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CLASSIFYING THE SET WHERE A BAIRE 1 FUNCTION IS APPROXIMATELY CONTINUOUS

The aim of this paper is to give full characterization of the set of points at which a Baire 1 function $f:(a,b)\to R$ is approximately continuous. We show that this characterization coincides with the characterization of the set of points at which a derivative is approximately continuous, which is the problem mentioned in monograph [2].

Throughout this paper $\mathcal{B}_1$ is the family of all Baire 1 functions, Δ the family of all derivatives and C the family of all continuous functions. The set A_f is the set of points at which the function f is not approximately continuous and $\lambda(E)$ the Lebesque measure of the set E.

We first need the concept of density of a set at a point. Let A be a measurable subset of R and let $x_0 \in R$. The number

$$\bar{d}(x_0,A) = \limsup_{h\to 0} \frac{1}{2h}\lambda(A\cap [x_0-h,x_0+h])$$

is called the upper density of A at x_0. The lower density $\underline{d}(x_0,A)$ is defined analogously. If $\bar{d}(x_0,A) = \underline{d}(x_0,A)$, we call this number the density of A at x_0 and denote it by $d(x_0,A)$.

It is easy to verify that the following lemmas hold.

Lemma 1. Let $\{I_n\}_{n=1}^{\infty}$ be a sequence of intervals such that $\lambda(I_n)\to 0$ and $x_0\in \bar{I}_n$ for every n=1,2,... . If for a measurable set B $\lambda(B\cap I_n)>\frac{1}{m}\lambda(I_n)$ for every n, then $\bar{d}(x_0,B)\geq\frac{1}{2m}$.

Lemma 2. Let $U = \bigcup_{n=1}^{\infty} I_n$, where I_n are open intervals,

$\lambda(U) < +\infty$ and B be a measurable set. If $\lambda(B \cap I_n) > \frac{1}{m}\lambda(I_n)$ for every n, then $\lambda(B \cap U) \geq \frac{1}{2m}\lambda(U)$.

Lemma 3. Let $\{I_n\}_{n=1}^{\infty}$ be a sequence of pairwise disjoint intervals such that $x_0 \notin \bigcup_{n=1}^{\infty} \bar{I}_n$ and $\bar{d}(x_0, \bigcup_{n=1}^{\infty} I_n) > 0$. If for a measurable set B $\lambda(B \cap I_n) > \frac{1}{m}\lambda(I_n)$ for every $n = 1, 2, \ldots$, then $\bar{d}(x_0, B) > 0$.

Definition 1 ([2]). A function $f:(a,b) \to R$ is said to be approximately continuous at an $x \in (a,b)$ if for each given $\mathcal{E} > 0$ the set $A(x, \mathcal{E}) = \{t, |f(t) - f(x)| < \mathcal{E}\}$ has the density 1 at x; that is $\underline{d}(x, A(x, \mathcal{E})) = \bar{d}(x, A(x, \mathcal{E})) = 1$.

We give Corollary 3.11 ([3]) as our Theorem 1.

Theorem 1. If a function $f \in \mathcal{B}_1$ and $\lambda(E) = 0$, then there exists an approximately continuous function g such that $f(x) = g(x)$ for every $x \in E$.

Because any Baire 1 function is approximately continuous almost everywhere ([2], page 19), without loss of generality we may assume that

(1) $$f(x) = 0 \text{ for every } x \in A_f.$$

Theorem 2. If the function $f \in \mathcal{B}_1$, then the set A_f is of type $G_{\delta\sigma}$.

Proof. Let $A_{n,k,l}$ be the set of all $x_0 \in (a,b)$ for which there exist an open interval $I^n(x_0)$ such that

(2) $$\lambda(\{x \in I^n(x_0), |f(x)| \geq \tfrac{1}{k}\}) > \tfrac{1}{l}\lambda(I^n(x_0)),$$

$x_0 \in I^n(x_0)$ and $\lambda(I^n(x_0)) < \frac{1}{n}$. We denote $N_f = \{x, f(x) = 0\}$. Using (1), (2) and Lemma 1 we obtain $A_f = \bigcup_{l=1}^{\infty} \bigcup_{k=1}^{\infty} (N_f \cap \bigcap_{n=1}^{\infty} A_{n,k,l})$ and moreover,

(3) $$\bar{d}(x_0, \{x, |f(x)| \geq \tfrac{1}{k}\}) \geq \tfrac{1}{2l} \text{ for every } x_0 \in N_f \cap \bigcap_{n=1}^{\infty} A_{n,k,l}.$$

Because $\bigcap_{n=1}^{\infty} A_{n,k,l} = \bigcap_{n=1}^{\infty} \bigcup\{I^n(x),\ x \in \bigcap_{n=1}^{\infty} A_{n,k,l}\}$ and the set N_f ([2], page 1) are sets of type G_δ, the set A_f is of type $G_{\delta\sigma}$.

