N^2\alpha_N)$, then $[x_o,x_o+h)\cap I_n = \emptyset$ for $n = 1,2, \ldots, 2^N-1$. Let k be the least index such that $[x_o,x_o+h)\cap I_k \neq \emptyset$ and suppose $I_k \equiv (a,b)$ is an interval of stage M. Then $M > N$. The definition of M and the construction of C imply that if $n \leq M$ and J is an interval of stage n such that $J \cap [x_o, x_o+h) \neq \emptyset$, then $J = I_k$. Consequently,

$$\frac{m([x_o,x_o+h)\cap G)}{p(h)} \leq \frac{m(I_k) + m(\bigcup_{i \geq 2^M - 1} I_i)}{p(a-x_o)}$$

$$\leq \frac{2(b-a)}{p(a-x_o)} \leq \frac{2\alpha_M}{M^2 \alpha_M} \leq \frac{2}{N^2} .$$

It follows that $m([x_o,x_o+h)\cap G) = o(p(h))$ as $h \to o$. Similarly, $m((x_o-h,x_o]\cap G) = o(p(h))$ as $h \to o$ and so D is superdense at each of its points.

REFERENCES

[1] William J. Gorman III, The homeomorphic transformation of c-sets into d-sets, Proc. Amer. Math. Soc. 17 (1966), 825-830.

[2] Jaroslav Lukeš, Jan Malý, and Ludek Zajíček, Fine topology methods in real analysis and potential theory, Lecture Notes in Mathematics, No. 1189 (1985), Springer-Verlag, Berlin - Heidelberg - New York.

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