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BAIRE MEASURES ON $[0,\Omega)$ AND $[0,\Omega]$

1. Baire and Borel Sets

Let X be the set of all ordinals less than the first uncountable ordinal Ω, and let $\bar{X}$ be $X \cup \{\Omega\}$, each with the order topology. Then X is a locally compact, noncompact, nonmetrizable, first countable T_4-space, and $\bar{X}$ is a compact, nonmetrizable Hausdorff space. Moreover, $\bar{X}$ is the one-point compactification of X and is not first countable at Ω. (See [2], [4].)

Let Y be either X or $\bar{X}$, and let $\mathcal{B}(Y)$ be the σ-algebra of Borel sets in Y, that is, the σ-algebra generated in Y by the open subsets of Y. Let $C(Y)$ be the system of continuous real-valued functions defined on Y, and let $C_c(Y)$ be the system of $f \in C(Y)$ with compact support. Let $\mathcal{B}_0(Y)$ be the σ-algebra generated in Y by $C(Y)$, called the σ-algebra of Baire sets in Y.

It follows from [3, Problem 10, p. 231], together with [1, Lemma 1, p 195], that

$\mathcal{B}(X) = \{A \subset X$: A or $X-A$ contains an unbounded closed subset of $X\}$,

$\mathcal{B}(\bar{X}) = \{A \subset \bar{X}$: A or $\bar{X}-A$ contains an unbounded closed subset of $X\}$.

Let $\mathcal{T},\mathcal{T}_0$ denote respectively the system of open and open F_σ subsets of X, and let $\bar{\mathcal{T}},\bar{\mathcal{T}}_0$ denote respectively the system of open and open F_σ subsets of $\bar{X}$. Let $\mathcal{F}_0,\bar{\mathcal{F}}_0$ be the system of closed G_δ subsets of X and $\bar{X}$, respectively. It is easy to see that $\mathcal{B}_0(X)$ and $\mathcal{B}_0(\bar{X})$

coincide with the σ-algebras generated by $\mathcal{T}_0$ and $\bar{\mathcal{T}}_0$, respectively. Let $\mathfrak{K}$ denote the system of all countable subsets of X, $\mathfrak{K}' = \{X-A: A \in \mathfrak{K}\}$, and $\mathfrak{K}^c = \{\bar{X}-A: A \in \mathfrak{K}\}$. It is straightforward to show that

$$\mathcal{B}_0(X) = \mathfrak{K} \cup \mathfrak{K}', \quad \mathcal{B}_0(\bar{X}) = \mathfrak{K} \cup \mathfrak{K}^c, \quad \mathcal{B}_0(X) \subsetneq \mathcal{B}(X), \quad \mathcal{B}_0(\bar{X}) \subsetneq \mathcal{B}(\bar{X})$$

and that $\mathcal{T}_0 = \mathcal{T} \cap \mathcal{B}_0(X)$, $\bar{\mathcal{T}}_0 = \bar{\mathcal{T}} \cap \mathcal{B}_0(\bar{X})$.

If $\mathcal{K}$ denotes the system of compact subsets of X, then $\mathcal{K} \subset \mathfrak{K}$ and $\mathcal{K} \subset \mathcal{F}_0 \cap \bar{\mathcal{F}}_0$. Note that $\mathcal{B}_0(X)$ coincides with the σ-algebra generated by $\mathcal{K}$ or, equivalently, $C_c(X)$.

2. Baire and Borel Measures on X

Let Y be either X or $\bar{X}$, let $\mathcal{A}$ be a σ-algebra of subsets of Y such that $\mathcal{B}_0(Y) \subset \mathcal{A}$, and let μ be a measure defined on $\mathcal{A}$. The measure μ is said to be *outer regular* or *inner regular* or *inner semi-regular* at $A \in \mathcal{A}$, depending on whether

$$\mu(A) = \inf\{\mu(G): A \subset G, \ G \text{ open}, \ G \in \mathcal{A}\}$$

or

$$\mu(A) = \sup\{\mu(C): C \subset A, \ C \text{ compact}, \ C \in \mathcal{A}\}$$

or

$$\mu(A) = \sup\{\mu(F): F \subset A, \ F \text{ closed}, \ F \in \mathcal{A}\}.$$