In the following part we shall deal with some properties of the set A_f.

Definition 2 ([1]). A function f is a member of the family $[C]$ if and only if there are functions $g_i \in C$, $i=1,2,\ldots$, and closed sets K_i such that $\bigcup_{i=1}^{\infty} K_i = R$ and $f(x) = g_i(x)$ for every $x \in K_i$.

The paper [1] contains Theorem 3 below.

Theorem 3. Let $f \in \mathcal{B}_1$. Then there are $f_n \in [C]$, $n=1,2,\ldots$, such that $f_n \to f$ uniformly.

For a given $f \in \mathcal{B}_1$ and for every $k=1,2,\ldots$ we choose a function $g_k \in [C]$ such that $|f(x) - g_k(x)| < \frac{1}{4k}$ for every x. Let K_i^k, $i=1,2,\ldots$, be closed sets, $\bigcup_{i=1}^{\infty} K_i^k = (a,b)$ and g_{k/K_i^k} is the continuous function. We set $A_{k,l}^i = \bigcap_{n=1}^{\infty} A_{n,k,l} \cap N_f \cap K_i^k$.

Lemma 4. Let $U \supset A_{k,l}^i$ be an open set. Then there is an open set U' such that $U \supset U' \supset A_{k,l}^i$ and for each of its component, T_s,

$$\lambda(T_s \cap (\sim \bar{A}_{k,l}^i)) \geq \lambda(T_s \cap \{x, |f(x)| \geq \tfrac{1}{k}\}) \geq \tfrac{1}{2l}\lambda(T_s). \tag{4}$$

Proof. The function g_k is continuous on the set $\bar{A}_{k,l}^i \subset K_i^k$ and $|f(x) - g_k(x)| < \frac{1}{4k}$ for every $x \in K_i^k$. Because f is 0 at all $x \in A_{k,l}^i$, the function g_k is bounded by the constant $\frac{1}{4k}$ on the set $\bar{A}_{k,l}^i$. Hence

$$|f(x)| \leq |f(x) - g_k(x)| + |g_k(x)| < \tfrac{1}{2k} \text{ for every } x \in \bar{A}_{k,l}^i. \tag{5}$$

From the definition of the set $A_{k,l}^i$ it is evident that for every $x \in A_{k,l}^i$ we may choose an open interval $I(x) \subset U$ such that $\lambda(\{t \in I(x), |f(t)| \geq \frac{1}{k}\}) > \frac{1}{l}\lambda(I(x))$. Let $U' = \bigcup\{I(x),\ x \in A_{k,l}^i\}$. If T_s is a component of the set U', then according to Lemma 2 $\lambda(\{t \in T_s, |f(t)| \geq \frac{1}{k}\}) \geq \frac{1}{2l}\lambda(T_s)$. Since by (5)

$\{t \in T_s, |f(t)| \geq \frac{1}{k}\} \subset T_s \cap (\sim \bar{A}^i_{k,l})$, we have (4).

