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Iterates of almost continuous functions and Sarkovskii's Theorem

A function $f : X \to Y$ is *almost continuous* if for each open set $D \subseteq X \times Y$ such that $f \subseteq D$ there exists a continuous $g : X \to Y$ such that $g \subseteq D$. (I make no distinction between a function and its graph.) f is a *connectivity function* if the restriction, $f|_C$, of f to C is connected whenever C is connected. In case $X = Y = \Re$, where $\Re$ denotes the real numbers, an almost continuous function must be a connectivity function and a function is a connectivity function if and only if its graph is connected.

Professor William Transue has asked me if Sarkovskii's Theorem might hold for connectivity functions. Sarkovskii's Theorem states that if $f : \Re \to \Re$ is continuous and f has a point of prime period k, then f also has points of prime period n, where n follows k in the Sarkovskii ordering given by $3, 5, \ldots, 6, 10, \ldots, \ldots 4, 2, 1$. Since Sarkovskii's Theorem is closely related to the fixed point property, one might well guess that this is true. Our Theorem 1 shows that it is in fact false.

It is well known that almost continuous and connectivity functions are ill-behaved relative to functional composition. In [**2**] I proved that if J and I are n-cells then there exist almost continuous functions $f : J \to I$ and $g : I \to J$ such that $g \circ f : J \to J$ has no fixed point, and hence is not almost continuous. In case J and I are both an interval, it follows that f and g are connectivity functions while $g \circ f$ is not. Ceder [**1**] has recently given a similar example. Theorem 2 of the present paper is a stronger result in that it shows that the composition of an almost continuous function with itself can fail to be a connectivity function. Note that the proof of Theorem 2 does not use the Continuum Hypothesis.

Suppose $f : \Re \to \Re$ and K is a closed subset of the plane such that $f \cap K = \emptyset$ and $g \cap K \neq \emptyset$ whenever $g : \Re \to \Re$ is continuous. The set K is called a *blocking set* of f. Clearly f is almost continuous if and only if it has no blocking set. In [**3**] it is proved that every blocking set of a function from $\Re$ into $\Re$ contains an irreducible blocking set and that the X-projection of an irreducible blocking set is non-degenerate and connected. Since a blocking set must intersect each constant function, its Y- projection must be $\Re$. Thus, if $f : \Re \to \Re$ intersects each closed set in the plane both of whose projections have the cardinality of $\Re$, then f is almost continuous.

THEOREM 1. *There exists an almost continuous function $f : \Re \to \Re$ which has a point of prime period 3 but for each x, either $x = f(x)$, f is of prime period 3 at x, or $x \neq f^n(x)$ for all $n > 1$.*

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PROOF: First let $f(0) = 1$, $f(1) = 2$ and $f(2) = 0$. Also for each integer $n > 2$, let $f(n) = n + 1$. We will complete the definition of f by transfinite induction.

Let $\mathcal{K}$ be the set of all closed subsets K of the plane such that both the X-projection and the Y-projection of K have the cardinality, c, of the real line. We will construct f in such a way that $f \cap K \neq \emptyset$ for each $K \in \mathcal{K}$.

By indexing $\mathcal{K}$ with the cardinal c, we can well-order
$\mathcal{K}$ as $K_1, K_2, \ldots, K_\alpha, \ldots$ so that each
K_α is preceded by fewer than c-many members of $\mathcal{K}$. Assume that for some $\alpha < c$ we have defined $f(x)$ for $\max\{\aleph_0, \alpha\}$-many points in such a way that:

a. If $f(x)$ has been defined, then $f^n(x)$ has been defined for all $n = 2, 3, \ldots$ so that either $x = f(x)$, $x = f^3(x)$ and $x \neq f^2(x)$, or $x \neq f^n(x)$ for all $n > 1$.

b. For each $\beta < \alpha$, f has been defined at a point x_β so that $(x_\beta, f(x_\beta)) \in K_\beta$.

Let D_α be the set of x for which $f(x)$ has been defined so far. We wish to place a point $(x_\alpha, f(x_\alpha))$ in K_α. We have two cases:

Case 1. Suppose that we can choose a point $(x_\alpha, y_\alpha) \in K_\alpha$ such that neither x_α nor y_α is in D_α. Let $f(x_\alpha) = y_\alpha$ and, unless $x_\alpha = y_\alpha$, let $f(y_\alpha) = 3$.

Case 2. If the point (x_α, y_α) cannot be chosen as in Case 1, it follows that for some y_α in D_α, K_α contains more than α-many points of the form (x, y_α). Let (x_α, y_α) be such a point where $x_\alpha \notin D_\alpha$ and let $f(x_\alpha) = y_\alpha$.

Finally, if for any x, $f(x)$ has not been defined by the above induction, let $f(x) = x$.

If $f : \Re \to \Re$ is almost continuous, then we know that f^2 must be a Darboux function and f^2 cannot be separated by the diagonal. Theorem 2 shows that f^2 can be separated by a non-horizontal line.

THEOREM 2. *There exists an almost continuous function $f : \Re \to \Re$ such that f^2 is separated by the line $y = x + 1$.*

PROOF: The proof is quite similar to that of Theorem 1. Let $\mathcal{K}$ be defined as above, indexed by c. Assume that for some $\alpha < c$ we have defined $f(x)$ for $\max\{\aleph_0, \alpha\}$-many points in such a way that:

a. If $f(x)$ has been defined, then $f^2(x)$ has been defined so that $f^2(x) \neq x + 1$.

b. For each $\beta < \alpha$, f has been defined at a point x_β so that $(x_\beta, f(x_\beta)) \in K_\beta$.

Let D_α be the set of x for which $f(x)$ has been defined so far. Again, we wish to place a point $(x_\alpha, f(x_\alpha))$ in K_α. We again have two cases:

Case 1. Suppose that we can choose a point $(x_\alpha, y_\alpha) \in K_\alpha$ such that neither x_α nor y_α is in D_α. Let $f(x_\alpha) = y_\alpha$ and, unless $x_\alpha = y_\alpha$, let $f(y_\alpha) = x_\alpha$.

Case 2. If the point (x_α, y_α) cannot be chosen as in Case 1, it follows that for some y_α in D_α, K_α contains more than α-many points of the form (x, y_α). Let (x_α, y_α) be such a point where $x_\alpha \notin D_\alpha$ and $x_\alpha \neq f(y_\alpha) + 1$. Let $f(x_\alpha) = y_\alpha$.

If for any x, $f(x)$ has not been defined by the above induction, let $f(x) = x$.

By the construction of f, $f^{-1}(y)$ is dense in $\Re$ for each y, from which it follows that f^2 is a dense subset of the plane. Thus f^2 is separated by $y = x + 1$.

Questions

1. Is every Darboux function $\Re \to \Re$ the composition of two almost continuous functions?

2. Does Sarkovskii's Theorem hold for Darboux functions of Baire Class 1?

References

1. J. Ceder, *On compositions with connected functions*, Real Analysis Exchange **11, No. 2** (1985-86), 380–389.
2. K. R. Kellum, *Almost continuous functions on I^n*, Fund. Math. **79** (1973), 213–215.
3. K. R. Kellum, *Sum and limits of almost continuous functions.*, Colloq. Math. **31** (1974), 125–128.

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