The measure μ is said to be *regular* or *semi-regular* at $A \in \mathcal{A}$ if it is inner and outer regular or inner semi-regular and outer regular at A. We say that μ is inner regular (outer regular, regular, semi-regular) if it is inner regular (outer regular, semi-regular) at every set in $\mathcal{A}$. A *Baire (Borel) measure* μ on Y is any measure on $\mathcal{B}_0(Y)$ $[\mathcal{B}(Y)]$ such that $\mu(C) < \infty$ for every compact subset C of Y. For any Baire (Borel) measure μ on $\bar{X}$, since $\mu(\bar{X}) < \infty$, the notions of outer regularity,

regularity and semi-regularity for μ coincide. Define

$$D(\mu) = \{x \in Y: \mu(\{x\}) > 0\}, \quad C(\mu) = \{x \in Y: \mu(\{x\}) = 0\} .$$

LEMMA 1. Let $\mathscr{A}$ be a σ-algebra of subsets of X such that $\mathscr{B}_0(X) \subset \mathscr{A}$, and let μ be a measure on $\mathscr{A}$. Then the following assertions are equivalent:

(i) $\mu(K) < \infty$ for every compact subset K of X.

(ii) $\mu(A) < \infty$ for every countable subset A of X.

(iii) The set $D(\mu)$ is countable and $\mu(D(\mu)) < \infty$.

PROOF. (i) $\Rightarrow$ (ii): Suppose (i) holds. For any $A \in \mathscr{R}$, let $a = \sup A$. Since the interval $[0,a]$ is compact and contains A, we get $\mu(A) < \infty$. Therefore (ii) holds.

(ii) $\Rightarrow$ (iii): Suppose that (ii) holds and that $D(\mu) \neq \phi$. To prove (iii), it is enough to show that for each $n=1,2,\ldots,$ the set $E_n = \{x: \mu(\{x\}) \geq 1/n\}$ is finite. Suppose E_n is an infinite set for some n. Let A be a countably infinite subset of E_n. Then $\mu(A) = \infty$. On the other hand, we have from (ii) that $\mu(A) < \infty$, a contradiction.

(iii) $\Rightarrow$ (i): Suppose (iii) holds. For any compact subset K of X we get $\mu(K) = \mu(K \cap D(\mu)) < \infty$. Therefore (i) holds. $\square$

Let μ be any Baire (Borel) measure on X. By Lemma 1 we get $D(\mu) \in \mathscr{R}$ and $C(\mu) = X - D(\mu) \in \mathscr{R}'$. The measure μ is called *purely discontinuous* if $\mu(X) = \mu(D(\mu)) > 0$ and *continuous* if $\mu(\{x\}) = 0$ for all $x \in X$. For each $x \in X$, let ϵ_x be the measure defined on the power set of X such that $\epsilon_x(A) = 1$ if $x \in A$ and $\epsilon_x(A) = 0$ if $x \notin A$. Let ν be the Baire (Borel) measure defined by $\nu(A) = \mu(A \cap D(\mu))$. If $D(\mu)$ is void, then $\nu=0$, the zero measure. Otherwise ν is a purely discontinuous measure of the form $\Sigma_n \mu(\{x_n\})\epsilon_{x_n}$, where $\{x_1, x_2, \ldots\}$ is an enumeration of $D(\mu)$ such that $x_i \neq x_j$ if $i \neq j$.

Let γ be the continuous Baire measure defined by $\gamma(A) = 0$ if $A \in \mathfrak{R}$ and $\gamma(A) = 1$ if $A \in \mathfrak{R}'$. For each $p \in [0,\infty]$, define the continuous Baire measure $p\gamma$ on X by setting $(p\gamma)(A) = p\gamma(A)$. We adopt the convention that $\infty \cdot 0 = 0$.

THEOREM 1. Every Baire measure μ on X is semi-regular and can be expressed in exactly one way in the form

$$\mu = \nu + p\gamma ,$$

where ν is either 0 or a purely discontinuous Baire measure and $0 \leq p \leq \infty$.