The set $A^i_{k,l}$ is of type G_δ and measure 0; that is, $A^i_{k,l} = \bigcap_{n=1}^{\infty} V_n$ where V_n are open sets, $\lambda(V_n) \to 0$. By Lemma 4, there is an open set V'_1 such that $A^i_{k,l} \subset V'_1 \subset V_1$ and for each of its component T^s_1 (4) holds. In every set $T^s_1 \cap (\sim \bar{A}^i_{k,l})$ choose a finite number of closed, pairwise disjoint intervals $J^{s,1}_1, \ldots, J^{s,l(s,1)}_1$ such that $\lambda(\bigcup_{r=1}^{l(s,1)} J^{s,r}_1) \geq \frac{1}{2}\lambda(T^s_1 \cap (\sim \bar{A}^i_{k,l}))$. Because the set $W_1 = V'_1 - \bigcup_s (\bigcup_{r=1}^{l(s,1)} J^{s,r}_1) = \bigcup_s (T^s_1 - \bigcup_{r=1}^{l(s,1)} J^{s,r}_1)$ is open, by Lemma 4 there is an open set V'_2 such that $A^i_{k,l} \subset V'_2 \subset V_2 \cap W_1$ and for each of its component T^s_2 (4) holds. In every set $T^s_2 \cap (\sim \bar{A}^i_{k,l})$ choose a finite number of closed, pairwise disjoint intervals $J^{s,1}_2, \ldots, J^{s,l(s,2)}_2$ such that $\lambda(\bigcup_{r=1}^{l(s,2)} J^{s,r}_2) \geq \frac{1}{2}\lambda(T^s_2 \cap (\sim \bar{A}^i_{k,l})) \geq \frac{1}{4l}\lambda(T^s_2)$. Inductively we may construct a sequence of open sets V'_n satisfing (4), $\bigcap_{n=1}^{\infty} V'_n = A^i_{k,l}$, $\lambda(V'_n) \to 0$ and a sequence of closed, pairwise disjoint intervals $J^{s,r}_n$, $n=1,2,\ldots$, $s=1,2,\ldots$, $r=1,2,\ldots,l(s,n)$ such that $\bigcup_{r=1}^{l(s,n)} J^{s,r}_n \subset T^s_n - V'_{n+1}$ and

$$\lambda\Big(\bigcup_{r=1}^{l(s,n)} J^{s,r}_n\Big) \geq \frac{1}{4l}\lambda(T^s_n) , \tag{6}$$

where T^s_n are components of the set V'_n.

Let $\mathcal{S}$ be a family of sets. Define
$\bar{d}^*(x, \mathcal{S}) = \bar{d}(x, \bigcup\{A \in \mathcal{S}, d(x,A) = 0\})$.
Further we show that the sequence of closed intervals $J^{s,r}_n$, $n=1,2,\ldots$, $s=1,2,\ldots$, $r=1,2,\ldots,l(s,n)$, constructed above satisfies the following Theorem 4.

Theorem 4. $A_f \supset \{x, \bar{d}^*(x, \{J^{s,r}_n\}) > 0\} \supset A^i_{k,l}$.

Proof. Let x_0 be a point of the set $A^i_{k,l}$. Then there is a sequence of components $T^{s(n)}_n$ of sets V'_n such that $x_0 \in \bigcap_{n=1}^{\infty} T^{s(n)}_n$.

Because $\lambda(T_n^{s(n)}) \to 0$, by Lemma 1, (6) implies that $\bar{d}^*(x_o, \{J_n^{s,r}\}) \geq \bar{d}(x_o, \bigcup_{n=1}^{\infty} \bigcup_{r=1}^{l(s(n),n)} J_n^{s(n),r}) \geq \frac{1}{8l}$, which proves the second inclusion.

Now let $\bar{d}^*(x_o, \{J_n^{s,r}\}) > 0$. We shall show that $x_o \in A_f$. Since $\bar{d}(x_o, J_n^{s,r}) > 0$ for one interval $J_n^{s,r}$ at most, we may assume that $\bar{d}(x_o, \bigcup_{n=1}^{\infty} \bigcup_{s=1}^{\infty} \bigcup_{r=1}^{l(s,n)} J_n^{s,r}) > 0$ and $x_o \notin J_n^{s,r}$ for every n,s,r. If for every n there is a component $T_n^{s(n)}$ of V_n' such that $x_o \in \bar{T}_n^{s(n)}$, then $x_o \in \bar{A}_{k,l}^i$ and by (5) $|f(x_o)| < \frac{1}{2k}$. Since for components $T_n^{s(n)}$, n=1,2,..., (4) holds and since $\lambda(T_n^{s(n)}) \to 0$, Lemma 1 implies $\bar{d}(x_o, \{x, |f(x)| \geq \frac{1}{k}\}) \geq \frac{1}{4l}$. From above it follows that $\bar{d}(x_o, \{x, |f(x) - f(x_o)| \geq \frac{1}{2k}\}) \geq \bar{d}(x_o, \{x, |f(x)| \geq \frac{1}{k}\}) \geq \frac{1}{4l}$, e.i. $x_o \in A_f$.