PROOF. For each $A \in \mathfrak{B}_0(X)$ let $\nu(A) = \mu(A \cap D(\mu))$ and $\lambda(A) = \mu(A \cap C(\mu))$. Then ν is either 0 or a purely discontinuous Baire measure, and λ is a continuous Baire measure. Since $\lambda(A) = 0$ for $A \in \mathfrak{R}$ and $\lambda(A) = \mu(C(\mu))$ for $A \in \mathfrak{R}'$, we get $\lambda = p\gamma$, where $p = \mu(C(\mu))$. Consequently $\mu = \nu + p\gamma$. It is obvious that such a decomposition of μ is unique.

Plainly ν is regular. To prove μ is semi-regular, it is enough to show that λ is semi-regular. Suppose $p > 0$. We show readily that λ is semi-regular but not inner regular at every $A \in \mathfrak{R}'$. Suppose $A \in \mathfrak{R}$ and $a = \sup A$. Since $\lambda(A) = 0$, λ is inner regular at A. Since $[0,a]$ is an open set such that $A \subset [0,a]$ and $\lambda([0,a]) = 0$, λ is outer regular at A. Consequently λ is regular at every $A \in \mathfrak{R}$ and hence λ is semi-regular. $\square$

For each $A \in \mathfrak{B}(X)$, let $\delta(A) = 1$ or 0, depending on whether A or

X-A contains an unbounded closed subset of X. It follows that δ is a Borel measure on X. For each $p \in [0,\infty]$, let $p\delta$ be the Borel measure on X defined by $(p\delta)(A) = p\delta(A)$.

LEMMA 2. Let μ be a measure on $\mathcal{B}(X)$. Then μ is continuous and semi-regular if and only if $\mu = p\delta$ for some $p \in [0,\infty]$.

PROOF. Let $p \in [0,\infty]$. It is straightforward to show that $p\delta$ is a continuous Borel measure which is semi-regular at every $A \in \mathcal{B}(X)$ with $\delta(A) = 1$ and is regular at every $A \in \mathcal{B}(X)$ with $\delta(A) = 0$.

Suppose μ is continuous and semi-regular. Plainly μ is a Borel measure. Let C be any unbounded closed subset of X. For every open set G with $C \subset G$, X-G is compact so that $\mu(X\text{-}G) = 0$ and $\mu(G) = \mu(X)$. Let $p = \mu(X)$. Since μ is outer regular, we get $\mu(C) = p$. Since $\mu(K) = 0$ for every compact (bounded closed) set K, we get $\mu(F) = p\delta(F)$ for every closed subset F of X. Since both μ and $p\delta$ are inner semi-regular, we get $\mu(A) = p\delta(A)$ for all $A \in \mathcal{B}(X)$. □

THEOREM 2. Every semi-regular Borel measure μ on X can be expressed in exactly one way in the form

$$\mu = \nu + p\delta \ ,$$

where ν is either 0 or a purely discontinuous Borel measure, and $0 \leq p \leq \infty$.

PROOF. Define the Borel measures ν and λ on X by setting $\nu(A) = \mu(A \cap D(\mu))$ and $\lambda(A) = \mu(A \cap C(\mu))$. Then ν is either 0 or purely discontinuous, and λ is continuous. To prove λ is semi-regular, let A be any Borel set and $a = \sup D(\mu)$. It follows that

$C(\mu) = C(\mu) \cap [0,a] \cup [a+1,\Omega)$ and $\lambda(A) = \mu(A \cap [a+1,\Omega))$. If $\lambda(A) = \infty$, then λ is outer regular at A. Suppose $\lambda(A) < \infty$. Since μ is outer regular, there is, for any $\epsilon > 0$, an open set U such that

$$A \cap [a+1,\Omega) \subset U \subset [a+1,\Omega), \qquad \mu(U) < \lambda(A) + \epsilon .$$

If we define $G = [0,a] \cup U$, then G is open, $A \subset G$, and

$$\lambda(G) = \mu(U) < \lambda(A) + \epsilon .$$

Consequently λ is outer regular at A. To prove λ is inner semi-regular at A, suppose $\lambda(A) > 0$. Since μ is inner semi-regular, there is, for any $t < \lambda(A)$, a closed subset F of X such that

$$F \subset A \cap [a+1,\Omega], \quad t < \mu(F) = \lambda(F).$$

Let C be the closed set defined by $C=F$ if $A \subset [a+1,\Omega)$ and $C = \{x\} \cup F$ if $A \cap [0,a] \neq \phi$, where $x \in A \cap [0,a]$. Then we have $C \subset A$ and $\lambda(C) = \lambda(F) > t$ so that λ is inner semi-regular at A. Consequently λ is semi-regular and hence by Lemma 2, $\lambda = p\delta$ where $p = \lambda(X)$. Therefore $\mu = \nu + p\delta$ and the uniqueness of such a decomposition of μ follows easily. $\square$

COROLLARY. For any nonzero Baire (Borel) measure μ on X, μ is regular if and only if μ is purely discontinuous.

LEMMA 3. Every continuous, finite Borel measure μ on X is semi-regular.

PROOF. By a minor modification of the proof of [1, Lemma 3, p 197] we get $\mu = p\delta$ where $0 \leq p < \infty$. By Lemma 2 the measure μ is semi-regular. $\square$

The next theorem follows from Theorem 2, together with Lemma 3.

THEOREM 3. Every finite Borel measure on X is semi-regular.

Since the Borel measure δ is an extension of the Baire measure γ, the following extension theorem follows from Theorems 1 and 2.

THEOREM 4. Every Baire measure on X is uniquely extended to a semi-regular Borel measure on X.

We close this section with two examples of non semi-regular infinite Borel measures.

EXAMPLES

1. For each $A \in \mathcal{B}(X)$, let $\tau(A) = 0$ if A is countable and $\tau(A) = \infty$ if A is uncountable. Plainly τ is a continuous, infinite Borel measure which is outer regular at each uncountable Borel set. It is easy to show that τ is regular at each countable Borel set. Let Y be the set of all limit ordinals in X, and let Z be the set of all nonlimit ordinals in X. Then Y is an uncountable closed set and Z is an uncountable open set. Since Z does not contain any unbounded closed subset of X and since $\tau(Z) = \infty$, τ is not inner semi-regular at Z. Therefore τ is outer regular but not inner semi-regular. Note that τ is a Borel extension of the Baire measure $\infty\,\gamma$.

2. Notation is as in Example 1. Let μ be the Borel measure on X defined by $\mu(A) = \delta(A \cap Y) + \tau(A \cap Z)$. Then μ is a continuous Borel measure such that $\mu(Y) = 1$ and $\mu(Z) = \infty$. For any open set G such that $Y \subset G$, the set X-G is compact and the set $G \cap Z$ is unbounded open so that $\mu(G) = \infty$. Consequently μ is not outer regular at Y. Plainly μ is not inner semi-regular at Z. Therefore μ is neither outer regular nor inner semi-regular.

3. Baire Measures on $\bar{X}$

Let $\bar{\gamma}$ be the Baire measure on $\bar{X}$ defined by $\bar{\gamma}(A) = 0$ if $A \in \mathfrak{K}$ and $\bar{\gamma}(A) = 1$ if $A \in \mathfrak{K}^c$. For each $p \in [0,\infty)$, let $(p\bar{\gamma})(A) = p\bar{\gamma}(A)$ for $A \in \mathfrak{B}_0(\bar{X})$. We show readily that each $p\bar{\gamma}$ is a regular Baire measure on $\bar{X}$.

THEOREM 5. Every Baire measure μ on $\bar{X}$ is regular and can be expressed in exactly one way in the form

$$\mu = \nu + p\bar{\gamma} ,$$

where ν is a Baire measure concentrated on a countable subset of X and $0 \le p < \infty$.