In the opposite case there is n_o such that $x_o \notin \bar{T}_{n_o}^s$ for every s. From the assumption it follows that

$\bar{d}(x_o, \bigcup_{n=n_o}^{\infty} \bigcup_{s=1}^{\infty} \bigcup_{r=1}^{l(s,n)} J_n^{s,r}) > 0$ or $\bar{d}(x_o, \bigcup_{s=1}^{\infty} \bigcup_{r=1}^{l(s,n)} J_{k_o}^{s,r}) > 0$ for some $k_o \in \{1,2,\ldots,n_o - 1\}$. Therefore $\bar{d}(x_o, \bigcup_{s=1}^{\infty} T_{n_o}^s) > 0$, $x_o \notin \bar{T}_{n_o}^s$ or there is s_o such that $\bar{d}(x_o, \bigcup_{s=s_o}^{\infty} T_{k_o}^s) > 0$, $x_o \notin \bar{T}_{k_o}^s$ for $s \geq s_o$. In both cases $x_o \in \bar{A}_{k,l}^i$ and $|f(x_o)| < \frac{1}{2k}$. Applying Lemma 3 from (4) we obtain $\bar{d}(x_o, \{x, |f(x)| \geq \frac{1}{k}\}) > 0$. Because $\bar{d}(x_o, \{x, |f(x) - f(x_o)| \geq \frac{1}{2k}\}) \geq \bar{d}(x_o, \{x, |f(x)| \geq \frac{1}{k}\})$, it follows that $x_o \in A_f$ which finishes the proof.

Corollary 1. For every set $A_{k,l}^i$ there is an open set V such that $A_f \supset \{x, \bar{d}^*(x, \{T^s\}_{s=1}^{\infty}) > 0\} \supset A_{k,l}^i$, where T^s are the components of the set V.

Proof. It is sufficient to put $V = \bigcup_{n=1}^{\infty} \bigcup_{s=1}^{\infty} \bigcup_{r=1}^{l(s,n)} \operatorname{int} J_n^{s,r}$. The statement of Corollary 1 is obvious, because $\bar{d}^*(x, \{J_n^{s,r}\}) = \bar{d}^*(x, \{\operatorname{int} J_n^{s,r}\})$.

We are now ready to state the main result of this paper.

Theorem 5. If f is a Baire 1 function, then there is a sequence of open sets V_n, n=1,2,..., such that

$$A_f = \bigcup_{n=1}^{\infty}\{x,\ \bar{d}^*(x,\{T_n^s\}_{s=1}^{\infty})>0\}, \tag{7}$$

where T_n^s are the components of V_n, and conversely, for every sequence of open sets V_n, n=1,2,..., there is a Baire 1 function f such that (7) holds.

Proof. Combining Corollary 1 with the fact that $A_f = \bigcup_{k=1}^{\infty}\bigcup_{l=1}^{\infty}\bigcup_{i=1}^{\infty} A_{k,l}^i$, we obtain the proof of the first part of Theorem 5.

Now let $\{V_n\}_{n=1}^{\infty}$ be a sequence of open sets and let T_n^s, s=1,2,..., be the sequence of components of the set V_n. For every n=1,2,... we shall define a function f_n in the following way:

$$f_n(x) = \begin{cases} 0, \text{ if } x\in \sim\bigcup_{s=1}^{\infty} (a_n^s,b_n^s) \\ \sin \dfrac{2^{s+1}\pi(x-a_n^s)}{b_n^s - a_n^s}, \text{ if } x\in(a_n^s,b_n^s), \end{cases}$$

where (a_n^s,b_n^s) is the middle open third of T_n^s. It is easy to compute that $f_n\in\Delta$ and evidently $A_{f_n} = \{x,\ \bar{d}^*(x,\{T_n^s\}_{s=1}^{\infty})>0\}$. The function $f = \sum_{n=1}^{\infty}\frac{1}{4^n}\cdot f_n$ is a derivative ([2],page 17) and therefore a Baire 1 function. Moreover $A_f = \bigcup_{n=1}^{\infty} A_{f_n} =$
$= \bigcup_{n=1}^{\infty}\{x,\ \bar{d}^*(x,\{T_n^s\}_{s=1}^{\infty})>0\}$, which completes the proof of Theorem 5.

In closing we observe that the function f of the proof of Theorem 5 is a derivative. This together with the argument $\Delta\subset\mathcal{B}_1$ yields the following result.

Corollary 2. Theorem 5 is true, if we replace the concept "a Baire 1 function" by "a derivative" .

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