PROOF. Let $Y = \{x \in X: \mu(\{x\}) > 0\}$ and $Z = \bar{X} - Y$. Since $\{\Omega\}$ is not a Baire subset of $\bar{X}$, we get $Y = D(\mu) \in \mathfrak{K}$, $Z \in \mathfrak{K}^c$, and $Z = \{x \in X: \mu(\{x\}) = 0\} \cup \{\Omega\}$. If $a = \sup Y$, then $Z = Z \cap [0,a] \cup [a+1,\Omega]$ and $\mu(Z) = \mu([a+1,\Omega])$. For each $A \in \mathfrak{B}_0(\bar{X})$, let $\nu(A) = \mu(A \cap Y)$ and $\lambda(A) = \mu(A \cap Z)$. Then ν is concentrated on the countable set Y. Plainly $\lambda(A) = 0$ for all $A \in \mathfrak{K}$. If $A \in \mathfrak{K}^c$, then $\lambda(A) = \mu(Z)$. Consequently $\lambda = p\bar{\gamma}$ where $p = \mu(Z)(<\infty)$. Therefore $\mu = \nu + p\bar{\gamma}$ and, since both ν and $p\bar{\gamma}$ are regular, μ is regular. It is clear that such a decomposition of μ is unique. □

For each $A \subset \bar{X}$, let $\epsilon_\Omega(A) = 1$ if $\Omega \in A$ and $\epsilon_\Omega(A) = 0$ if $\Omega \notin A$.

THEOREM 6. Every Baire measure μ on $\bar{X}$ is uniquely extended to a regular Boarel measure σ on $\bar{X}$. Moreover, the measure σ is of the form

$$\sigma = \Sigma_n \mu(\{x_n\})\epsilon_{x_n} + p\epsilon_\Omega ,$$

where $\{x_1, x_2, \ldots\}$ is a countable subset of X, and $0 \le p < \infty$.

PROOF. Let $\mu = \nu + p\bar{\gamma}$ be the decomposition of μ as in Theorem 5. Then there exists a countable subset $\{x_1, x_2, \ldots\}$ of X such that $\nu = \Sigma_n \mu(\{x_n\})\epsilon_{x_n}$. It follows readily that $p\epsilon_\Omega$ is a regular Borel measure on $\bar{X}$ which extends the Baire measure $p\bar{\gamma}$. Consequently $\sigma = \Sigma_n \mu(\{x_n\})\epsilon_{x_n} + p\ \epsilon_\Omega$ is a regular Borel measure which extends μ.

Suppose τ is any regular Borel measure on $\bar{X}$ which extends μ. Let $Y = \{x \in X: \mu(\{x\}) > 0\}$ and $Z = \bar{X}\text{-}Y$. Define $\tau_1(A) = \tau(A \cap Y)$, $\tau_2(A) = \tau(A \cap Z)$ where $A \in \mathcal{B}(\bar{X})$. Since $\tau(\{x\}) = \mu(\{x\})$ for all $x \in X$, we get $\tau_1 = \nu$. By a minor modification of an argument given in the proof of Theorem 2, we show that τ_2 is outer regular. Let G be any open subet of $\bar{X}$ such that $\Omega \in G$. By definition $G = \bar{X}\text{-}K$, where K is a compact subset of X. It follows that $\tau(K \cap Z) = \mu(K \cap Z) = 0$ so that $\tau_2(G) = \tau_2(\bar{X}) = p$, where $p = \mu(Z)$. Since τ_2 is outer regular, we get $\tau_2(\{\Omega\}) = p$ so that $\tau_2 = p\epsilon_\Omega$. Therefore $\sigma = \tau$. □

For each $A \in \mathcal{B}(\bar{X})$, let $\bar{\delta}(A) = 1$ or 0, depending on whether A or $\bar{X}\text{-}A$ contains an unbounded closed subset of X. Then $\bar{\delta}$ is a Borel measure on $\bar{X}$ which is not outer regular at Ω (see [3, p. 231]). Plainly $\bar{\delta}$ extends the Baire measure $\bar{\gamma}$. Consequently every Baire measure μ on $\bar{X}$ can be extended to a Borel measure ν of the form

$$\Sigma_n \mu(\{x_n\})\epsilon_{x_n} + p\bar{\delta} ,$$

where $\{x_1, x_2, \ldots\}$ is a countable subset of X, and $0 \le p < \infty$. Note that ν is not regular if $p > 0$. It is known [1, Theorem, p. 197] that every Borel measure μ on $\bar{X}$ is of the form

$$\nu + p\bar{\delta} \ ,$$

where ν is a Borel measure concentrated on a countable subset of $\bar{X}$, and $0 \leq p < \infty$.

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